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NEWTON-PRODUCT INTEGRATION FOR A TWO-PHASE STEFAN PROBLEM WITH KINETICS

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Abstract. We reduce the two phase Stefan problem with kinetic to a system of nonlinear Volterra integral equations of second kind and apply Newton's method to linearize it. We provide the product integration solution of the linear form. Sufficient conditions for convergence of the numerical method are given and their applicability is illustrated with an example.

1. Introduction

We consider the modified two-phase Stefan problem in one spatial variable

$$u_t = u_{xx} - \gamma u, \qquad x \neq s(t), \tag{1.1}$$

$$u(x,0) = u^0(x) \geq 0, \tag{1.2}$$

$$g(u(s(t),t)) = V(t), \qquad t > 0, \tag{1.3}$$

$$[u_x(s(t),t)] := u_x^+(s(t),t) - u_x^-(s(t),t) = V(t), \qquad t > 0. \tag{1.4}$$

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Here $u(x,t)$ is temperature and $\gamma \geq 0$ is damping term due to the volumetric heat losses. The two boundary conditions above determine the problem and allow us to find the free boundary denoted by $s(t)$, and $V(t) = \dot{s}(t)$ is the interface velocity.
Assume that $g(t)$ is monotonically decreasing differentiable function on $[0, \infty)$ with $|g^{'}| \leq C$ and satisfying

$$-V_0 \leq g(t) \leq -v_0 \quad \text{for some } v_0 > 0, V_0 > 0.$$

The free-boundary problem (1.1)-(1.4) arises naturally as a mathematical model of a variety of exothermic phase transition type processes, such as solid combustion [7] also known as Self-propagating High-temperature Synthesis or SHS [8], solidification with undercooling [6], laser induced evaporation [5], rapid crystallization in thin films [10] and etc. These processes are characterized by production of heat at the interface, and their dynamics is determined by the feedback mechanism between the heat release due to the kinetics and the heat dissipation by the medium. In addition to its theoretical interest, SHS has industrial applications as a method of synthesizing certain technologically advanced materials for high-temperature semiconductors, nuclear safety devices, fuel cells etc. (see [8], [11] and also [12] for a popular expositions). SHS propagates through mixtures of fine elemental reactant powders (e.g., Ti + C, Ti + 2B), resulting in the synthesis of compounds.
In Section 2 a local existence condition is obtained. In Section 3 the Stefan problem with kinetics reduced to a system of nonlinear Volterra integral equations of second kind and Newton's method is applied to linearize it. A convergence analysis of Newton's method for the problem is provided in the Subsections of Section 3. Product integration solution of the linear form is obtained in Section 4. Convergence of product integration method is given in Subsection 4.1. Finally in Section 5, numerical results of test problem solved by the proposed method is reported.

2. **Existence and Uniqueness of Local Classical Solutions**

A short-time solution of the free boundary problem (1.1)-(1.4) will be sought in the form of a superposition of heat potentials
(2.1)

$$u(x,t) = e^{-\gamma t}\int_{-\infty}^{\infty} G(x,\xi,t)u^0(\xi)d\xi - \int_0^t G(x,s(\tau),t-\tau)e^{-\gamma(t-\tau)}V(\tau)d\tau,$$

where G is the fundamental solution of the heat equation

$$G(x,\xi,t-\tau)=\exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right][4\pi(t-\tau)]^{-\frac{1}{2}}.$$

Taking the limit of (2.1) as $x\to s(t)$ and using the kinetics condition (1.3) we obtain an integral equation in terms of V only [4]

$$V=K(V), \tag{2.2}$$

where the nonlinear operator K is defined as follows

$$\begin{aligned} K(V)(t) \quad = \quad & g\Bigg\{e^{-\gamma t}\int_{-\infty}^{\infty}G(s(t),\xi,t)u^0(\xi)d\xi \\ & -\int_0^t G(s(t),s(\tau),t-\tau)e^{-\gamma(t-\tau)}V(\tau)d\tau\Bigg\}, \end{aligned}$$

and here as usual,

$$s(t)=\int_0^t V(\tau)d\tau.$$

Now $u(x,t),V(t)$ form a classical solution of (1.1)-(1.4) if

(i) $u(x,t)$ and $V(t)$ are continuous for $t\geq 0$;
(ii) u_{xx} and u_t are continuous for $x\neq s(t),t>0$;
(iii) equations (1.1)-(1.4) are satisfied.

