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2-QUASIRECOGNIZABILITY OF THE SIMPLE GROUPS $B_n(p)$ AND $C_n(p)$ BY PRIME GRAPH

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ABSTRACT. Let G be a finite group and let $GK(G)$ be the prime graph of G. We assume that n is an odd number. In this paper, we show that if $GK(G) = GK(B_n(p))$, where $n \geq 9$ and $p \in \{3, 5, 7\}$, then G has a unique nonabelian composition factor isomorphic to $B_n(p)$ or $C_n(p)$. As consequences of our result, $B_n(p)$ is quasirecognizable by its spectrum and also by a new proof, the validity of a conjecture of W. J. Shi for $B_n(p)$ is obtained.

1. Introduction

If G is a finite group, then we denote by $\pi(G)$ the set of all prime divisors of $|G|$ and the *spectrum* $\omega(G)$ of G is the set of elements orders of G, i.e., a natural number n is in $\omega(G)$ if there is an element of order n in G. The *Gruenberg-Kegel graph* (or *prime graph*) $GK(G)$ of G is the graph with vertex set $\pi(G)$ where two distinct vertices p and q are adjacent by an edge (briefly, adjacent) if $pq \in \omega(G)$, in which case, we write $(p, q) \in GK(G)$.

A finite group G is called *recognizable by its spectrum* (briefly, *recognizable*) if every finite group H with $\omega(G) = \omega(H)$ is isomorphic to G. A

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finite simple nonabelian group P is called *quasirecognizable by its spectrum*, if each finite group G with $\omega(G) = \omega(P)$ has a unique nonabelian composition factor isomorphic to P [2].

A finite group G is called *recognizable by its prime graph*, if every finite group H with $GK(G) = GK(H)$ is isomorphic to G. A finite simple nonabelian group P is called *quasirecognizable by its prime graph*, if each finite group G with $GK(G) = GK(P)$ has a unique nonabelian composition factor isomorphic to P [7]. We say that a finite simple nonabelian group P is *2-quasirecognizable by its prime graph*, if each finite group G with $GK(G) = GK(P)$ has a unique nonabelian composition factor isomorphic to P or another simple group Q with $GK(Q) = GK(P)$.

Finite groups G satisfying $GK(G) = GK(H)$ have been determined, where H is one of the following groups: a sporadic simple group [6], a CIT simple group [14], $PSL(2, q)$ where $q = p^\alpha < 100$ [16], $PSL(2, p)$ where $p > 3$ is a prime [15], $G_2(7)$ [26], ${}^2G_2(q)$ where $q = 3^{2m+1} > 3$ [7, 26], $PSL(2, q)$ [8, 10], $L_{16}(2)$ [13, 27]. Also, the quasirecognizability of the following simple nonabelian groups by their prime graphs have been obtained: Alternating group A_p where p and $p-2$ are primes [12], $L_{10}(2)$ [9], ${}^2F_4(q)$ where $q = 2^{2m+1}$ for some $m \geq 1$ [1], ${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime [11], $C_n(2)$ where $n \neq 3$ is odd [4].

Prime graphs of the stated groups have more than two connected components, except the groups $G_2(7)$, $C_n(2)$ where n is an odd prime number and some sporadic simple groups which have two connected components, and the groups $L_{10}(2)$, $L_{16}(2)$ and $C_n(2)$ where n is an odd non-prime number which have connected prime graphs. In this paper, we show that the simple groups $B_n(3)$, $B_n(5)$ and $B_n(7)$ are 2-quasirecognizable by their prime graphs. In fact, we have the following Main Theorem:

Main Theorem. *Let n be an odd number. The simple groups $B_n(p)$, where $n \geq 9$ and $p \in \{3, 5, 7\}$, are 2-quasirecognizable by their prime graphs.*

Since $GK(B_n(p))$ and $GK(C_n(p))$ are coincide (see [24, Proposition 7.5]), the conclusion of the Main Theorem is obtained for the group $C_n(p)$ as well. Moreover, it is worthy to mention that $GK(B_n(5))$ and $GK(B_n(7))$ are always connected and if n is an odd non-prime, then $GK(B_n(3))$ is connected as well and if n is an odd prime, then $GK(B_n(3))$ has two connected components.

It is obvious that $\omega(G)$ determines $GK(G)$ and hence, as the first result of the Main Theorem, we have the following corollary:

Corollary. *Let n be an odd positive integer. The simple groups $B_n(3)$, $B_n(5)$ and $B_n(7)$, where $n \geq 9$, are quasirecognizable by their spectra.*

Of course, for the spacial case, i.e., when n is a prime number, the quasirecognizability of the group $B_n(3)$ by its spectrum is obtained ([17]).
W. J. Shi in [18], put forward the following conjecture:

Conjecture. *Let G be a finite group and let M be a finite simple group. Then $G \cong M$ if and only if*
(i) $|G| = |M|$, and
(ii) $\omega(G) = \omega(M)$.

A series of papers proved that this conjecture is valid for most of finite simple groups (see a survey in [19]) and the last step of the proof of this conjecture is to prove that the conjecture holds for the simple groups $B_n(q)$ and $C_n(q)$. Also, Mazurov and his students just proved that this conjecture is valid for these groups as well and hence, Shi's conjecture is now proved positively [22, 23]. As another corollary of the Main Theorem, by a new proof the validity of this conjecture is obtained for the groups under study.

2. Preliminaries

Throughout this paper, we use the following notations: By $[x]$ we denote the integer part of x and by $gcd(a_1, a_2, \cdots, a_n)$ we denote the greatest common divisor of numbers $a_1, a_2, \cdots, a_n$. A set of vertices of a graph is called a coclique (or independent), if its elements are pairwise nonadjacent. We denote by $\rho(G)$ and $\rho(r, G)$ a coclique of maximal size in $GK(G)$ and a coclique of maximal size, containing r, in $GK(G)$, respectively. Also, we put $t(G) = |\rho(G)|$ and $\mathrm{t}(r, G) = |\rho(r, G)|$.

Lemma 2.1. [20, Proposition 1] *Let G be a finite group, $t(G) \geq 3$, and let K be the maximal normal soluble subgroup of G. Then for every subset ρ of primes in $\pi(G)$ such that $|\rho| \geq 3$ and all primes in ρ are* *pairwise nonadjacent in $GK(G)$, the intersection $\pi(K) \cap \rho$ contains at most one number. In particular, G is insoluble.*

Lemma 2.2. [21, Theorem 1] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the followings hold:*

(1) *There exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/K \leq Aut(S)$ for the maximal normal soluble subgroup K of G.*
(2) *For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K|.|\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*
(3) *One of the following holds:*
  (a) *every prime $r \in \pi(G)$ nonadjacent to 2 in $GK(G)$ does not divide the product $|K|.|\bar{G}/S|$; in particular, $t(2, S) \geq t(2, G)$;*
  (b) *there exists a prime $r \in \pi(K)$ nonadjacent to 2 in $GK(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong A_7$ or $A_1(q)$ for some odd q.*

Lemma 2.3. [24, Proposition 1.1] *Let $G = A_n$ be an alternating group of degree n.*

(1) *Let $r, s \in \pi(G)$ be odd primes. Then r and s are nonadjacent iff $r + s > n$.*
(2) *Let $r \in \pi(G)$ be an odd prime. Then 2 and r are nonadjacent iff $r + 4 > n$.*

Lemma 2.4. *Let G be a finite group. If H is a subgroup of G and N is a normal subgroup of G, then:*

(1) *If $(p, q) \in GK(H)$, then $(p, q) \in GK(G)$;*
(2) *If $(p, q) \in GK(\frac{G}{N})$, then $(p, q) \in GK(G)$;*
(3) *If $(p, q) \in GK(G)$ and $\{p, q\} \cap \pi(N) = \emptyset$, then $(p, q) \in GK(\frac{G}{N})$.*

Proof. The proof is straightforward. □

Let s be a prime and let m be a natural number. The s-part of m is denoted by m_s, i.e., $m_s = s^t$ if $s^t \mid m$ and $s^{t+1} \nmid m$. If q is a natural number, r is an odd prime and $gcd(r, q) = 1$, then by $e(r, q)$ we denote the smallest natural number m such that $q^m \equiv 1 \pmod r$. Obviously by Fermat's little theorem it follows that $e(r, q) \mid (r - 1)$. Also, if $q^n \equiv 1 \pmod r$, then $e(r, q) \mid n$. Therefore, we can use the following function in GAP [5], to compute $e(r, q)$:

```
e:=function(r,q)
local i,a;
      a:=DivisorsInt(r-1);
      for i in a do
           if (q^i-1) mod r=0 then
                               return i;
           fi;
      od;
end;
```

If q is odd, we put $e(2,q)=1$ if $q \equiv 1 \pmod 4$, and $e(2,q)=2$ otherwise.

