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UNIQUENESS OF MEROMORPHIC FUNCTIONS DEALING WITH MULTIPLE VALUES IN AN ANGULAR DOMAIN

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ABSTRACT. This paper uses the Tsuji's characteristic to investigate the uniqueness of transcendental meromorphic functions with shared values in an angular domain dealing with the multiple values which improve a result of J. Zheng.

1. **Introduction**

In this paper, we shall use the standard notations of the Nevanlinna's value distribution theory (see, e.g., [3, 12]), such as $\delta(a, f)$ to denote the Nevanlinna deficiency of $f(z)$ with respect to $a \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. A transcendental meromorphic function is meromorphic function in the complex plane $\mathbb{C}$ which is not rational. For the sake of convenience, we also give the following notations (see, e.g., [10, 12]). Let X be a subset of $\mathbb{C}_\infty$. An $a \in \mathbb{C}_\infty$ is called an IM (ignoring multiplicities) shared value in X of two functions $f(z)$ and $g(z)$ in X provided that $f(z) = a$ if and only if $g(z) = a$, and, a CM (counting multiplicities) shared value in X if $f(z)$ and $g(z)$ assume a at the same points in X with the same multiplicities. We also use $\overline{E}_{k)}(a, X, f)$ to denote the set of zeros of

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$f(z)-a$ in X, with multiplicities not greater than k, in which each zero is counted only once.

In 1920s Nevanlinna [6] proved that for two nonconstant meromorphic functions f, g, if f and g have five distinct IM shared values in $\mathbb{C}$, then $f\equiv g$. Furthermore, if f and g have four distinct CM shared values in $\mathbb{C}$, then $f=L(g)$, where L denotes a suitable Möbius transformation. For the past nine decades, the so called uniqueness theory of meromorphic functions and its related topics have been zealously pursued by complex analysts throughout the world (see, e.g., [1] and [11]). However, it is well known that four shared values are not sufficient to uniquely determine a meromorphic function. Hence, one considers some additional conditions (see, e.g., [2] et al).

In 2003, Zheng [13, 14] investigated the uniqueness of a meromorphic function in a precise subset of $\mathbb{C}_\infty$ in terms of Nevanlinna characteristic for angular domains and proved the following theorem.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be transcendental meromorphic functions, and let $f(z)$ be of finite lower order μ and for some $a\in\mathbb{C}_\infty,\delta=\delta(a,f)>0$. Given an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0<\beta-\alpha\le 2\pi$ and*

$$\beta-\alpha>\max\{\frac{\pi}{\sigma},2\pi-\frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\},$$

where $\mu\le\sigma\le\rho$ and $\sigma<\infty$, we assume that $f(z)$ and $g(z)$ have four distinct IM shared values in Ω, then $f(z)\equiv g(z)$.

Following Zheng [13], [14] the uniqueness of meromorphic function in angular domains have been pursed by Lin [5], Xuan [10] and the present author [8],[9] et al. Most recently, Zheng [12] proved the following theorem by using Tsuji's characteristic.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be both meromorphic functions in an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$ and $f(z)$ be transcendental in Tsuji's sense. Assume that $f(z)$ and $g(z)$ have four distinct IM shared values $a_j(j=1,2,3,4)$ in Ω. If for some $a\in\mathbb{C}_\infty-\{a_j:j=1,2,3,4\}$, a is a Tsuji deficient value of $f(z)$ in Ω or*

$$\begin{aligned}\rho_\Omega(a)&=\limsup_{r\to\infty}\frac{\log N(r,\Omega,f=a)}{\log r}\\&<\lim_{\varepsilon\to 0}\limsup_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega_\varepsilon,f)}{\log r}=\lim_{\varepsilon\to 0}\lambda_{\Omega_\varepsilon}(f),\end{aligned}$$

where $\mathcal{T}(r,\Omega_\varepsilon,f)$ is the Ahfors-Shimizu characteristic of $f(z)$ for the angular domain $\Omega_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$ and $N(r,\Omega,f=a)=\int_1^r\frac{n(t,\Omega,f=a)}{t}$ with $n(t,\Omega,f=a)$ being the number of zeros of $f-a$ in $\Omega\cap\{z:1<|z|\le r\}$, then $f(z)\equiv g(z)$.