The following theorem is stated in [4]:

Theorem 2.1. *Suppose that the kinetic function g satisfies the following assumptions:*
($\mathbf{A_1}$) *$g(u)$ is a continuously differentiable, monotone decreasing, negative function on $(0,\infty)$ with $g(0)=-v_0$ for some velocity $-v_0<0$;*
($\mathbf{A_2}$) *$g(u)$ is sublinear: $\lim_{u\to\infty} g(u)/u=0$;*
and that the initial data $u^0(x)\geq 0$ are bounded.
Then there exists one and only one classical solution $u(x,t)>0$ and $V(t)$ of the free interface problem (1.1)-(1.4). This solution is uniformly bounded for all $t>0$.

3. Application of the Newton's Method

For applying Newton's method to linearize the problem (2.2), define

$$S:=\left\{V\middle|V\in C[0,b]\right\}, \tag{3.1}$$

for some $b > 0$. Then S is a Banach space with the maximum norm of $C[0, b]$. Introducing an operator $T : S \to S$ through the formula

$$T(V) = V - K(V), \qquad V \in S. \tag{3.2}$$

Problem (2.2) can be written in the form

$$T(V) = 0. \tag{3.3}$$

Now Newton's method for finding roots of (3.3) is

$$V_{n+1} = V_n - [T'(V_n)]^{-1}T(V_n),$$

where $n = 0, 1, 2 \cdots$. Putting

$$\delta_{n+1} := V_{n+1} - V_n,$$

gives the following linear system

$$[T'(V_n)]\delta_{n+1} = -T(V_n). \tag{3.4}$$

To set the numerical process it is sufficient to evaluate derivative of T. Let $U, V \in C[0, b]$, then

(3.5)

$$\begin{aligned}
T'(V)U =& \lim_{h\to 0} h^{-1}[T(V + hU) - T(V)] \\
=& U - \lim_{h\to 0} h^{-1}[K(V + hU) - K(V)] \\
=& U - \lim_{h\to 0} h^{-1}\Bigg[g\Bigg\{e^{-\gamma t}\int_{-\infty}^{\infty} G((s + h\sigma)(t), \xi, t)u^0(\xi)d\xi \\
&- \int_0^t G((s + h\sigma)(t), (s + h\sigma)(\tau), t - \tau)e^{-\gamma(t-\tau)}(V + hU)(\tau)d\tau\Bigg\} \\
&-g\Bigg\{e^{-\gamma t}\int_{-\infty}^{\infty} G(s(t), \xi, t)u^0(\xi)d\xi \\
&- \int_0^t G(s(t), s(\tau), t - \tau)e^{-\gamma(t-\tau)}V(\tau)d\tau\Bigg\}\Bigg] \\
=& \; U - \lim_{h\to 0} h^{-1}[g(\alpha(V + hU; t)) - g(\alpha(V; t))] \\
=& \; U - \lim_{h\to 0} h^{-1}[\Delta\alpha g'(\alpha_1)] \\
=& \; U - g'(\alpha(V; t)) \lim_{h\to 0} h^{-1}\Delta\alpha,
\end{aligned}$$

where

$$\alpha(V;t) = e^{-\gamma t}\int_{-\infty}^{\infty} G(s(t),\xi,t)u^0(\xi)d\xi - \int_0^t G(s(t),s(\tau),t-\tau)e^{-\gamma(t-\tau)}V(\tau)d\tau, \tag{3.6}$$

$$\Delta\alpha(t) = \alpha(V+hU;t) - \alpha(V;t), \quad \sigma(t) = \int_0^t U(\tau)d\tau,$$

$$\alpha_1 = \theta\alpha(V+hU;t) - (1-\theta)\alpha(V;t), \quad \text{for some } \theta \in (0,1).$$

The last term in (3.5) is

(3.7)