Lemma 2.5. [25, Corollary to Zsigmondy's theorem] *Let q be a natural number greater than 1. For every natural number m there exists a prime r with $e(r,q)=m$, except for the cases $q=2$ and $m=1$, $q=3$ and $m=1$, and $q=2$ and $m=6$.*

The prime r with $e(r,q)=m$ is called a *primitive prime divisor* of q^m-1. It is obvious that q^m-1 can have more than one primitive prime divisor. We denote by $r_m(q)$ some primitive prime divisor of q^m-1. If there is no ambiguity, we write r_m instead of $r_m(q)$.
We write $A_n^\varepsilon(q)$ and $D_n^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$, and $A_n^+(q) = A_n(q)$, $A_n^-(q) = {}^2A_n(q)$, $D_n^+(q) = D_n(q)$, $D_n^-(q) = {}^2D_n(q)$. Also, $\nu(n)$ and $\eta(n)$ for an integer n, are defined in [24] as follow:

$$\nu(n) = \begin{cases} n \ if \ n \equiv 0 \ (mod\ 4); \\ \frac{n}{2} \ if \ n \equiv 2 \ (mod\ 4); \\ 2n \ if \ n \equiv 1 \ (mod\ 2). \end{cases} \quad , \ \eta(n) = \begin{cases} n \ if \ n \ is \ odd; \\ \frac{n}{2} \ otherwise. \end{cases}$$

Lemma 2.6. *Let G be a finite simple group of Lie type over a field of order q with characteristic p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$.*

(1) *If $G = A_{n-1}(q)$ and $2 \leq k \leq l$, then r and s are nonadjacent if and only if $k + l > n$ and k does not divide l;*
(2) *If $G = {}^2A_{n-1}(q)$ and $2 \leq \nu(k) \leq \nu(l)$, then r and s are nonadjacent if and only if $\nu(k) + \nu(l) > n$ and $\nu(k)$ does not divide $\nu(l)$;*
(3) *If $G = B_n(q)$ or $C_n(q)$ and $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $\eta(k) + \eta(l) > n$ and $\frac{l}{k}$ is not an odd natural number.*
(4) *If $G = D_n^\varepsilon(q)$ and $1 \leq \eta(k) \leq \eta(l)$, then r and s are nonadjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$ and $\frac{l}{k}$ is not an*

odd natural number and, if $\varepsilon = +$, then the chain of equalities $n = l = 2\eta(l) = 2\eta(k) = 2k$, is not true.

(5) *If $G = E_7(q)$ and $1 \leq k \leq l$, then r and s are nonadjacent if and only if $k \neq l$ and either $l = 5$ and $k = 4$, or $l = 6$ and $k = 5$, or $l \in \{14, 18\}$ and $k \neq 2$, or $l \in \{7, 9\}$ and $k \geq 2$, or $l = 8$ and $k \geq 3$, $k \neq 4$, or $l = 10$ and $k \geq 3$, $k \neq 6$, or $l = 12$ and $k \geq 4, k \neq 6$.*

Proof. See [24, Propositions 2.1 and 2.2] and [25, Propositions 2.4; 2.5 and 2.7(5)]. □

Lemma 2.7. [24, Proposition 3.1] *Let G be a finite simple classical group of Lie type of characteristic p and let $r \in \pi(G)$ and $r \neq p$. Then r and p are nonadjacent if and only if one of the following holds:*

(1) *$G = A_{n-1}(q), r$ is odd, and $e(r, q) > n - 2$;*
(2) *$G = {}^2A_{n-1}(q)$, r is odd, and $\nu(e(r, q)) > n - 2$;*
(3) *$G = C_n(q)$, $\eta(e(r, q)) > n - 1$;*
(4) *$G = B_n(q)$, $\eta(e(r, q)) > n - 1$;*
(5) *$G = D^{\varepsilon}_n(q)$, $\eta(e(r, q)) > n - 2$;*
(6) *$G = A_1(q), r = 2$;*
(7) *$G = A^{\varepsilon}_2(q), r = 3$ and $(q - \varepsilon 1)_3 = 3$.*

Lemma 2.8. [24, Proposition 4.1] *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q - 1$, and s be an odd prime number not equal to the characteristic of G. Put $k = e(s, q)$. Then s and r are nonadjacent if and only if one of the following holds:*

(1) *$k = n$, $n_r \leq (q - 1)_r$, and if $n_r = (q - 1)_r$, then $2 < (q - 1)_r$;*
(2) *$k = n - 1$ and $(q - 1)_r \leq n_r$.*

Lemma 2.9. *Let $G = {}^2A_{n-1}(q)$ be a finite simple group of Lie type, r be a prime divisor of $q + 1$, and s be an odd prime number not equal to the characteristic of G. Put $k = e(s, q)$. Then s and r are nonadjacent if and only if one of the following holds:*

(1) *$\nu(k) = n$, $n_r \leq (q + 1)_r$, and if $n_r = (q + 1)_r$, then $2 < (q + 1)_r$;*
(2) *$\nu(k) = n - 1$ and $(q + 1)_r \leq n_r$.*

Lemma 2.10. *Let G be a finite simple group of Lie type over a field of order q with odd characteristic p. Let r be an odd prime divisor of $|G|$, $r \neq p$, and $k = e(r, q)$.*

(1) *If $G = B_n(q)$ or $C_n(q)$, then r and 2 are nonadjacent if and only if $\eta(k) = n$ and one of the following holds:*

(a) *n is odd and $k = (3 - e(2, q))n$;*
(b) *n is even and $k = 2n$.*

(2) *If $G = D^{\varepsilon}_n(q)$, then r and 2 are nonadjacent if and only if one of the following holds:*
(a) *$\eta(k) = n$ and $(4, q^n - \varepsilon 1) = (q^n - \varepsilon 1)_2$;*
(b) *$\eta(k) = k = n - 1$, n is even, $\varepsilon = +$, and $e(2, q) = 2$;*
(c) *$\eta(k) = \frac{k}{2} = n - 1$, $\varepsilon = +$, and $e(2, q) = 1$;*
(d) *$\eta(k) = \frac{k}{2} = n - 1$, n is odd, $\varepsilon = -$, and $e(2, q) = 2$.*

(3) *If $G = E_7(q)$, then r and 2 are nonadjacent if and only if either $k \in \{7, 9\}$ and $e(2,q)=2$ or $k \in \{14, 18\}$ and $e(2, q) = 1$;*

(4) *If $G = E_8(q)$, then r and 2 are nonadjacent if and only if $k \in \{15, 20, 24, 30\}$.*

Proof. See [24, Propositions 4.3; 4.4 and 4.5(5,6)]. □

Remark 2.11. *In order to facilitate the reader, we state the orders of some simple groups and their outer automorphism groups in the following table: (We assume that $q = p^{\alpha}$)* [3]

Table 1

G	d	$\lvert G\rvert$	$\lvert Out(G)\rvert$
J_4	1	$2^{21}.3^3.5.7.11^3.23.29.31.37.43$	1
F_1	1	$2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71$	1
F_2	2	$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47$	1
$A_n(q)$ $n \geqslant 1$	$gcd(n+1, q-1)$	$\frac{1}{d}q^{\frac{n(n+1)}{2}}\prod_{i=1}^{n}(q^{i+1}-1)$	$2d\alpha$, if $n \geqslant 2$; $d\alpha$, if $n = 1$
${}^2A_n(q)$ $n \geqslant 1$	$gcd(n+1, q+1)$	$\frac{1}{d}q^{\frac{n(n+1)}{2}}\prod_{i=1}^{n}(q^{i+1}-(-1)^{i+1})$	$2d\alpha$, if $n \geqslant 2$; $d\alpha$, if $n = 1$
$B_n(q)$ $n \geqslant 2$	$gcd(2, q-1)$	$\frac{1}{d}q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$	$d\alpha$, if $n \geqslant 3$; 2α, if $n = 2$
$C_n(q)$ $n \geqslant 2$	$gcd(2, q-1)$	$\frac{1}{d}q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$	$d\alpha$, if $n \geqslant 3$; 2α, if $n = 2$
$D_n(q)$ $n \geqslant 4$	$gcd(4, q^n-1)$	$\frac{1}{d}q^{n(n-1)}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$2d\alpha$, if $n \neq 4$; $6d\alpha$, if $n = 4$
${}^2D_n(q)$ $n \geqslant 4$	$gcd(4, q^n+1)$	$\frac{1}{d}q^{n(n-1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$2d\alpha$
$E_7(q)$	$gcd(2, q-1)$	$\frac{1}{d}q^{63}\prod_{i\in\{2,6,8,10,12,14,18\}}(q^i-1)$	$d\alpha$
$E_8(q)$	1	$q^{120}\prod_{i\in\{2,8,12,14,18,20,24,30\}}(q^i-1)$	α

3. Proof of the main theorem

Assume that $p \in \{3, 5, 7\}$ and n is an odd number, where $n \geq 9$. By Tables 6 and 8 in [24], we have $t(B_n(p)) = [\frac{3n+5}{4}]$, $t(2, B_n(p)) = 2$ and $\rho(2, B_n(5)) = \{2, r_{2n}(5)\}$ and if $p \in \{3, 7\}$, then $\rho(2, B_n(p)) = \{2, r_n(p)\}$. Hence, if G is a finite group with $GK(G) = GK(B_n(p))$ and the maximal normal soluble subgroup K, then Lemma 2.2 implies that G has a unique nonabelian composition factor S, in which case $S \leq \bar{G} = G/K \leq Aut(S)$, $t(S) \geq t(B_n(p)) - 1$ and $t(2, S) \geq 2$. Also, if $p \in \{3, 7\}$, then $r_n(p) \in \pi(S)$ and if $p = 5$, then $r_{2n}(p) \in \pi(S)$. Since S is a finite nonabelian simple group, it follows by classification theorem of finite simple groups that S is a sporadic simple group, an alternating group or a simple group of Lie type. We prove that $S \cong B_n(p)$ or $S \cong C_n(p)$, by a sequence of lemmas.