However, it was not discussed whether there are similar results dealing with multiple values in an angular domain. In this paper we investigate this problem and obtain the following general result of which Theorem 1.2 appears as a particular case.

Theorem 1.3. *Let $f(z)$ and $g(z)$ be meromorphic functions in an angular domain $\Omega=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$ and let $f(z)$ be transcendental in Tsuji's sense. Assume that $a_j(j=1,2,\cdots,q)$ are q distinct complex numbers, and let $k_j(j=1,2,\cdots,q)$ be positive integers or ∞ satisfying*

$$k_1\ge k_2\ge\cdots\ge k_q, \tag{1.1}$$

$$\sum_{j=3}^{q}\frac{k_j}{k_j+1}=2, \tag{1.2}$$

$$\overline{E}_{k_j)}(a_j,\Omega,f)=\overline{E}_{k_j)}(a_j,\Omega,g). \tag{1.3}$$

If for some $a\in\mathbb{C}_\infty-\{a_j:j=1,2,\cdots,q\}$, a is a Tsuji deficient value of $f(z)$ in Ω or $\rho_\Omega(a)<\lim\limits_{\varepsilon\to 0}\lambda_{\Omega_\varepsilon}(f)$, then $f(z)\equiv g(z)$.

Remark 1.4. *In Theorem 1.2, $q=4$, $k_j=\infty$ $(j=1,2,3,4)$ and $\sum\limits_{j=3}^{4}\frac{k_j}{k_j+1}=2$. So Theorem 1.2 is a special case of Theorem 1.3.*

2. **Preliminaries**

Our proof requires the Tsuji's characteristic (see [4], [12]): if $f(z)$ is meromorphic in an angular domain Ω and $\omega=\frac{\pi}{\beta-\alpha}$, we define

$$\mathfrak{M}_{\alpha,\beta}(r,f)=\frac{1}{2\pi}\int_{\arcsin(r^{-\omega})}^{\pi-\arcsin(r^{-\omega})}\log^+|f(re^{i(\alpha+\omega^{-1}\theta)}\sin^{\omega^{-1}}\theta)|\frac{1}{r^\omega\sin^2\theta}d\theta,$$

$$\mathfrak{N}_{\alpha,\beta}(r,f) = \sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}} \left(\frac{\sin\omega(\beta_n-\alpha)}{|b_n|^{\omega}} - \frac{1}{r^{\omega}}\right)$$
$$= \omega\int_1^r \frac{\mathfrak{n}_{\alpha,\beta}(t,f)}{t^{\omega+1}}dt,$$

where b_n's are the poles of $f(z)$ in $\Xi(\alpha,\beta,r) = \{z = te^{i\theta} : \alpha < \theta < \beta, 1 < t \le r(\sin(\omega(\theta-\alpha)))^{\omega^{-1}}\}$ appearing often according to their multiplicities and $\mathfrak{n}_{\alpha,\beta}(t,f)$ is the number of poles of $f(z)$ in $\Xi(\alpha,\beta,r)$. When a pole b_n occurs in the sum $\sum\limits_{1<|b_n|<r}(\sin(\omega(\beta_n-\alpha)))^{\omega^{-1}}$ only once, we denote it by $\overline{\mathfrak{N}}(r,f)$. Setting $\mathfrak{T}_{\alpha,\beta}(r,f) = \mathfrak{M}_{\alpha,\beta}(r,f) + \mathfrak{N}_{\alpha,\beta}(r,f)$, we have the following properties of the Tsuji's characteristic. If $f(z)$ is nonconstant then for all complex numbers a,

$$\mathfrak{T}_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = \mathfrak{T}_{\alpha,\beta}(r,f) + O(1). \tag{2.1}$$