$$\begin{aligned}
&\lim_{h\to 0} h^{-1}\Delta\alpha \\
= \;& \lim_{h\to 0} h^{-1}\Big[e^{-\gamma t}\int_{-\infty}^{\infty}\{G((s+h\sigma)(t),\xi,t) - G(s(t),\xi,t)\}u^0(\xi)d\xi \\
& - \int_0^t \{G((s+h\sigma)(t),(s+h\sigma)(\tau),t-\tau)(V+hU)(\tau) \\
& - G(s(t),s(\tau),t-\tau)V(\tau)\}e^{-\gamma(t-\tau)}d\tau\Big] \\
= \;& e^{-\gamma t}\int_{-\infty}^{\infty}\lim_{h\to 0} h^{-1}\delta G(\xi,t)u^0(\xi)d\xi \\
& - \int_0^t \lim_{h\to 0} h^{-1}\Delta G(t,\tau)V(\tau)e^{-\gamma(t-\tau)}d\tau \\
& - \int_0^t G(s(t),s(\tau),t-\tau)U(\tau)e^{-\gamma(t-\tau)}d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\lim_{h\to 0} h^{-1}\delta G(\xi,t) &:= \lim_{h\to 0} h^{-1}\{G((s+h\sigma)(t),\xi,t) - G(s(t),\xi,t)\} \\
&= \sigma(t)\frac{\xi - s(t)}{2t}G(s(t),\xi,t), \\
\lim_{h\to 0} h^{-1}\Delta G(t,\tau) &:= \lim_{h\to 0} h^{-1}\big\{G((s+h\sigma)(t),(s+h\sigma)(\tau),t-\tau) \\
&\quad - G(s(t),s(\tau),t-\tau)\big\} \\
&= (\sigma(\tau)-\sigma(t))\frac{s(t)-s(\tau)}{2(t-\tau)}G(s(t),s(\tau),t-\tau).
\end{aligned}$$

Substitution the above results in (3.7) gives

$$\lim_{h\to 0} h^{-1}\Delta\alpha = \sigma(t)I_1(V;t) + I_2(V,U;t) + I_3(V,U;t), \tag{3.8}$$

where

$$I_1(V;t) = \frac{e^{-\gamma t}}{2t}\int_{-\infty}^{\infty}(\xi - s(t))G(s(t),\xi,t)u^0(\xi)d\xi,$$

$$I_2(V,U;t) = -\int_0^t \lim_{h\to 0} h^{-1}\Delta G(t,\tau)V(\tau)e^{-\gamma(t-\tau)}d\tau$$

$$= \int_0^t U(\tau)\left[\int_0^\tau \frac{s(t)-s(\widehat{\tau})}{2(t-\widehat{\tau})}G(s(t),s(\widehat{\tau}),t-\widehat{\tau})V(\widehat{\tau})e^{-\gamma(t-\widehat{\tau})}d\widehat{\tau}\right]d\tau,$$

$$I_3(V,U;t) = -\int_0^t U(\tau)G(s(t),s(\tau),t-\tau)e^{-\gamma(t-\tau)}d\tau.$$

Using (3.5) and (3.8) gives

$$T'(V)U = U(t) - g'(\alpha(V;t))(\sigma(t)I_1(V;t) + I_2(V,U;t) + I_3(V,U;t)). \tag{3.9}$$

3.1. **Convergence of Newton's Method.** The operator T in (3.2) satisfies the hypothesis of the following theorem.

Theorem 3.1. *(Kantorovich) Let X, Y be two Banach spaces and suppose that*
(a) $T : D(T) \subseteq X \to Y$ is differentiable on an open convex set $D(T)$, and the derivative is Lipschitz continuous

$$\|T'(U) - T'(V)\| \leq L\|U - V\| \quad \forall U, V \in D(T).$$

(b)For some $V_0 \in D(T), [T'(V_0)]^{-1}$ exists and is a continuous operator from X to Y, and such that $h = acL \leq 1/2$ for some $a \geq \|[T'(V_0)]^{-1}\|$ and $c \geq \|[T'(V_0)]^{-1}T(V_0)\|$. Denote

$$t^* = \frac{1-(1-2h)^{1/2}}{aL}, \quad t^{**} = \frac{1+(1-2h)^{1/2}}{aL}.$$

(c) V_1 is chosen so that $\overline{B}(V_1, r) \subseteq D(T)$, where $r = t^* - c$.
Then the equation (3.3) has a solution $V^* \in \overline{B}(V_1, r)$ and the solution is unique in $\overline{B}(V_0, t^{**}) \cap D(T)$; the sequence $\{V_n\}$, generated by Newton's method, converges to V^*, and the error estimate is given by

$$\|V_n - V^*\| \leq \frac{(1-(1-2h)^{1/2})^{2^n}}{2^n aL}, \ n = 0, 1, ...$$

Proof. See Theorem 4.4.2 [1]. □

By choosing $D(T) = S$ in (3.1), it is clear that condition (c) is automatically satisfied. For condition (b), if we let $V_0 \in S$ such that $\|T(V_0)\| \leq 1$, then

$$\|[T'(V_0)]^{-1}(T(V_0))\| \leq \|[T'(V_0)]^{-1}\| \leq a.$$

Thus putting $c = a$; it is sufficient to show that $a^2 L \leq 1/2$. By (3.9) one can write