Lemma 3.1. *S can not be isomorphic to a sporadic simple group.*

Proof. If $n \geq 17$, then by Lemma 2.2(2), $t(S) \geq t(B_{17}(p)) - 1 \geq 13$ and hence, the conclusion immediately holds by Table 2 in [24]. Otherwise, we have $n \in \{9, 11, 13, 15\}$ and since $t(S) \geq t(B_9(p)) - 1 \geq 7$, it follows by Table 2 in [24] that S can be isomorphic to one of the groups F_1, F_2 or J_4. Since $r_n(p) \in \pi(S)$, if $p \in \{3, 7\}$, and $r_{2n}(p) \in \pi(S)$, if $p = 5$, by computing the numbers $r_9(3) = 757$, $r_{11}(3) = 3851$, $r_{13}(3) = 797161$, $r_{15}(3) = 4561$, $r_9(7) = 37, 1063$, $r_{11}(7) = 1123, 293459$, $r_{13}(7) = 16148168401$, $r_{15}(7) = 31, 159871$, $r_{18}(5) = 5167$, $r_{22}(5) = 5281$, $r_{26}(5) = 5227$, $r_{30}(5) = 7621$ by GAP [5] and also by considering $\pi(F_1)$, $\pi(F_2)$ and $\pi(J_4)$ in Table 1, we can easily get a contradiction and the proof is complete. □

Lemma 3.2. *S can not be isomorphic to an alternating group.*

Proof. If $S \cong A_m$, where $m \geq 5$, then by considering the cases $n \geq 17$ and $9 \leq n \leq 15$ separately, we get a contradiction.

Case 1. If $n \geq 17$, then $t(S) \geq 13$ hence $|\pi(A_m)| \geq 13$. Thus according to the set $\pi(A_m)$, we can assume that $m \geq 41$ and it implies that $\{17, 19\} \subseteq \pi(S)$. First, we find an upper bound for $t(17, A_m)$ and $t(19, A_m)$. If $x \in \rho(17, A_m) \setminus \{17\}$, then by Lemma 2.3, $x \neq 2$ and $x + 17 > m$. Also, since $x \in \pi(A_m)$, we conclude that $x \in \{s |\ s$ is a prime, $m - 16 \leq s \leq m\}$. Hence, since $m \geq 41$, there exist at most six choices for x. Thus $t(17, A_m) \leq 7$. Also, by the same procedure, we can see that $t(19, A_m) \leq 8$. Since $S \leq G/K$ and $\pi(G) = \pi(B_n(p))$, we have $\pi(S) \subseteq \pi(B_n(p))$ hence $\{17, 19\} \subseteq \pi(B_n(p))$. Now we find a lower bound for $t(17, B_n(5))$, $t(17, B_n(7))$ and $t(19, B_n(3))$. Since $e(17, 5) = e(17, 7) = 16$, n is an odd number and $n \geq 17$, it follows by Lemma 2.6(3) that the set $\tau = \{17, r_n, r_{2n}, r_{n-2}, r_{2(n-2)}, r_{n-4}, r_{2(n-4)}, r_{n-6}, r_{2(n-6)}\}$ is a coclique of $GK(B_n(5))$ and $GK(B_n(7))$ and also, since 8 divides exactly one of the numbers $n-1, n-3, n-5, n-7$, we can add three elements of the set $\{r_{2(n-1)}, r_{2(n-3)}, r_{2(n-5)}, r_{2(n-7)}\}$ to the set τ. Therefore, $t(17, B_n(5)) \geq 12$ and $t(17, B_n(7)) \geq 12$. By the same argument, since $e(19,3)=18$, we can use the coclique

$$\tau'=\{19,r_n,r_{2(n-1)},r_{n-2},r_{2(n-3)},r_{n-4},r_{2(n-5)},r_{n-6},r_{2(n-7)},r_{n-8}\}$$

of $GK(B_n(3))$ and conclude that $t(19,B_n(3))\geq 10$. By using Lemma 2.2(2) for the sets $\rho(17,B_n(5))$, $\rho(17,B_n(7))$ and $\rho(19,B_n(3))$, we have

$$|\rho(17,B_n(5))\bigcap\pi(A_m)|\geq t(17,B_n(5))-1\geq 11,$$

$$|\rho(17,B_n(7))\bigcap\pi(A_m)|\geq t(17,B_n(7))-1\geq 11,$$

$$|\rho(19,B_n(3))\bigcap\pi(A_m)|\geq t(19,B_n(3))-1\geq 9.$$

On the other hand, since $S\leq G/K$, it follows by Lemma 2.4(1,2) that

$$|\rho(17,B_n(5))\bigcap\pi(A_m)|\leq t(17,A_m),$$

$$|\rho(17,B_n(7))\bigcap\pi(A_m)|\leq t(17,A_m),$$

$$|\rho(19,B_n(3))\bigcap\pi(A_m)|\leq t(19,A_m)$$

hence $11\leq t(17,A_m)\leq 7$ and $9\leq t(19,A_m)\leq 8$, which is impossible.
Case 2. $9\leq n\leq 15$. We know that $r_n(p)\in\pi(A_m)$, if $p\in\{3,7\}$, and $r_{2n}(p)\in\pi(A_m)$, if $p=5$. Thus according to the numbers $r_n(3)$, $r_{2n}(5)$ and $r_n(7)$ which are obtained in Lemma 3.1 we conclude that $79\in\pi(A_m)$ and hence, since $\pi(A_m)\subseteq\pi(B_n(p))$, we have $79\in\pi(B_n(p))$. But, $e(79,3)=e(79,7)=78$ and $e(79,5)=39$. This is a contradiction considering $|B_n(q)|$, where $n\leq 15$ (see Table 1). The proof is now complete. □

Lemma 3.3. *S can not be isomorphic to a finite simple group of Lie type of characteristic different from p.*

Proof. Assume that S is isomorphic to a finite simple group of Lie type of characteristic s, where $s\neq p$. We get a contradiction by considering two parts A and B, as follows.
Part A. $n\geq 17$. In this part, since $t(S)\geq 13$, by Table 4 in [25], we conclude that S can not be an exceptional group of Lie type . Thus S is one of the classical groups $A^{\varepsilon}_{m-1}(q)$, $D^{\varepsilon}_m(q)$, $C_m(q)$ or $B_m(q)$, where $q=s^{\alpha}$. We get a contradiction case by case:
Case 1. If $S\cong A_{m-1}(q)$, then since $gcd(s,p)=1$, we can assume that $t=e(s,p)$. If t is an odd number not equal to 1, 3, let

$$\rho=\{s,r_{2n},r_{2(n-1)},r_{2(n-2)},r_{2(n-3)}\}.$$

Since $GK(G) = GK(B_n(p))$, by Lemma 2.6(3), we can see that ρ is a coclique of $GK(G)$, containing s and hence, by Lemmas 2.2(2) and 2.4(1,2), we conclude that $t(s, S) \geq |\rho| - 1$. On the other hand, since $t(S) \geq 13$, by Table 8 in [24], we can assume that $m \geq 25$ and hence, Table 4 in [24] implies that $t(s, S) = 3$. Thus $3 \geq |\rho| - 1 = 4$, which is impossible. Also, if t is an even number except 2,6, where $\frac{t}{2}$ is odd, then it is enough to replace ρ with the coclique $\{s, r_n, r_{2(n-1)}, r_{n-2}, r_{2(n-3)}\}$ of $GK(B_n(p))$ and get a contradiction. If t and $\frac{t}{2}$ are even numbers and $t \neq 4$, then by replacing ρ with the coclique $\{s, r_n, r_{2n}, r_{n-2}, r_{2(n-2)}\}$ of $GK(B_n(p))$ in the previous argument, we can get a contradiction. Therefore, we should only consider different cases for t, where $t \in \{1, 2, 3, 4, 6\}$. Since $t = e(s, p)$ and $p \in \{3, 5, 7\}$, by Lemma 2.5, we can see that $s \in \{2, 5, 7, 13\}$, if $p = 3$, and $s \in \{2, 3, 7, 13, 31\}$, if $p = 5$, and $s \in \{2, 3, 5, 19, 43\}$, if $p = 7$. Since $m \geq 25$, according to $|A_{m-1}(q)|$ (see Table 1), we have $r_7 \in \pi(S)$ and if $q \neq 2, 3$, then $r_1 \in \pi(S)$. For considering the remaining cases, first we find an upper bound for $t(r_1, S)$ and $t(r_7, S)$. If $r_1 \in \pi(S)$, then Lemma 2.8 implies that $t(r_1, S) \leq 3$. We claim that $t(r_7, S) = 7$:

By Lemmas 2.8 and 2.7(1), we can see that $(2, r_7), (r_1, r_7), (s, r_7) \in GK(S)$. Thus if $x \in \rho(r_7, S) \setminus \{r_7\}$, then $x \notin \{2, s, r_1\}$ and if $e(x, s^\alpha) = l$, then by Lemma 2.6(1) we conclude that $l + 7 > m$ and $7 \nmid l$. Also, according to $|S|$, we have $l \leq m$ and hence, $l \in \{m - 6, m - 5, \cdots, m\}$ and $7 \nmid l$. Since $m - 6, m - 5, \cdots, m$ are seven consecutive numbers, so 7 divides exactly one of them and we have exactly six choices for l and hence, $t(r_7, S) = 7$. For getting a contradiction, we consider the cases $p = 3$, $s \in \{2, 5, 7, 13\}$ and $p = 5$, $s \in \{2, 3, 7, 13, 31\}$ and $p = 7$, $s \in \{2, 3, 5, 19, 43\}$ separately:

Subcase a. $p = 3$. If $s = 2$, then since $r_7(2) = 127$, we have $127 \in \{r_1(2^\alpha), r_7(2^\alpha)\} \subseteq \pi(S)$ and by the above statements we conclude that $t(127, S) \leq 7$. On the other hand, since $\pi(S) \subseteq \pi(B_n(3))$, we have $127 \in \pi(B_n(3))$. Also, we know that $e(127, 3) = 126$ and hence, according to $|B_n(3)|$, we conclude that $n \geq 63$. Moreover, since n is an odd number, it follows by Lemma 2.6(3) that the set $\tau \bigcup \{127\}$ is a coclique of $GK(B_n(3))$, where

$$\tau = \{r_i | \ n - 14 \leq i \leq n, \ i \equiv 1 (\mathrm{mod} 2)\} \bigcup \{r_{2i} | \ n - 15 \leq i \leq n - 1, \ i \equiv 0 (\mathrm{mod} 2)\}$$

and hence, $t(127, B_n(3)) \geq 17$. Also, since $S \leq G/K$, it follows by Lemma 2.4(1,2) that $|\rho(127, B_n(3)) \bigcap \pi(S)| \leq t(127, S)$. By using Lemma 2.2(2) for $\rho(127, B_n(3))$, we have $7 \geq t(127, S) \geq t(127, B_n(3)) - 1 \geq 16$, which is impossible. If $s = 5$ or $s = 13$, then $r_7(s) = 19531$ and $r_7(s) = 5229043$, respectively. Also, since $e(19531, 3) = 6510$ and $e(5229043, 3) = 249002$, Lemma 2.6(3) implies that the set $\tau \bigcup \{r_7(s)\}$ is a coclique of $GK(B_n(3))$ and hence, $t(r_7(s), B_n(3)) \geq 17$ and similar to the previous argument we can get a contradiction. If $s = 7$, then $r_7(7) = 4733$ and similar to the case $s = 2$, we conclude that $t(4733, S) \leq 7$ and $4733 \in \pi(B_n(3))$. Also, since $e(4733, 3) = 676$, Lemma 2.6(3) implies that the set $\tau' \bigcup \{4733\}$, where $\tau' = \{r_i, r_{2i} | \; n-14 \leq i \leq n, \; i \equiv 1 \pmod 2\}$ is a coclique of $GK(B_n(3))$ and hence, $t(4733, B_n(3)) \geq 17$. Now similar to the previous arguments, we can get a contradiction.

Subcase b. $p = 5$. If $s = 13$, then $r_7(s) = 5229043$. Also, we can see that $e(5229043, 5) = 1743014$ and hence, similar to the previous subcase by using the coclique $\tau \bigcup \{r_7(s)\}$, we can get a contradiction. If $s \in \{3, 7\}$, then $r_7(s) \in \{1093, 4733\}$. Also, we have $e(1093, 5) = 1092$ and $e(4733, 5) = 4732$ and hence, similar to the previous subcase by using the coclique $\tau' \bigcup \{r_7(s)\}$, we can get a contradiction. If $s = 2$, then $r_7(s) = 127$ and since $e(127, 5) = 42$, according to $|B_n(5)|$, we conclude that $n \geq 21$ and hence, Lemma 2.6(3) implies that the set $\tau'' \bigcup \{127\}$ is a coclique of $GK(B_n(5))$, where $\tau'' = \{r_i | \; n-10 \leq i \leq n, \; i \equiv 1 \pmod 2\} \bigcup \{r_{2i} | \; n-9 \leq i \leq n-1, \; i \equiv 0 \pmod 2\}$. Moreover, since 21 divides at most one of the numbers $n-10, n-9, \cdots, n$, by Lemma 2.6(3), we can add at least five elements of the set $\{r_{2(n-10)}, r_{2(n-8)}, r_{2(n-6)}, r_{2(n-4)}, r_{2(n-2)}, r_{2n}\}$ to $\tau'' \bigcup \{127\}$. Therefore, $t(127, B_n(5)) \geq 16$ and we can get a contradiction by similar argument in the previous subcase. If $s = 31$, then $r_7(s) = 917087137$ and since $e(917087137, 5) = 917087136$, by using the coclique $\tau' \bigcup \{917087137\}$ and similar argument in the previous subcase, we can get a contradiction.

Subcase c. $p = 7$. If $s \in \{2, 5, 43\}$, then $r_7(s) \in \{127, 19531, 5839\}$. Also, we know that $e(127, 7) = 126$, $e(19531, 7) = 3906$ and $e(5839, 7) = 1946$ and hence, similar to the subcase a, case 1, part A, we can use the coclique $\tau \bigcup \{r_7(s)\}$ and get a contradiction. If $s \in \{3, 19\}$, then $r_7(s) \in \{1093, 701\}$. Also, we have $e(1093, 7) = 273$ and $e(701, 7) = 175$. Thus Lemma 2.6(3) implies that the set $\{r_{2i} | \; n-15 \leq i \leq n\} \bigcup \{r_7(s)\}$ is a coclique of $GK(B_n(7))$ and hence, $t(r_7(s), B_n(7)) \geq 17$. Now similar to the previous arguments, we can get a contradiction.

Case 2. $S \cong {}^2A_{m-1}(q)$. Since $t(S) \geq 13$, by Table 8 in [24] we can see that $m \geq 25$ and hence, $t(s, S) = 3$, by Table 4 in [24]. Thus similar to the case 1, it is enough to consider $s \in \{2, 5, 7, 13\}$, if $p = 3$, and $s \in \{2, 3, 7, 13, 31\}$, if $p = 5$, and $s \in \{2, 3, 5, 19, 43\}$, if $p = 7$. Since $m \geq 25$, according to $|^2A_{m-1}(q)|$, we have $r_7 \in \pi(S)$ and if $q \neq 2, 3$, then $r_1 \in \pi(S)$. We want to find an upper bound for $t(r_7, S)$. Since $m \geq 25$, by Lammas 2.9 and 2.7(2), we have $(2, r_7), (r_2, r_7), (s, r_7) \in GK(S)$. Thus if $x \in \rho(r_7, S) \setminus \{r_7\}$, then $x \notin \{2, s, r_2\}$ and if $e(x, s^\alpha) = l$, then by Lemma 2.6(2), we have $\nu(l) + 14 > m$ and $14 \nmid \nu(l)$. Furthermore, by $|^2A_{m-1}(q)|$, we can see that $\nu(l) \leq m$. Thus $\nu(l) \in \{m - 13, m - 12, \cdots, m\}$ and $14 \nmid \nu(l)$. Moreover, since $m-13, m-12, \cdots, m$ are fourteen consecutive numbers, so 14 divides exactly one of them and hence, we have thirteen choices for l. Therefore, $t(r_7, S) = 14$. If $r_1 \in \pi(S)$, by the same procedure, we can show that $t(r_1, S) = 2$. Hence, since $r_7(s) \in \{r_1(s^\alpha), r_7(s^\alpha)\}$, we have $t(r_7(s), S) \leq 14$. Now we can use all the statements in the subcases a, b and c, case 1, part A to get a contradiction.