Also

$$(q-2)\mathfrak{T}_{\alpha,\beta}(r,f) \le \sum_{j=1}^{q} \overline{\mathfrak{N}}\left(r, \frac{1}{f-a_j}\right) + Q_{\alpha,\beta}(r,f), \tag{2.2}$$

for q distinct points $a_j \in \mathbb{C}_\infty$ and

$$Q_{\alpha,\beta}(r,f) = O(\log^+ \mathfrak{T}_{\alpha,\beta}(r,f) + \log r), r \notin E. \tag{2.3}$$

where E denotes a set with finite linear measure. It is not necessarily the same for every occurrence in the context. For the sake of simplicity, we omit the subscript in all notations and use $\mathfrak{M}(r,f)$, $\mathfrak{N}(r,f)$, $Q(r,f)$ and $\mathfrak{T}(r,f)$ instead of $\mathfrak{M}_{\alpha,\beta}(r,f)$, $\mathfrak{N}_{\alpha,\beta}(r,f)$, $Q_{\alpha,\beta}(r,f)$ and $\mathfrak{T}_{\alpha,\beta}(r,f)$, respectively.

Definition 2.1. *A meromorphic function f in an angular domain Ω is called transcendental in the Tsuji's dense if*

$$\limsup_{r\to\infty} \frac{\mathfrak{T}(r,f)}{\log r} = \infty.$$

Definition 2.1 was first given by Zheng in [12]. In [12], Zheng introduce the Tsuji deficiency of $f(z)$ meromorphic in Ω and transcendental in the sense of Tsuji. Set

$$\delta_T(a,f) = \liminf_{r\to\infty} \frac{\mathfrak{M}\left(r, \frac{1}{f-a}\right)}{\mathfrak{T}\left(r, \frac{1}{f-a}\right)} = 1 - \liminf_{r\to\infty} \frac{\mathfrak{N}\left(r, \frac{1}{f-a}\right)}{\mathfrak{T}\left(r, \frac{1}{f-a}\right)},$$

then $\delta_T(a,f)$ is called the Tsuji deficiency of $f(z)$ at a and if $\delta_T(a,f)>0$, then a is said to be a Tsuji deficient value of $f(z)$. It is also obvious that the total sum of all Tsuji deficiencies does not exceed 2, and there are at most a countable number of Tsuji deficient values.

In order to prove Theorem 1.3, we establish the following key Lemma by referring to Lo Yang's method in dealing with the multiple values problem. Let $f(z)$ denote a nonconstant meromorphic function in an angular domain Ω, a an arbitrary complex number, and k a positive integer. We use $\overline{\mathfrak{N}}^{k)}\left(r,\frac{1}{f-a}\right)$ to denote the zeros of $f(z)-a$ in Ω whose multiplicities are not greater than k and are counted only once. Likewise, we use $\overline{\mathfrak{N}}^{(k+1}\left(r,\frac{1}{f-a}\right)$ to denote the zeros of $f(z)-a$ in Ω whose multiplicities are greater than k and are counted only once.

Lemma 2.2. *Let $f(z)$ be a nonconstant meromorphic function in an angular domain Ω, $a_j\in\mathbb{C}_\infty(j=1,2,\cdots,q)$ $q(\geq 3)$ distinct complex numbers, and $k_j(j=1,2,\cdots,q)$ q positive integers. Then*

$$(2.4)\qquad \begin{aligned}(q-2)\mathfrak{T}(r,f) &< \sum_{j=1}^{q}\tfrac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}\left(r,\tfrac{1}{f-a_j}\right)\\ &\quad+\sum_{j=1}^{q}\tfrac{1}{k_j+1}\mathfrak{N}\left(r,\tfrac{1}{f-a_j}\right)+Q(r,f),\end{aligned}$$

$$(2.5)\qquad \begin{aligned}\left(q-2-\sum_{j=1}^{q}\tfrac{1}{k_j+1}\right)\mathfrak{T}(r,f) &< \sum_{j=1}^{q}\tfrac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}\left(r,\tfrac{1}{f-a_j}\right)\\ &\quad+Q(r,f),\end{aligned}$$

where the term $\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)$ will be replaced by $\overline{\mathfrak{N}}^{k_j)}(r,f)$ when $a_j=\infty$.