$$T'(V) = I - A(V),$$

where

$$A(V)U = g'(\alpha(V;t))(\sigma(t)I_1(V;t) + I_2(V,U;t) + I_3(V,U;t)).$$

If $\delta := \|A(V_0)\| < 1$, then from the geometric series theorem [1], $[T'(V_0)]^{-1}$ is a linear continuous operator on S, and $\|[T'(V_0)]^{-1}\| \leq \frac{1}{1-\delta}$. Take $a = \frac{1}{1-\delta}$. Then it follows that for condition (b) it is sufficient to show that

(1) $\|T(V_0)\| \leq 1$ is valid for some $V_0 \in S$ and $g \in C[0,\infty)$,
(2) $\delta := \|A(V_0)\| < 1$ is valid for some $b > 0, V_0 \in S$,
(3) $\frac{L}{(1-\delta)^2} \leq \frac{1}{2}$ is valid for some $b > 0$.

These plus condition (a) will prove the Theorem. The proofs of (1), (2) and (3) are given on 3.3-3.5.

3.2. **The Derivative Operator is Lipschitz Continuous.** It is sufficient to show that for all $b > 0$ there exists $L = L(b) > 0$ such that for $V, \widetilde{V} \in C[0,b]$,

$$\|T'(V) - T'(\widetilde{V})\| \leq L\|V - \widetilde{V}\|,$$

where all norms are the standard norm in $C[0,b]$. Let $\widehat{V} \in C[0,b]$ and $\|\widehat{V}\| \leq 1$ be arbitrary. Then

$$\|(T'(V) - T'(\widetilde{V}))\widehat{V}\| = \|(I - AV)\widehat{V} - (I - A\widetilde{V})\widehat{V}\| = \|(AV - A\widetilde{V})\widehat{V}\|.$$

Thus it is sufficient to show that

$$\|A(V)\widehat{V} - A(\widetilde{V})\widehat{V}\| \leq L\|V - \widetilde{V}\|.$$

Let $\alpha = \alpha(V;t), \beta = \beta(V,\widehat{V};t), \widetilde{\alpha} = \alpha(\widetilde{V};t), \widetilde{\beta} = \beta(\widetilde{V},\widehat{V};t)$, where α is defined in (3.6) and

$$\beta(V,\widehat{V};t) = \widehat{s}(t)I_1(V;t) + I_2(V,\widehat{V};t) + I_3(V,\widehat{V};t), \widehat{s}(t) = \int_0^t \widehat{V}(\tau)d\tau.$$

Since g' is bounded and Lipschitz continuous, hence

$$\exists C_1 > 0, \exists L_1 > 0 \text{ such that } |g'(x)| \leq C_1, |g'(x) - g'(y)| \leq L_1|x - y|,$$
$$\forall x, y \in [0, \infty).$$

Thus

(3.10)
$$\begin{aligned} |A(V)\widehat{V} - A(\widetilde{V})\widehat{V}| &= |g'(\alpha)\beta - g'(\widetilde{\alpha})\widetilde{\beta}| \\ &= |g'(\alpha)\beta - g'(\alpha)\widetilde{\beta} + g'(\alpha)\widetilde{\beta} - g'(\widetilde{\alpha})\widetilde{\beta}| \\ &\leq |g'(\alpha)||\beta - \widetilde{\beta}| + |\widetilde{\beta}||g'(\alpha) - g'(\widetilde{\alpha})| \\ &\leq C_1|\beta - \widetilde{\beta}| + L_1|\widetilde{\beta}||\alpha - \widetilde{\alpha}|. \end{aligned}$$

The following gives some upper bounds for the right hand side quantities in (3.10).