Case 3. S is isomorphic to one of the groups $B_m(q), C_m(q)$ or $D_m(q)$. Since $t(S) \geq 13$, by Table 8 in [24] and Appendix in [25], we can see that $m \geq 16$ and hence, $t(s, S) \leq 3$, by Table 4 in [24]. Thus similar to the case 1, it is enough to consider $s \in \{2, 5, 7, 13\}$, if $p = 3$, and $s \in \{2, 3, 7, 13, 31\}$, if $p = 5$, and $s \in \{2, 3, 5, 19, 43\}$, if $p = 7$. If $S \cong B_n(q)$ or $C_n(q)$, then since $m \geq 16$, according to $|S|$, we can see that $r_7 \in \pi(S)$ and if $q \neq 2, 3$, then $r_1 \in \pi(S)$. By Lemmas 2.7(3,4) and 2.10, we have $(2, r_7), (s, r_7) \in GK(S)$. Thus if $x \in \rho(r_7, S) \setminus \{r_7\}$, then $x \notin \{2, s\}$ and if $e(x, s^\alpha) = l$, then by Lemma 2.6(3), we have $\eta(l) + 7 > m$. Furthermore, according to the order of $B_m(q)$ and $C_m(q)$, we can see that $\eta(l) \leq m$. Thus $\eta(l) \in \{m - 6, m - 5, \cdots, m\}$ and by the definition of $\eta(l)$, there are at most eleven choices for l. Therefore, $t(r_7, S) \leq 12$. Also, if $r_1 \in \pi(S)$, then by the same argument, we can show that $t(r_1, S) \leq 3$. If $S \cong D_m(q)$, then by Lemmas 2.6(4), 2.7(5), 2.10(2) and the previous argument, we conclude that $t(r_7, S) \leq 13$ and if $r_1 \in \pi(S)$, then $t(r_1, S) \leq 4$. Now we can use all the statements in the subcases a, b and c, case 1, part A to get a contradiction.

Case 4. $S \cong {}^2D_m(q)$. Since $t(S) \geq 13$, by Table 8 in [24] we can see that $m \geq 16$ and hence, $t(s, S) \leq 4$, by Table 4 in [24]. By using similar argument in the case 1, if $t = e(s, p)$ and t is an odd number except 1,3, it is enough to replace ρ with the coclique $\rho \bigcup \{r_{2(n-4)}\}$ of $GK(B_n(p))$, and if t is an even number except 2,6, where $\frac{t}{2}$ is odd, we replace ρ with the coclique $\{s, r_n, r_{2(n-1)}, r_{n-2}, r_{2(n-3)}, r_{n-4}\}$ of $GK(B_n(p))$ and if $t \notin \{4, 8\}$ and t and $\frac{t}{2}$ are even numbers, we replace ρ with the coclique $\{s, r_n, r_{2n}, r_{n-2}, r_{2(n-2)}, r_{n-4}\}$ of $GK(B_n(p))$. Thus in this case, if $p = 3$, $p = 5$ and $p = 7$, then we should consider $s \in \{2, 5, 7, 13, 41\}$, $s \in \{2, 3, 7, 13, 31, 313\}$ and $s \in \{2, 3, 5, 19, 43, 1201\}$, respectively. By the same procedure in the case 3 for $S \cong D_m(q)$, we can prove that $t(r_7, S) \le 12$ and if $r_1 \in \pi(S)$, then $t(r_1, S) \le 3$. If $p = 3$, $s \in \{2, 3, 7, 13\}$ or $p = 5$, $s \in \{2, 3, 7, 13, 31\}$, or $p = 7$, $s \in \{2, 3, 5, 19, 43\}$, then by using all the statements in the subcases a, b and c, case 1, part A, we can get a contradiction. If $p = 3$, $s = 41$ or $p = 5$, $s = 313$ or $p = 7$, $s = 1201$, then since

$$r_7(41) = 113229229, e(113229229, 3) = 56614614,$$

$$r_7(313) = 29, 32528030679467, e(32528030679467, 5) = 32528030679466,$$

$$r_7(1201) = 29429, e(29429, 7) = 1051,$$

we use the cocliques $\tau \bigcup \{113229229\}$ and $\tau \bigcup \{32528030679467\}$ and $\{r_{2i} |\ n - 15 \le i \le n\} \bigcup \{29429\}$, of $GK(B_n(3))$ and $GK(B_n(5))$ and $GK(B_n(7))$, respectively. Hence, by the same argument in the subcases a, case 1, part A, we can get a contradiction.

Part B. $9 \le n \le 15$. In this part, by Table 8 in [24], we have:

1. $\rho(B_9(p)) = \{r_5, r_7, r_9, r_{10}, r_{12}, r_{14}, r_{16}, r_{18}\}$;
2. $\rho(B_{11}(p)) = \{r_7, r_9, r_{11}, r_{12}, r_{14}, r_{16}, r_{18}, r_{20}, r_{22}\}$;
3. $\rho(B_{13}(p)) = \{r_7, r_9, r_{11}, r_{13}, r_{14}, r_{16}, r_{18}, r_{20}, r_{22}, r_{24}, r_{26}\}$;
4. $\rho(B_{15}(p)) = \{r_9, r_{11}, r_{13}, r_{15}, r_{16}, r_{18}, r_{20}, r_{22}, r_{24}, r_{26}, r_{28}, r_{30}\}$.

Also, since $t(S) \ge t(B_9(p)) - 1 = 7$, Tables 8 in [24] and 4 in [25] imply that in addition to classical groups of Lie type , S can be isomorphic to exceptional groups of Lie type $E_7(q)$ and $E_8(q)$. Moreover, by Tables 4 and 5 in [24] we conclude that $t(s, S) \le 5$. If $t = e(s, p)$, then since $\pi(S) \subseteq \pi(B_n(p))$ and according to $|B_n(p)|$, we conclude that $t \le 30$. Thus by considering the cases "t is odd" and "t is even" separately and according to the coclique $\rho(B_n(p))$, it is easy to check that if $t \notin \{1, 2, 3, 4, 6, 8\}$, then we can find some seven-element coclique, containing s, in $GK(B_n(p))$ and conclude that $t(s, B_n(p)) \ge 7$. To be short, we omit the details. Hence, similar to the case 1, part A, by using Lemmas 2.2(2) and 2.4(1,2), we can get a contradiction. If $t \in \{1, 2, 3, 4, 6, 8\}$, then since $t = e(s, p)$ and $p \in \{3, 5, 7\}$, by Lemma 2.5, we can see that $s \in \{2, 5, 7, 13, 41\}$, if $p = 3$, and $s \in \{2, 3, 7, 13, 31, 313\}$, if $p = 5$, and $s \in \{2, 3, 5, 19, 43, 1201\}$, if $p = 7$. we consider these three cases separately.

Case 1. $p = 3$. If $s \in \{5, 7, 13, 41\}$, then by checking $|S|$ in different cases, we conclude that $\{r_1(s^\alpha), r_5(s^\alpha)\} \subseteq \pi(S)$ and hence, $r_5(s) \in$

$\pi(S) \subseteq \pi(B_n(3))$. Also, it is easy to check that $e(r_5(s), 3) > 30$. On the other hand, according to $|B_n(3)|$, if $x \in \pi(B_n(3))\backslash\{3\}$, then $e(x, 3) \leq 2n$. Thus we get a contradiction, because $n \leq 15$. If $s = 2$, then by checking $|S|$ in different cases, we can see that if $S \ncong {}^2A_{m-1}(2^\alpha)$, then $r_7(2^\alpha) \in \pi(S)$, and if $\alpha \neq 1$, then $r_1(2^\alpha) \in \pi(S)$. Since $r_7(2) = 127$, thus $127 \in \{r_1(2^\alpha), r_7(2^\alpha)\} \subseteq \pi(S) \subseteq \pi(B_n(3))$, but $e(127, 3) = 126 > 30$ and similar to the previous argument, we can get a contradiction. If $S \cong {}^2A_{m-1}(2^\alpha)$, then since $t(S) \geq 7$, by Table 8 in [24], we can see that $m \geq 13$ and $\{r_2(2^\alpha), r_{14}(2^\alpha)\} \subseteq \pi(S)$. If α is an odd number, then $43 = r_{14}(2) \in \{r_2(2^\alpha), r_{14}(2^\alpha)\} \subseteq \pi(S) \subseteq \pi(B_n(3))$, but $e(43, 3) = 42 > 30$, which is impossible. Otherwise, there exists a natural number β such that $q = 4^\beta$ and since $r_8(4) = 257$ and according to $|{}^2A_{m-1}(4^\beta)|$, we have $257 \in \{r_1(q), r_2(q), r_4(q), r_8(q)\} \subseteq \pi(S) \subseteq \pi(B_n(3))$, but $e(257, 3) = 256 > 30$, which is impossible.