Proof. Note that

$$\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)=\overline{\mathfrak{N}}^{k)}\left(r,\frac{1}{f-a}\right)+\overline{\mathfrak{N}}^{(k+1}\left(r,\frac{1}{f-a}\right),$$

and

$$\overline{\mathfrak{N}}^{(k+1}\left(r,\frac{1}{f-a}\right)\leq\frac{1}{k+1}\mathfrak{N}^{(k+1}\left(r,\frac{1}{f-a}\right).$$

Then

$$\begin{aligned}\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right) &\leq \frac{k}{k+1}\overline{\mathfrak{N}}^{k)}\left(r,\frac{1}{f-a}\right)+\frac{1}{k+1}\overline{\mathfrak{N}}^{k)}\left(r,\frac{1}{f-a}\right)\\ &+\frac{1}{k+1}\overline{\mathfrak{N}}^{(k+1}\left(r,\frac{1}{f-a}\right)\\ &\leq \frac{k}{k+1}\overline{\mathfrak{N}}^{k)}\left(r,\frac{1}{f-a}\right)+\frac{1}{k+1}\mathfrak{N}\left(r,\frac{1}{f-a}\right)\end{aligned}$$

By (2.3), we have

$$\begin{aligned}(q-2)\mathfrak{T}(r,f) &\leq \sum_{j=1}^{q}\overline{\mathfrak{N}}\left(r,\frac{1}{f-a_j}\right)+Q(r,f)\\ &\leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{q}\frac{1}{k_j+1}\mathfrak{N}\left(r,\frac{1}{f-a_j}\right)+Q(r,f),\end{aligned}$$

and (2.4) follows. Furthermore, $\mathfrak{N}\left(r,\frac{1}{f-a_j}\right)<\mathfrak{T}_{\alpha,\beta}(r,f)$, and combining this with (2.4), we get (2.5). □

The Ahlfors-Shimizu characteristic of $f(z)$ for an angle is important and applicable in this paper. For $\overline{\Omega}=\{z:\alpha\leq\arg z\leq\beta\}$, define (see [7], [12])

$$\mathcal{T}(r,\Omega,f)=\int_0^r\frac{\mathcal{A}(t,\Omega,f)}{t}dt,$$

$$\mathcal{A}(t,\Omega,f)=\frac{1}{\pi}\int_\alpha^\beta\int_0^r\left(\frac{|f'(te^{i\phi})|}{(1+|f(te^{i\phi})|^2)}\right)^2 tdtd\phi.$$

Lemma 2.3. [12]. *Let $f(z)$ be a transcendental and meromorphic function in Ω. Set*

$$\lambda_{\Omega_\varepsilon}(f)=\limsup_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega_\varepsilon,f)}{\log r}\leq\lambda_\Omega(f)=\limsup_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega,f)}{\log r}.$$

Then there exist at most two $a\in\mathbb{C}_\infty$ such that $\rho_\Omega(f,a)<\lim_{\varepsilon\to 0}\lambda_{\Omega_\varepsilon}(f)$.