$$\begin{aligned} &|\alpha - \widetilde{\alpha}| = \\ &\Big| e^{-\gamma t} \textstyle\int_{-\infty}^{\infty} (s(t) - \widetilde{s}(t)) \frac{\partial G}{\partial x}(\overline{s}(t), \xi, t) u^0(\xi) d\xi \\ &- \textstyle\int_0^t \big[G(s(t) - s(\tau), 0, t - \tau) V(\tau) - G(s(t) - s(\tau), 0, t - \tau) \widetilde{V}(\tau) \\ &+ G(s(t) - s(\tau), 0, t - \tau) \widetilde{V} G(\widetilde{s}(t) - \widetilde{s}(\tau), 0, t - \tau) \widetilde{V}(\tau) \big] \\ &\quad e^{-\gamma(t-\tau)} d\tau \Big| \\ &\leq \tfrac{e^{-\gamma t}}{\sqrt{4\pi t}} |s(t) - \widetilde{s}(t)| \textstyle\int_{-\infty}^{\infty} \frac{\xi - \overline{s}(t)}{2t} \exp\left\{ -\frac{(\overline{s}(t) - \xi)^2}{4t} \right\} u^0(\xi) d\xi \\ &+ \textstyle\int_0^t e^{-\gamma(t-\tau)} G(s(t) - s(\tau), 0, t - \tau) |V(\tau) - \widetilde{V}(\tau)| d\tau \\ &+ \textstyle\int_0^t e^{-\gamma(t-\tau)} |\widetilde{V}(\tau)| |G(s(t) - s(\tau), 0, t - \tau) - G(\widetilde{s}(t) \\ &\quad - \widetilde{s}(\tau), 0, t - \tau)| d\tau \\ &\leq \left(\sqrt{\tfrac{t}{\pi}} e^{-\gamma t} \|u^0\| + \tfrac{Erf(\sqrt{\gamma t})}{2\sqrt{\gamma}} \right) \|V - \widetilde{V}\| \\ &+ \tfrac{\|\widetilde{V}\|}{\sqrt{4\pi}} \|V - \widetilde{V}\| \textstyle\int_0^t e^{-\gamma(t-\tau)} \left| \frac{\underline{s}(t,\tau)}{\sqrt{4(t-\tau)}} \right| \exp\left\{ - \left| \frac{\underline{s}(t,\tau)}{\sqrt{4(t-\tau)}} \right|^2 \right\} d\tau \\ &\leq \left(\sqrt{\tfrac{t}{\pi}} e^{-\gamma t} \|u^0\| + \tfrac{Erf(\sqrt{\gamma t})}{2\sqrt{\gamma}} + \tfrac{\|\widetilde{V}\|}{\gamma\sqrt{8e\pi}} (1 - e^{-\gamma t}) \right) \|V - \widetilde{V}\| \\ &=: B(t) \|V - \widetilde{V}\|, \end{aligned}$$

where $\overline{s}\,(t)$ and $\underline{s}(t,\tau)$ are intermediate values obtained from the mean value theorem. Similar evaluations yield

(3.11)
$$\begin{aligned}|\beta-\widetilde{\beta}| &\leq |\widehat{s}(t)[I_1(V;t)-I_1(\widetilde{V};t)]|+|I_2(V,\widehat{V};t)-I_2(\widetilde{V},\widehat{V};t)| \\ &\quad +|I_3(V,\widehat{V};t)-I_3(\widetilde{V},\widehat{V};t)| \\ &\leq (B_1+B_2+B_3)(t)\|V-\widetilde{V}\|,\end{aligned}$$

where

$$\begin{aligned}B_1(t) &= \frac{\|u^0\|}{2}e^{-\gamma t}\Bigg[\left(\frac{1}{2\sqrt{e\pi}}\exp\left\{\frac{-\overline{s}(t)^2}{8t}\right\}+1\right)t \\ &\quad +2\sqrt{2\pi}\overline{s}(t)\sqrt{t}Erf\left(\frac{\overline{s}(t)}{\sqrt{8t}}\right)+|\widetilde{s}(t)|\sqrt{\frac{2t}{e}}\Bigg],\end{aligned}$$

$$B_2(t)=\frac{1-e^{-\gamma t}}{\gamma\sqrt{8e\pi}}+\frac{\|\widetilde{V}\|}{8\gamma\sqrt{\gamma\pi}}\left(\sqrt{\pi}Erf(\sqrt{\gamma t})-2e^{-\gamma t}\sqrt{\gamma t}\right),$$

$$B_3(t)=\frac{1-e^{-\gamma t}}{\gamma\sqrt{8e\pi}}.$$

Thus (3.10)-(3.11) imply $\|AV-A\widetilde{V}\|\leq L(b)\|V-\widetilde{V}\|$ where

$$L(t):=C_1(B_1+B_2+B_3)(t)+L_1|\widetilde{\beta}|B(t),$$

and

$$\lim_{t\to 0}L(t)=0. \tag{3.12}$$

3.3. $\|T(V_0)\|\leq 1$ **is valid for some** $V_0\in S$ **and** $g\in C[0,\infty)$**.** If $V_0\equiv 0$ then $s\equiv 0$. On the other hand the function $-g$ is monoton increasing and for $t\in[0,b]$

$$\begin{aligned}T(0)(t) &= K(0)(t)=-g\left(\frac{e^{-\gamma t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left\{\frac{-\xi^2}{4t}\right\}u^0(\xi)d\xi\right) \\ &\leq -g\left(e^{-\gamma t}\|u^0\|\right)\leq -g\left(\|u^0\|\right).\end{aligned}$$