Case 2. $p = 5$. If $s \in \{7, 13, 31, 313\}$, then it is easy to check that $e(r_5(s), 5) > 30$ and similar to the previous case we can get a contradiction. Also, if $s = 2$, then since $e(43, 5) = 42 > 30$ and $e(257, 5) = 256 > 30$, similar to the previous argument we get a contradiction. If $s = 3$, then by checking $|S|$ in different cases, we conclude that $\{r_1(s^\alpha), r_2(s^\alpha), r_3(s^\alpha), r_4(s^\alpha), r_6(s^\alpha), r_{12}(s^\alpha)\} \subseteq \pi(S)$ and hence, $r_{12}(s) \in \pi(S) \subseteq \pi(B_n(5))$. Also, we can see that $r_{12}(3) = 73$ and $e(73, 5) = 72 > 30$, which is impossible.

Case 3. $p = 7$. By checking $|S|$ in different cases, we can see that $\{r_1(s^\alpha), r_2(s^\alpha), r_5(s^\alpha), r_{10}(s^\alpha)\} \subseteq \pi(S)$. If $s \in \{5, 19, 43, 1201\}$, then we can check that $e(r_5(s), 7) > 30$ and hence, similar to the case 1, part B, we get a contradiction. Also, if $s = 3$, then we replace $r_5(s)$ with $r_{10}(3) = 61$ and since $e(61, 7) = 60 > 30$, we get a contradiction. If $s = 2$ and $S \ncong {}^2A_{m-1}(2^\alpha)$, where α is an odd number, then all the statements in the case 1, part B is true. Otherwise, i.e., $S \cong {}^2A_{m-1}(2^\alpha)$, where α is an odd number, since $m \geq 13$, by checking $|{}^2A_{m-1}(2^\alpha)|$, we have $\{r_2(2^\alpha), r_{26}(2^\alpha)\} \subseteq \pi(S)$ and hence, $2731 = r_{26}(2) \in \{r_2(2^\alpha), r_{26}(2^\alpha)\} \subseteq \pi(S) \subseteq \pi(B_n(7))$. But $e(2731, 7) = 2730 > 30$, which is impossible. The proof is now complete. □

Lemma 3.4. *If S is isomorphic to a finite simple group of Lie type of characteristic p, then $S \cong B_n(p)$ or $S \cong C_n(p)$.*

Proof. Assume that S is isomorphic to a finite simple group of Lie type over a field of order p^α and $p \in \{3, 7\}$. By Lemma 2.2(3-a), $r_n(p) \in \pi(S)$.

Put $e_n = e(r_n(p), p^\alpha)$. Since $r_n(p)$ divides $p^{\alpha e_n} - 1$, we get that n divides αe_n. Suppose that $\alpha e_n > n$. Then a prime r with $e(r, p) = \alpha e_n$ divides the order of S and hence, r divides the order of $B_n(p)$ and by $|B_n(p)|$, we conclude that $\alpha e_n \leq 2n$. Consequently, $\alpha e_n \in \{n, 2n\}$. Now to prove the lemma, we consider classical and exceptional groups of Lie type separately:

Part A. If S is a classical group of Lie type of characteristic p, then S is isomorphic to one of the groups $A^\varepsilon_{m-1}(p^\alpha)$, $D^\varepsilon_m(p^\alpha)$, $C_m(p^\alpha)$ or $B_m(p^\alpha)$. Now with a case by case analysis, we prove that $S \cong B_n(p)$ or $S \cong C_n(p)$:

Case 1. $S \cong A_{m-1}(p^\alpha)$. Since $(r_n(p), 2) \notin GK(S)$ and $e_n = e(r_n(p), p^\alpha)$, it follows by Lemma 2.8 that $e_n \in \{m, m-1\}$. Moreover, since $\alpha e_n \in \{n, 2n\}$, we have the following four subcases:

Subcase a. $\alpha(m-1) = 2n$. In this case, since $\alpha m > \alpha(m-1) = 2n$ and we know that $p^{\alpha m} - 1$ divides the order of S, we conclude that $r_{\alpha m}(p) \in \pi(S)$. Also, since $\pi(S) \subseteq \pi(B_n(p))$, we have $r_{\alpha m}(p) \in \pi(B_n(p))$. But, $\alpha m > 2n$ and we can get a contradiction by $|B_n(p)|$.

Subcase b. $\alpha m = 2n$. First, we claim that $\{r_{2(n-1)}(p), r_{2(n-2)}(p)\} \bigcap \pi(S) = \emptyset$:

If $r_{2(n-1)}(p) \in \pi(S)$, then according to $|S|$ there exists an integer k such that $0 \leq k \leq m-2$ and $r_{2(n-1)}(p) \mid (p^{\alpha(m-k)} - 1)$ and hence, $2(n-1) \mid \alpha(m-k) = \alpha m - \alpha k = 2n - \alpha k$. Thus $\alpha k = 2$ and this implies that $\alpha \in \{1, 2\}$. If $\alpha = 1$, then $m = 2n$ and according to $|S|$, we can see that $r_{\alpha(m-1)}(p) \in \pi(S)$ and hence, $r_{2n-1}(p) \in \pi(S) \subseteq \pi(B_n(p))$, which is impossible according to $|B_n(p)|$. If $\alpha = 2$, then $m = n$ and according to $|S|$, we have $r_{2n}(p) \in \pi(S)$. Since n is odd, it is easy to check that $e(r_{2n}(p), p^2) = e(r_n(p), p^2) = n$ and hence, by using Lemmas 2.6(1,3), we have $(r_{2n}(p), r_n(p)) \in GK(S)$ and $(r_{2n}(p), r_n(p)) \notin GK(B_n(p))$. But by Lemma 2.4(1,2), it is impossible. Thus $r_{2(n-1)}(p) \notin \pi(S)$. Also, by the same procedure, since n is an odd number, we can show that $r_{2(n-2)}(p) \notin \pi(S)$. Now by using the coclique $\rho = \{r_{2n}, r_{2(n-1)}, r_{2(n-2)}\}$ of $GK(B_n(p))$ and Lemma 2.2(2), we can get a contradiction.

Subcase c. $\alpha m = n$. In this case, we claim that $\rho \bigcap \pi(S) = \emptyset$, where ρ is the coclique which is stated above and hence, similar to the previous case, we can get a contradiction. If $r_{2n}(p) \in \pi(S)$, according to the order of S, there is a natural number k such that $2 \leq k \leq m$ and $r_{2n}(p) \mid (p^{\alpha k} - 1)$ and hence, $2n \mid \alpha k$. Also, since $(p^{\alpha k} - 1) \mid |S|$ and $|S| \mid |B_n(p)|$, according to $|B_n(p)|$, we conclude that $\alpha k \leq 2n$. Thus $\alpha k = 2n$. But, $k \leq m$ and hence, $2n = \alpha k \leq \alpha m = n$, which is impossible. By the same procedure, since $n \geq 9$, we can prove that

$\{r_{2(n-1)}, r_{2(n-2)}\} \bigcap \pi(S) = \emptyset$.

Subcase d. $\alpha(m-1) = n$. Similar to the previous case, we use the set ρ and prove that $|\rho \bigcap \pi(S)| \leq 1$ and then we get a contradiction. If $r_{2n}(p) \in \pi(S)$, then similar to the subcase b, $r_{2n}(p) \mid (p^{\alpha(m-k)} - 1)$, where $0 \leq k \leq m-2$ and hence, $2n \mid \alpha(m-k)$. If $k \geq 1$, then $m - k \leq m - 1$. Thus $2n \leq \alpha(m-k) \leq \alpha(m-1) = n$, which is impossible. Therefore, $k = 0$ and $r_{2n}(p) \mid (p^{\alpha m} - 1)$. On the other hand, since $(p^{\alpha m} - 1) \mid |S|$ and $|S| \mid |B_n(p)|$, we conclude that $\alpha m = 2n$ and hence, $e(r_{2n}(p), p^{\alpha}) = m$. Similarly, since $n \geq 9$ we can prove that if $r_{2(n-1)}(p) \in \pi(S)$ or $r_{2(n-2)}(p) \in \pi(S)$, then $e(r_{2(n-1)}(p), p^{\alpha}) = m$ or $e(r_{2(n-2)}(p), p^{\alpha}) = m$, respectively. Now if $|\rho \bigcap \pi(S)| \geq 2$, then similar to the subcase b, by Lemmas 2.6(1,3) and 2.4(1,2), we can get a contradiction. Therefore, $|\rho \bigcap \pi(S)| \leq 1$ and we can get a contradiction by Lemma 2.2(2). Thus $S \ncong A_{m-1}(p^{\alpha})$.

Case 2. $S \cong {}^2A_{m-1}(p^{\alpha})$. Since $(r_n(p), 2) \notin GK(S)$ and $e_n = e(r_n(p), p^{\alpha})$, by Lemma 2.9 and the definition of $\nu(m)$, we have $e_n \in \{m, 2m, 2(m-1), \frac{m}{2}\}$ and since $\alpha e_n \in \{n, 2n\}$ and n is odd, we have the following four subcases:

Subcase a. $\alpha m = 4n$. According to $|S|$, we have $p^{\alpha m} - (-1)^m \mid |S|$ and hence, $r_{\alpha m}(p)$ or $r_{2\alpha m}(p)$ belongs to $\pi(S)$. But, since $\alpha m = 4n$ and $\pi(S) \subseteq \pi(B_n(p))$, according to $|B_n(p)|$ we can get a contradiction.