Lemma 2.4. [12]. *Let $f(z)$ be a meromorphic function in Ω. Then for $\varepsilon>0$, we have*

$$\mathfrak{N}(r,f)\geq\omega c^\omega\frac{N(cr,\Omega_\varepsilon,f)}{r^\omega}.$$

3. Proof of Theorem 1.3

Suppose that $f(z) \not\equiv g(z)$. By using Lemma 2.2 to f we have

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)\mathfrak{T}(r,f) < \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right) + Q(r,f). \tag{3.1}$$

The equality (1.3) implies that

$$1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2}.$$

This and (3.1) yield

$$\begin{aligned}&\left(\sum_{j=1}^{q}\frac{k_j}{k_j+1}-2\right)\mathfrak{T}(r,f)\\ &\leq \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+\left(\frac{k_1}{k_1+1}-\frac{k_2}{k_2+1}\right)\overline{\mathfrak{N}}^{k_1)}\left(r,\frac{1}{f-a_1}\right)+Q(r,f)\\ &\leq \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+\left(\frac{k_1}{k_1+1}-\frac{k_2}{k_2+1}\right)\mathfrak{T}(r,f)+Q(r,f).\end{aligned}$$

Therefore,

$$\left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{2k_2}{k_2+1}-2\right)\mathfrak{T}(r,f) < \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right) + Q(r,f). \tag{3.2}$$

But (1.2) and (3.2) mean

$$2\mathfrak{T}(r,f) \leq \sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+Q(r,f). \tag{3.3}$$

From(1.3) and (2.2), it follows that

$$\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right) \leq \mathfrak{N}\left(r,\frac{1}{f-g}\right) \leq \mathfrak{T}(r,f)+\mathfrak{T}(r,g)+O(1). \tag{3.4}$$

Then (3.3) and (3.4) yield

$$\mathfrak{T}(r,f) \leq \mathfrak{T}(r,g)+Q(r,f). \tag{3.5}$$

Similarly, we have

$$\mathfrak{T}(r,g) \leq \mathfrak{T}(r,f)+Q(r,g). \tag{3.6}$$

Thus (3.3), (3.4) and (3.6) lead to

$$2\mathfrak{T}(r,f)\le \sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+Q(r,f)\le 2\mathfrak{T}(r,f)+Q(r,f).$$

Hence,

$$\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)=2\mathfrak{T}(r,f)+Q(r,f). \tag{3.7}$$

Assume without any loss of generality that $a\in\mathbb{C}$ and indeed the same argument is available to complete the proof for the case when $a=\infty$.Using Lemma 2.2 to f we have

$$\begin{aligned}\left(q+1-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)\mathfrak{T}(r,f)\quad &\le \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)\\ &+Q(r,f).\end{aligned}$$

This yields

$$\begin{aligned}\left(\sum_{j=1}^{q}\frac{k_j}{k_j+1}-1\right)\mathfrak{T}(r,f)\quad &\le \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)\\ &+\left(\frac{k_1}{k_1+1}-\frac{k_2}{k_2+1}\right)\overline{\mathfrak{N}}^{k_1)}\left(r,\frac{1}{f-a_1}\right)\\ &+\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)+Q(r,f)\\ &\le \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)\\ &+\left(\frac{k_1}{k_1+1}-\frac{k_2}{k_2+1}\right)\mathfrak{T}(r,f)+\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)+Q(r,f).\end{aligned}$$

Hence,

$$\begin{aligned}\left(\sum_{j=3}^{q}\frac{k_j}{k_j+1}+\frac{2k_2}{k_2+1}-1\right)\mathfrak{T}(r,f)\quad &\le \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)\\ &+Q(r,f).\end{aligned}$$

Combining the above formulas with (1.2) we have

$$\begin{aligned}\left(\frac{2k_2}{k_2+1}+1\right)\mathfrak{T}(r,f)\quad &\le \frac{k_2}{k_2+1}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}\left(r,\frac{1}{f-a_j}\right)+\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)\\ &+Q(r,f).\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) gives,

$$\left(\frac{2k_2}{k_2+1}+1\right)\mathfrak{T}(r,f)\le \frac{k_2}{k_2+1}2\mathfrak{T}(r,f)+\overline{\mathfrak{N}}\left(r,\frac{1}{f-a}\right)+Q(r,f).$$