Therefor, $\|T(0)\|\leq -g\left(\|u^0\|\right)$, and a sufficient condition for $\|T(0)\|\leq 1$ is

$$-g\left(\|u^0\|\right)\leq 1.$$

3.4. $\delta := \|A(V_0)\| < 1$ **is valid for some** $b > 0, V_0 \in S$**.** For $V_0 \equiv 0$, let $\delta := \|A(0)\| < 1$. Put $b > 0$, $V \in C[0,b]$, and $\|V\| \leq 1$. Using definition of $\alpha, \beta, I_1, I_2, I_3$ from Section 3 it follows

$$\begin{aligned} |A(0)V|(t) &= \left|g^{'}(\alpha(0;t))\beta(0,V;t)\right| \\ &\leq C_1\big(|s(t)I_1(0;t)| + |I_2(0,V;t)| + |I_3(0,V;t)|\big) \\ &\leq C_1\left(4\|u^0\|te^{-\gamma t} + \frac{Erf(\sqrt{\gamma t})}{2\sqrt{\gamma}}\right) =: C_1B_4(t). \end{aligned}$$

Since $\lim_{t\to 0} B_4(t) = 0$, then there exists $b > 0$ such that
$\delta := \|A(V_0)\|_{C[0,b]} < 1$.

3.5. $\frac{L}{(1-\delta)^2} \leq \frac{1}{2}$ **is valid for some** $b > 0$**.** This is a consequence of equation (3.12).

4. The Product Integration Technique Applied to One Step of Newton's Method

The weakly singular integral equation (3.4) will be solved by the method described in [9]. This method allows us to overcome the difficulty caused by the poor behavior of the solution $U(t)$ at the initial point $t = 0$.
Given a relatively short interval $[0, b]$ we first solve the problem

$$U(t) = F(t) + \int_0^t K(t,\tau)U(\tau)d\tau \qquad t \in [0,b], \tag{4.1}$$

by a Nystrom-type method based upon a whole-interval product integration rule of interpolation type, witch integrates exactly the kernel $p(t,\tau)$. Here $K(t,\tau) = p(t,\tau)\widetilde{K}(t,\tau)$ where $p(t,\tau)$ is weakly singular kernel and $\widetilde{K}(t,\tau)$ is continuous.
After the initial interval, the bad behavior of the derivative of U is of less significance. Now by a step-by-step method the following can be solved

$$U(t) = U_1(t) + \int_b^t K(t,\tau)U(\tau)d\tau \qquad t \in [b,\infty),$$

with

$$U_1(t) = F(t) + \int_0^b K(t,\tau)U(\tau)d\tau.$$

Since the computation of $U_1(t)$ depends on the starting approximation of $U(t)$, $t \in [0,b]$, the two methods have to be regarded as paired.
Now a Nystrom-type method is used to solve equation (4.1) numerically. Having chosen $N+1$ distinct points $\{t_n\}_{n=0}^N$ in the interval $[0,b]$, we collocate the equation (4.1) at the nodes $\{t_n\}_{n=0}^N$

$$U(t_n) = F(t_n) + \int_0^{t_n} \widetilde{K}(t_n,\tau)p(t_n,\tau)U(\tau)d\tau \quad n = 0,1,2,...,N.$$

Substituting Lagrange interpolation polynomial

$$L_N(f;t) = \sum_{j=0}^{N} l_{N,j}(t)f(t_j),$$

we approximate $U(\tau)$. The following algorithm is set up

$$U_{N,n} = F(t_n) + \int_0^{t_n} \sum_{j=0}^{N} l_{N,j}(\tau)U_{N,j}\widetilde{K}(t_n,\tau)p(t_n,\tau)d\tau, \tag{4.2}$$

from now on where $n = 0,1,2,...,N$. The equation (4.2) is equivalent to

$$U_{N,n} = F(t_n) + \sum_{j=0}^{N} \omega_j(t_n)U_{N,j}, \tag{4.3}$$

where

$$\omega_j(t) = \int_0^t l_{N,j}(\tau)p(t,\tau)\widetilde{K}(t,\tau)d\tau.$$

By solving the linear system (4.3) we obtain $U_N(t)$ as a Nystrom approximation for $U(t)$:

$$U_N(t) = F(t) + \sum_{j=0}^{N} \omega_j(t)U_{N,j}.$$

Now we are ready to give the convergence of product integration method.