Subcase b. $\alpha m = 2n$. If m is odd, then according to $|S|$, we have $p^{\alpha m} + 1 = p^{\alpha m} - (-1)^m \mid |S|$ and hence, $r_{2\alpha m}(p) \in \pi(S) \subseteq \pi(B_n(p))$. But, $2\alpha m > 2n$ and this is impossible according to $|B_n(p)|$. If m is even, then according to $|S|$, $r_{2\alpha(m-1)}(p) \in \pi(S) \subseteq \pi(B_n(p))$ and hence, $2\alpha(m-1) \leq 2n$ and this implies that $2n - \alpha = \alpha(m-1) \leq n$. Thus $n \leq \alpha$. Moreover, since n is odd, m is even and $\alpha m = 2n$, we have $\alpha | n$. Therefore, $\alpha = n$ and this implies that $m = 2$, which is impossible, according to Table 1.

Subcase c. $\alpha m = n$. First, we claim that $\{r_{2(n-1)}(p), r_{2(n-3)}(p)\} \bigcap \pi(S) = \emptyset$. If $r_{2(n-1)}(p) \in \pi(S)$, according to $|S|$, there exists an integer number k such that $0 \leq k \leq m-2$ and $r_{2(n-1)}(p) \mid (p^{\alpha(m-k)} - (-1)^{m-k})$ and hence, $r_{2(n-1)}(p) \mid (p^{2\alpha(m-k)} - 1)$. Thus $(n-1) \mid \alpha(m-k) = n - \alpha k$ and this implies that $\alpha k = 1$ and hence, $m = n$. But, since n is odd, we have $p^{\alpha(m-k)} - (-1)^{m-k} = p^{n-1} - 1$ and hence, $r_{2(n-1)}(p) \mid (p^{n-1} - 1)$, which is impossible. Thus $r_{2(n-1)}(p) \notin \pi(S)$. Similarly, we can prove that $r_{2(n-3)}(p) \notin \pi(S)$. Therefore, $\{r_{2(n-1)}(p), r_{2(n-3)}(p)\} \bigcap \pi(S) = \emptyset$. Now by using the cocliuqe $\rho = \{r_{2n}, r_{2(n-1)}, r_{2(n-3)}\}$ of $GK(B_n(p))$ and

Lemma 2.2(2), we can get a contradiction.
Subcase d. $\alpha(m-1)=n$. Similar to the previous subcase, we can prove that $\{r_{2(n-1)}(p), r_{2(n-3)}(p)\} \bigcap \pi(S)=\emptyset$ and get a contradiction. Therefore, $S \ncong {}^2A_{m-1}(p^\alpha)$.
Case 3. $S \cong {}^2D_m(p^\alpha)$. Since $(r_n(p),2) \notin GK(S)$ and $e_n=e(r_n(p),p^\alpha)$, by Lemma 2.10(2), we have $e_n \in \{2m, 2(m-1)\}$. Moreover, we know that $\alpha e_n \in \{n, 2n\}$ and n is odd and hence, there are the following two subcases:
Subcase a. $\alpha m=n$. Since $t(S) \geq t(G)-1=t(B_n(p))-1$, by Table 8 in [24], we have $[\frac{3m+4}{4}] \geq [\frac{3n+5}{4}]-1=[\frac{3n+1}{4}]$. Also, since n is odd, we have α is odd as well. If $\alpha \geq 3$, then $n=\alpha m \geq 3m$ and since $n \geq 9$, we conclude that $3n+1 \geq 3m+4$. Therefore, $[\frac{3m+4}{4}]=[\frac{3n+1}{4}]$ and this implies that $3n+1-(3m+4)<4$ and hence, $n-m \leq 2$. On the other hand, $n \geq 3m$ and hence, $2m \leq n-m \leq 2$, which implies that $m=1$ and this is impossible according to Table 1. Thus $\alpha=1$ and $S \cong {}^2D_n(p)$. Since n is odd, according to $|B_n(p)|$ and $|{}^2D_n(p)|$, we can see that $r_n \in \pi(B_n(p)) \backslash \pi({}^2D_n(p))$. Also, since $S \leq \bar{G}=G/K \leq Aut(S)$ and $Out({}^2D_n(p))$ is a 2-group, we conclude that $r_n \in \pi(K)$. Hence, using the cocliques $\rho=\{r_n, r_{2n}, r_{2(n-2)}\}$ and $\tau=\{r_n, r_{2n}, r_4\}$ of $GK(B_n(p))$ in Lemma 2.1, implies that $\{r_4, r_{2(n-2)}\} \bigcap \pi(K)=\emptyset$. By Lemma 2.6(3), we can see that $(r_4, r_{2(n-2)}) \in GK(B_n(p))$. Therefore, by Lemma 2.4(3), we conclude that $\bar{G}$ has an element g of order $r_4.r_{2(n-2)}$. On the other hand, since $\bar{G}/S \leq Out(S)$ and $Out({}^2D_n(p))$ is a 2-group, we can assume that $g \in S$ and hence, $(r_4, r_{2(n-2)}) \in GK(S)$. But by Lemma 2.6(4), it is impossible.
Subcase b. $\alpha(m-1)=n$. Similar to the previous argument, we can show that $\alpha=1$ and hence, $S \cong {}^2D_{n+1}(p)$. But, since $p^{n+1}+1$ divides the order of ${}^2D_{n+1}(p)$, we have $r_{2(n+1)}(p) \in \pi(S)$, which is impossible, because $\pi(S) \subseteq \pi(B_n(p))$ and according to $|B_n(p)|$, $r_{2(n+1)}(p) \notin \pi(B_n(p))$. Therefore, $S \ncong {}^2D_m(p^\alpha)$.
Case 4. $S \cong D_m(p^\alpha)$. Similar to the previous case, Lemma 2.10(2) imposes some restrictions on e_n and we have $e_n \in \{2(m-1), m-1, m\}$. Also, since $\alpha e_n \in \{n, 2n\}$ and n is odd, there are the following four subcases:
Subcase a. $\alpha(m-1)=2n$. Since $p^{\alpha m}-1$ divides the order of S, we have $r_{\alpha m}(p) \in \pi(S)$ and hence, $r_{\alpha m}(p) \in \pi(B_n(p))$. But, since $\alpha m > \alpha(m-1)=2n$, according to $|B_n(p)|$, we get a contradiction.
Subcase b. $\alpha(m-1)=n$. Since n is odd, we have α is odd as well. If $\alpha=1$, then $S \cong D_{n+1}(p)$. Since n is odd, according to

$|D_{n+1}(p)|$, we have $\{r_{n-1}(p), r_{n+3}(p)\} \subseteq \pi(D_{n+1}(p))$. Also, since $n \geq 9$, Lemma 2.6(3,4) implies that $(r_{n-1}(p), r_{n+3}(p)) \in GK(D_{n+1}(p))$ and $(r_{n-1}(p), r_{n+3}(p)) \notin GK(B_n(p))$. But, $S \leq G/K$ and we can get a contradiction by Lemma 2.4(1,2). Thus $\alpha \geq 3$. We know that $t(S) \geq t(B_n(p))-1$ and by Appendix in [25], $t(D_m(p)) \in \{\frac{3m+3}{4}, [\frac{3m+1}{4}]\}$ and by Table 8 in [25], $t(B_n(p)) = [\frac{3n+5}{4}]$ and hence, if $t(S) = [\frac{3m+1}{4}]$, then $[\frac{3m+1}{4}] \geq [\frac{3n+1}{4}]$. Moreover, $\alpha \geq 3$ implies that $n \geq 3(m-1)$ and since $n \geq 9$, we conclude that $\frac{3n+1}{4} \geq \frac{3m+1}{4}$. Therefore, $[\frac{3m+1}{4}] = [\frac{3n+1}{4}]$, which implies that $3n+1-(3m+1) < 4$ and hence, $m \in \{n, n-1\}$. This is impossible, because $\alpha(m-1) = n \geq 9$. If $t(S) = \frac{3m+3}{4}$, then similar to the above argument we can get a contradiction.

Subcase c. $\alpha m = 2n$. If $\alpha \geq 3$, then similar to the subcase b, we can get a contradiction. Thus we should consider the cases $\alpha = 1$ and $\alpha = 2$. If $\alpha = 1$, then $m = 2n$ and since $r_{2(m-1)} \in \pi(S)$, we have $r_{2(m-1)} \in \pi(B_n(p))$, but $2(m-1) = 2(2n-1) > 2n$, which is impossible, according to $|B_n(p)|$. If $\alpha = 2$, then $m = n$ and $S \cong D_n(p^2)$. Thus $(p^2)^{2(n-1)} - 1$ divides the order of S and hence, $r_{4(n-1)}(p) \in \pi(B_n(p))$, which is impossible, because $4(n-1) > 2n$.