Thus

$$\mathfrak{M}\left(r, \tfrac{1}{f-a}\right) = Q(r, f), \tag{3.9}$$

and further a cannot be a Tsuji deficient value of $f(z)$. The following method comes from [12]. Now suppose $\rho_\Omega(a) < \lim_{\varepsilon\to 0} \lambda_{\Omega_\varepsilon}(f)$ and so for some $\varepsilon > 0, \rho_\Omega(a) < \lambda_{\Omega_\varepsilon}(f)$. Then there is a σ with $\sigma < \lambda_{\Omega_\varepsilon}(f)$ such that $\mathfrak{n}_{\alpha,\beta}(r, f = a) < K_1 r^\sigma$ for $r \geq 1$. If $\sigma \leq \omega$, then we have

$$\mathfrak{N}\left(r, \frac{1}{f-a}\right) = \omega \int_1^r \frac{\mathfrak{n}_{\alpha,\beta}(t, f = a)}{t^{\omega+1}} dt \leq \omega K_1 \log r.$$

This implies that

$$\mathfrak{T}(r, f) = \mathfrak{T}\left(r, \frac{1}{f-a}\right) + O(1) = Q(r, f),$$

and so a contradiction to $f(z)$ to be transcendental in Tsuji's sense. Therefore, we have $\omega < \sigma$ and

$$\mathfrak{N}\left(r, \frac{1}{f-a}\right) = \omega \int_1^r \frac{\mathfrak{n}_{\alpha,\beta}(t, f = a)}{t^{\omega+1}} dt < K_1 \frac{\omega}{\sigma - \omega} r^{\sigma-\omega}.$$

This together with (3.9) yield that

$$\mathfrak{T}(r, f) = \mathfrak{T}\left(r, \frac{1}{f-a}\right) + O(1) \leq K_1 \frac{\omega}{\sigma - \omega} r^{\sigma-\omega} + Q(r, f).$$

Thus using (2.2) and (2.4) yields

$$\mathfrak{T}\left(r, \frac{1}{f-b}\right) \leq K_2 r^{\sigma-\omega} + O(1),$$

for each $b \in \mathbb{C}_\infty$ and a positive constant K_2. In virtue of Lemma 2.4, the following implication is clear:

$$\mathfrak{T}\left(r, \frac{1}{f-b}\right) \geq \mathfrak{N}\left(r, \frac{1}{f-b}\right) \geq \omega c^\omega \frac{N(cr, \Omega_{\varepsilon/2}, f = b)}{r^\omega}$$

for some $0 < c < 1$. This implies that

$$N(cr, \Omega_{\varepsilon/2}, f = b) \leq K r^\sigma$$

for a positive constant K, and so $\rho_{\Omega_{\varepsilon/2}}(b) \leq \sigma$. In view of Lemma 2.3, $\lambda_{\Omega_\varepsilon}(f) \leq \sigma$. Thus a contradiction is derived, and Theorem 1.3 follows.

In fact, from the above proof of Theorem 1.3, we get:

Corollary 3.1. *Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions in an angular domain Ω. Assume that $a_j(j = 1, 2, \cdots, q)$ are q distinct complex numbers, and let $k_j(j = 1, 2, \cdots, q)$ be positive integers or ∞ satisfying* (1.1)*,* (1.2) *and* (1.3)*. Then*

$$\mathfrak{T}(r, f) = \mathfrak{T}(r, g) + Q(r, g), \quad \mathfrak{T}(r, g) = \mathfrak{T}(r, f) + Q(r, g);$$

$$\sum_{j=1}^{q} \overline{\mathfrak{N}}^{k_j)}\left(r, \frac{1}{f - a_j}\right) = 2\mathfrak{T}(r, f) + Q(r, f);$$

$$\sum_{j=1}^{q} \overline{\mathfrak{N}}^{k_j)}\left(r, \frac{1}{g - a_j}\right) = 2\mathfrak{T}(r, g) + Q(r, g).$$

Remark 3.2. *The corresponding result of Corollary 3.1 in the whole complex plane was obtained by C. C. Yang and H. Yi in* [11]*.*

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