4.1. **Convergence of Product Integration for Solving Weakly Singular Integral Equation.** In convergence analysis we examine the linear test equation

$$U(t) = F(t) + \int_0^t p(t,\tau)U(\tau)d\tau, \qquad t \in [0,b]. \tag{4.4}$$

And assume that the forcing function $F \in C[0,b]$ and p is defined by $p(t,\tau) = (t-\tau)^{-\alpha}, \alpha \in (0,1)$ or $p(t,\tau) = \log|t-\tau|$. Then the test

equation (4.4) has a unique solution $U \in C[0, b]$ that may be expected to have unbounded derivatives at the end point $t = 0$.
If for a given mesh $\{t_j\}_{j=0}^{N}$ we apply the method of Section 4 to the test equation (4.4) to obtain $U_N(t)$ in the following Nystrom interpolant as approximate solution

$$U_N(t) = F(t) + \sum_{j=0}^{N} \omega_j(p;t) U_N(t_j),$$

where

$$\omega_j(p;t) = \int_0^t p(t,\tau) l_{N,j}(\tau) d\tau.$$

In order to examine the uniform convergence of the approximate solution $U_N(t)$ to the exact solution $U(t)$ of (4.4), note that

$$U(t) - U_N(t) = \sum_{j=0}^{N} \omega_j(p;t)\{U(t_j) - U_N(t_j)\} + t_N(p, U, t).$$

Where $t_N(p, U, t)$ is the local truncated error defined by

$$t_N(p, U, t) := \int_0^t p(t,\tau) U(\tau) d\tau - \sum_{j=0}^{N} \omega_j(p;t) U(t_j).$$

Hence we obtain

$$\|U - U_N\|_\infty \leq \left\|(I - A_N)^{-1}\right\|_\infty \|t_N\|_\infty , \tag{4.5}$$

where A_N is the linear operator defined by

$$\begin{cases} A_N : C[0, b] \to C[0, b], \\ A_N U(t) = \sum_{j=0}^{N} \omega_j(p;t) U(t_j), \quad U \in C[0, b], t \in [0, b]. \end{cases} \tag{4.6}$$

Convergence properties of the underling product quadrature rule is provided by:

Lemma 4.1. *Let $\{p_i\}_{i=1}^{N}$ be a sequence of orthogonal polynomials on $[-1, 1]$ with respect to the weight function $\omega(t)$. Then $\{q_i\}_{i=1}^{N}$ is a sequence of orthogonal polynomials on $[a, b]$ with respect the weight function $\widetilde{\omega}(t)$ where*

$$q_i(t) = p_i(\frac{2}{b-a}[t - \frac{b+a}{2}]), \qquad t \in [a, b],$$

$$\widetilde{\omega}(t) = \omega(\frac{2}{b-a}[t - \frac{b+a}{2}]), \qquad t \in [a, b].$$

Proof. Put $x = \frac{2}{b-a}[t - \frac{b+a}{2}]$. Then for $i, j \in \{1, ..., N\}$ and $i \neq j$

$$\int_a^b q_i(t)q_j(t)\widetilde{\omega}(t)dt = \frac{b-a}{2}\int_{-1}^1 p_i(x)p_j(x)\omega(x)dx = 0.$$

□

Theorem 4.2. *Let $\{t_j\}_{j=0}^N$ be the zeros of the $(N+1)$st-degree member of a set of polynomials that are orthogonal on $[0, b]$ with respect to the weight function*

$$\omega(t) = u(\frac{2t}{b} - 1)(2 - \frac{2t}{b})^\alpha(\frac{2t}{b})^\beta, \quad -1 < \alpha \leq \frac{3}{2}, \beta > -\frac{1}{2}. \tag{4.7}$$

Here $u(t)$ is positive and continuous on $[0, b]$ and the modulus of continuity φ of u satisfies $\int_0^1 \varphi(u, \delta)\frac{d\delta}{\delta} < \infty$. Let $L_N(f; t)$ denote the interpolating polynomial of degree $\leq N$ that coincides with the function f at the nodes $\{t_j\}_{j=0}^N$. Then, for every vector function f containing only endpoint singularity of the type $s^\sigma, \sigma > -1$ (not an integer), and in particular for every function $f \in C[0, b]$ there holds

$$\lim_{N\to\infty} \|t_N(p, f, t)\|_\infty = 0.$$

More particularly, we have the bounds

$$\left\|t_N(|t-\tau|^{-\frac{1}{2}}, f, t)\right\|_\infty = O\{(N+1)^{-2\sigma-1}\log(N+1)\}. \tag{4.8}$$