Subcase d. $\alpha m = n$. Since n is odd, we have α is odd as well. If $\alpha \geq 3$, similar to the subcase b, we can get a contradiction. If $\alpha = 1$, then $m = n$ and $S \cong D_n(p)$. In this case, since n is odd, we have $Out(D_n(p))$ is a 2-group. Also, the order of $D_n(p)$ implies that $r_{2n} \in \pi(B_n(p)) \setminus \pi(D_n(p))$ and since $\bar{G}/S \leq Out(S)$, we can conclude that $r_{2n} \in \pi(K)$. Thus it is enough to replace the set ρ in the subcase a, case 3, with the set $\{r_n, r_{2n}, r_{n-2}\}$ and conclude that $(r_4, r_{n-2}) \in GK(\bar{G})$, then use the same procedure to get a contradiction. Therefore $S \ncong D_m(p^\alpha)$.

Case 5. $S \cong B_m(p^\alpha)$ or $S \cong C_m(p^\alpha)$. Since $(r_n(p), 2) \notin GK(S)$ and $e_n = e(r_n(p), p^\alpha)$, by Lemma 2.10(1), we have $e_n \in \{m, 2m\}$. Moreover, we know that $\alpha e_n \in \{n, 2n\}$ and n is odd and hence, there are the following two subcases:

Subcase a. $\alpha m = 2n$. In this case, by considering the order of $B_m(p^\alpha)$ which equals the order of $C_m(p^\alpha)$ we can see that $r_{2\alpha m}(p) \in \pi(S)$ and hence, $r_{2\alpha m}(p) = r_{4n}(p) \in \pi(B_n(p))$, which is impossible by considering the order of $B_n(p)$.

Subcase b. $\alpha m = n$. In this case, it is enough to show that $\alpha = 1$. If not, then set $\rho = \{r_{2(n-1)}(p), r_{2(n-2)}(p), r_{2(n-4)}(p)\}$. We claim that $\rho \bigcap \pi(S) = \emptyset$. If $r_{2(n-1)}(p) \in \pi(S)$, then by considering the order of S, there exists an integer number k such that $0 \leq k \leq m-1$ and $r_{2(n-1)}(p) \mid (p^{2\alpha(m-k)} - 1)$. Thus, $n-1 \mid (\alpha m - \alpha k) = n - \alpha k$ and this implies that $\alpha k = 1$ and hence, $\alpha = 1$, which is a contradiction. Therefore, $r_{2(n-1)}(p) \notin \pi(S)$. Also, by the same argument, we can see that $r_{2(n-2)}(p), r_{2(n-4)}(p) \notin \pi(S)$. Thus $\rho \bigcap \pi(S) = \emptyset$. On the other hand, by Lemma 2.6(3), ρ is a coclique of $GK(B_n(p))$. Now we can get a contradiction by Lemma 2.2(2). Thus $\alpha = 1$ and $S \cong B_n(p)$ or $S \cong C_n(p)$.

Part B. If S is isomorphic to a finite simple exceptional group of Lie type of characteristic p, then by Table 4 in [25], we can see that $t(S) \leq 12$. Since $t(S) \geq t(B_n(p)) - 1 = [\frac{3n+5}{4}] - 1$ and $n \geq 9$ and n is odd, we conclude that $n \in \{9, 11, 13, 15\}$ and $t(S) \geq t(B_9(p)) - 1 = 7$. Therefore, $S \cong E_7(p^\alpha)$ or $S \cong E_8(p^\alpha)$ (see Table 4 in [25]). We consider these two cases separately:

Case 1. $S \cong E_7(p^\alpha)$. Since $(r_n(p), 2) \notin GK(S)$ and $e_n = e(r_n(p), p^\alpha)$, by Lemma 2.10(3), we have $e_n \in \{7, 9\}$ or $e_n \in \{14, 18\}$. Also, since $n \in \{9, 11, 13, 15\}$ and $\alpha e_n \in \{n, 2n\}$, by checking all different cases, we conclude that $n = 9$ and $\alpha \in \{1, 2\}$. Thus $S \cong E_7(p)$ or $S \cong E_7(p^2)$, when $GK(G) = GK(B_9(p))$. If $S \cong E_7(p^2)$, then by checking $|E_7(p^2)|$, we can see that $r_{18}(p^2) = r_{36}(p) \in \pi(S) \subseteq \pi(B_n(p))$, which is impossible. If $S \cong E_7(p)$, then according to $|B_9(p)|$ and $|E_7(p)|$, we can see that $r_{16}(p) \in \pi(B_9(p)) \setminus \pi(E_7(p))$. Also, since $S \leq \bar{G} \leq Aut(S)$ and $Out(E_7(p))$ is a 2-group, we conclude that $r_{16}(p) \in \pi(K)$. Hence, using the cocliques $\rho = \{r_7, r_{16}, r_{18}\}$ and $\tau = \{r_4, r_{16}, r_{18}\}$ of $GK(B_9(p))$ in Lemma 2.1, implies that $\{r_4, r_7\} \bigcap \pi(K) = \emptyset$. Also, by Lemma 2.6(3,5), we can see that $(r_4, r_7) \in GK(B_9(p))$ and $(r_4, r_7) \notin GK(S)$. Now by the same argument in the subcase a, case 3, part A, we can get a contradiction.

Case 2. $S \cong E_8(p^\alpha)$. Since $(r_n(p), 2) \notin GK(S)$ and $e_n = e(r_n(p), p^\alpha)$, by Lemma 2.10(4), we have $e_n \in \{15, 20, 24, 30\}$, and similar to the previous case, by checking all different cases, we conclude that $n = 15$ and $\alpha \in \{1, 2\}$. Thus $S \cong E_8(p)$ or $S \cong E_8(p^2)$, when $GK(G) = GK(B_{15}(p))$. If $S \cong E_8(p^2)$, then by checking the order of $E_8(p^2)$, we can see that $p^{60} - 1$ divides the order of $E_8(p^2)$, and hence, $r_{60}(p) \in \pi(B_{15}(p))$, which is impossible. If $S \cong E_8(p)$, then the order of $E_8(p)$ implies that the coclique $\rho = \{r_{13}, r_{22}, r_{26}\}$ of $GK(B_{15}(p))$ has an empty intersection with $\pi(E_8(p))$ and we can get a contradiction by Lemma 2.2(2). Therefore, S can not be isomorphic to an exceptional groups of Lie type of characteristic p, where $p \in \{3, 7\}$. Moreover, if $p = 5$, then $r_{2n}(p) \in \pi(S)$ and if we put $e_{2n} = e(r_n(p), p^\alpha)$, then we can see that $\alpha e_{2n} = 2n$ and hence, by omitting the subcases c and d, case 1, part A

and the subcase d, case 4, part A and using the remaining statements in part A and part B, we can conclude that S can not be isomorphic to a finite simple group of Lie type of characteristic p, and the proof is now complete.
Hence by Lemmas 3.1, 3.2, 3.3 and 3.4 the Main Theorem is proved. □

Corollary 3.5. *Let n be an odd number. The simple group $B_n(p)$, where $n \geq 9$ and $p \in \{3, 5, 7\}$ are quairecognizable by its spectrum.*

Proof. Let G be a finite group with $\omega(G) = \omega(B_n(p))$. Therefore, $GK(G) = GK(B_n(p))$ and hence, if S is a unique nonabelian composition factor of G, then by using the Main Theorem, we conclude that S is isomorphic to $B_n(p)$ or $C_n(p)$. If $S \cong C_n(p)$, then $\omega(C_n(p)) \subseteq \omega(G) = \omega(B_n(p))$ and this is impossible, because $p(p^{n-1} + 1) \in \omega(C_n(p)) \setminus \omega(B_n(p))$ (see [19, Proposition]). Therefore, the simple groups $B_n(3)$, $B_n(5)$ and $B_n(7)$ are quasirecognizable by their spectra.

□

Corollary 3.6. *Let n be an odd number and $n \geq 9$ and $p \in \{3, 5, 7\}$. If G is a finite group with $|G| = |B_n(p)|$ and $\omega(G) = \omega(B_n(p))$, then $G \cong B_n(p)$.*

Proof. Since $\omega(G) = \omega(B_n(p))$, so $GK(G) = GK(B_n(p))$ and hence, by Lemma 2.2(1), there exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/K \leq Aut(S)$ for the maximal normal soluble subgroup K of G and according to the Corollary 3.5, we have $S \cong B_n(p)$. Moreover, since $|G| = |B_n(p)|$, $|S| = |B_n(p)|$ and $S \leq G/K$, we conclude that $S \cong G$ and hence, $G \cong B_n(p)$. Therefore, the Shi conjecture is true for the simple groups $B_n(3)$, $B_n(5)$ and $B_n(7)$.

□

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