Proof. See [9, Theorem 1]. Apply Lemma 4.1 for balance of interval of orthogonality. The bound (4.8) is an immediate consequence of Theorem 5 in [3]. □

Now we investigate the behavior of the first term $\left\|(I - A)^{-1}\right\|_\infty$ on the right hand side of (4.5).

Theorem 4.3. *Let the operator A_N be defined as in (4.6) and the nodes $\{t_j\}_{j=1}^N$ chosen as in Theorem 4.2. Then for all N sufficiently large, there exists a constant $C > 0$ independent of N such that*

$$\left\|(I - A)^{-1}\right\|_\infty \leq C.$$

Proof. Conditions of Lemmas 1,2 of [9] are satisfied and hence by Theorem 2 of [9] the result follows. □

Theorem 4.4. *Let U be the exact solution of the equation (4.4). Let U_N be the approximate solution obtained by discretizing the integral term of (4.4) by a product quadrature rule of interpolatory type constructed on a set of distinct nodes $\{t_j\}_{j=0}^N$. If the nodes $\{t_j\}_{j=0}^N$ are the zeros of the*

$(N+1)$st-degree member of a set of polynomials that are orthogonal on $[0,b]$ with respect to the weight function (4.7) with $-\frac{1}{2}<\alpha,\beta<\frac{3}{2}$. Then U_N converges uniformly to U. Moreover, the rate of convergence of U_N to U coincides with that of the basic quadrature rule to approximate the integral term of (4.4).

The proof follows immediately from the estimate (4.5) together with Theorems 4.2 and 4.3. The bound (4.8) supply an estimate of the rate of convergence.
Another approach which is based on compactness of sequence A_N, is discussed in [2]

5. Numerical Examples and Discussion

Consider the following integral equation

(5.1)
$$V(t)=g\left(f(t)-\frac{1}{\sqrt{4\pi}}\int_0^t\frac{V(\tau)}{\sqrt{t-\tau}}\exp\left\{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}-\gamma(t-\tau)\right\}d\tau\right),$$

where

$$f(t)=-\log(2-2\alpha)-\frac{\alpha}{\sqrt{\alpha^2+4\gamma}}\operatorname{erf}\left(\sqrt{(\alpha^2/4+\gamma)t}\right),$$

$$g(u)=\frac{1}{2}e^{-u}-1,$$

and $\gamma\geq 0$ is the constant value in (1.1), $1/2<\alpha<1$ and as usual $s(t)=\int_0^t V(\tau)d\tau$. Equation (5.1) has the exact solution $V(t)=-\alpha$. This equation is equivalent with (1.1)-(1.4) where u^0 satisfies

$$\frac{\exp\left(-\sqrt{(\alpha^2/4+\gamma)t}\right)}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\xi^2}{4t}-\frac{1}{2}\alpha\xi\right\}u^0(\xi)d\xi=f(t).$$

In numerical result we put $b=0.01, \alpha=0.6$ and $V_0(t)=-0.54, t\in[0,b]$. The nodal points are zeros of $p_6(t)=1-4200t+42\times10^5t^2-168\times10^7t^3+315\times10^9t^4-2772\times10^{10}t^5+924\times10^{12}t^6$, where $\{p_n\}_{n=0}^{\infty}$ are orthogonal polynomials on $[0,b]$ with respect to the weight function $\omega(t)=1$. The initial relative error at all points is $|(V-V_0)/V|(t)=0.1$. This means that the relative error is 10%. After one step of Newton's method, the maximum error is 0.00101772, that is reasonable. This shows further steps of Newton's method will give better approximations if applied to the problem. Table 1 shows the relative errors at the points

i	$\left|\frac{V-V_1}{V}(t_i)\right|$
1	0.000338024
2	0.000475247
3	0.000578661
4	0.000664345
5	0.000738572
6	0.000804649
7	0.000864481
8	0.000919377
9	0.000970233
10	0.00101772

TABLE 1. After one step of Newton's method

$t_i = 0.001i, i = 1, ..., 10$. V_1 is the value of V after one step of Newton's method.

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