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A SURVEY OF INVERTIBILITY AND SPECTRUM PRESERVING LINEAR MAPS

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Abstract. We survey some results about invertibility and spectrum preserving linear maps on the algebra of all bounded operators on a Banach space or more generally on Banach algebras.

1. **Introduction**

Throughout, F will denote a field, $\mathcal{M}_n(F)$ the set of all $n \times n$ matrices with entries in F, and $\mathcal{T}_n(F)$ the set of all $n \times n$ upper triangular matrices over F. The algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$ (respectively, complex Hilbert space $\mathcal{H}$) will be denoted by $\mathcal{B}(\mathcal{X})$ (respectively, $\mathcal{B}(\mathcal{H})$). For an operator T in $\mathcal{B}(\mathcal{X})$, the spectrum of T will be denoted by $\sigma(T)$ and its spectral radius by $r(T)$.

Let $\mathcal{S}$ be a linear space of matrices or operators. A linear map φ: $\mathcal{S} \to \mathcal{S}$ is said to preserve

(1) the property P defined on $\mathcal{S}$, if $\varphi(T)$ has property P whenever T does;
(2) the function f defined on $\mathcal{S}$ if $f(\varphi(T)) = f(T)$, for all T in $\mathcal{S}$;
(3) the subset Ω of $\mathcal{S}$ if $\varphi(\Omega) \subseteq \Omega$;
(4) the relation $\approx$ defined on $\mathcal{S}$ if $\varphi(A) \approx \varphi(A)$, whenever $A \approx B$.

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The general Linear Preserver Problem (LPP): *Given a linear map φ satisfying one of the (overlapping) conditions (1)-(4), describe the structure of φ.*

The LPP has a relatively long history. The first result goes back to 1897, when G. Frobenius [19] described the structure of determinant-preserving linear maps.

Theorem 1.1. [19]. *A linear map φ: $\mathcal{M}_n(\mathbb{C}) \longrightarrow \mathcal{M}_n(\mathbb{C})$ satisfies det $(\varphi(T))$ = det (T) for all T if and only if there are $n \times n$ invertible matrices P and Q with the property det$(PQ) = 1$ such that either*

$$\varphi(T) = PTQ, \text{ for all } T; \text{ or } \varphi(T) = PT^tQ, \text{ for all } T,$$

where T^t is the transpose of T.

In this result, φ satisfies condition (1) above with $f(T) = \det(T)$. For almost forty five years, except for a couple of results by G. Polya [46] in 1913 and I. Schur [48] in 1925, no significant or directly related work was done on LPP. The subject gained attention in the 1940's with the work of K. Morita [41], [42], L. K. Hua [28] - [30], and J. Dieudonné [16]. The research has been continued since then, and in particular it has been very extensive in the last three decades. The partial list of references in a special survey issue of Linear and Multilinear Algebra (volume 33, No.1-2 (1992), pp.121-129), dedicated to this topic, contains more than 200 articles. Finally, before giving the general discussion, a few words about some applications of LPP are in order. In the matrix model in Systems Theory, one is interested in the structure of linear operators on spaces of matrices that preserve controllable systems or observable systems. Once the structure is known, one can use the linear map (which does not affect the system's nature) to transform a complex system into a simpler one (see for example [20] and its references). Also, for a quantum system and the corresponding matrix model, the entropy is related to determinant of the matrix. So, the above result of Frobenius is useful in finding linear transformations of the system that do not change the entropy.

2. **Some problems and background**

In what follows, a survey of some LPPs will be given. The general theme of the discussion is *Invertibility or Spectrum Preserving Linear Maps.* It is not an all-inclusive survey.

The first result on this kind of LPP is implicit in the work of J. Dieudonné [16].

Theorem 2.1. [16]. *An invertible linear map φ: $\mathcal{M}_n(F) \longrightarrow \mathcal{M}_n(F)$ preserves the set $\mathcal{S}$ of singular matrices if and only if there are $n \times n$ invertible matrices P and Q such that either*

$$\varphi(T) = PTQ, \text{ for all } T; \text{ or } \quad \varphi(T) = PT^tQ, \text{ for all } T.$$

Indeed, Dieudonné's work was on "semilinear" maps over an arbitrary field. Note that by the hypothesis of Theorem 2.1, the inverse of the map φ preserves invertibility, and so it is related to the next result of Marcus and Purves [38].

Theorem 2.2. [38]. *Let F be an algebraically closed field and φ: $\mathcal{M}_n(F) \longrightarrow \mathcal{M}_n(F)$ be a linear map.*

(1) *The map φ preserves invertibility; i.e., $\varphi(T)$ is invertible, whenever T is, if and only if it has one of the following forms*

$$\varphi(T) = PTQ, \text{ for all } T, \qquad (i)$$

or

$$\varphi(T) = PT^tQ, \text{ for all } T, \qquad (ii)$$

where P and Q are invertible matrices.

(2) *The map φ preserves the set of eigenvalues and their multiplicities if and only if it is of the form (i) or (ii) above with $Q = P^{-1}$.*

Jafarian and Sourour [31], extended the above result to algebras of operators on Banach spaces.

Theorem 2.3. [31]. *Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces and φ: $\mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{Y})$ be a surjective spectrum preserving (i.e., $\sigma(\varphi(T)) = \sigma(T)$) linear map. Then, either*

(1) *there is a bounded invertible operator A: $\mathcal{X} \longrightarrow \mathcal{Y}$ such that $\varphi(T) = ATA^{-1}$, for all T in $\mathcal{B}(\mathcal{X})$, or*

(2) *there is a bounded invertible operator B: $\mathcal{X}^* \longrightarrow \mathcal{Y}$ such that $\varphi(T) = BT^*B^{-1}$, for all T in $\mathcal{B}(\mathcal{X})$, where $\mathcal{X}^*$ is the dual of $\mathcal{X}$.*

Notes:
(1) The surjectivity assumption of φ is required. This can be seen from the example φ: $\mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X} \oplus \mathcal{Y})$ defined by $\varphi(T) = T \oplus T$.

(2) The proof of Theorem 2.3 shows that a surjective spectrum preserving linear map φ must be injective and hence bijective. The conclusion shows that φ is unital (i.e., $\varphi\ (I) = I$), multiplicative, and continuous as well.

(3) If φ satisfies the conditions of Theorem 2.3, then the conclusion implies that it is either an isomorphism or anti-isomorphism. The converse is a well-known result of Eidelheit [17].

One of the basic tools employed in the proof of Theorem 2.3 is a characterization of rank-1 operators. This was used to prove a key step, which is commonly used in proofs of many linear preserver problems, namely that *the set of rank-1 operators is mapped onto itself.* Theorem 2.3 has been extended in several directions, including versions for Banach and C*-algebras (see for example, [2] - [10], [15], [18], [24] - [27], [34], [35], [39], [40], [43], [44], and [49] - [55]). Some of these results will be quoted here for easy reference. In the first one, Omladič and Šemrl [43] proved the following result, by generalizing the mentioned characterization of rank-1 operators. Note that such an *additive* map φ turns out to be indeed linear.

Theorem 2.4. [43]. *The conclusion of Theorem 2.3 is valid under the less restrictive condition that φ is only additive.*

Since every spectrum preserving linear map φ preserves the spectral radius, i.e., $r(\varphi(T)) = r(T)$, the next result of Brešar and Šemrl [8] gives an extension of Theorem 2.3 in another direction.

Theorem 2.5. [8]. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are complex Banach spaces and ϕ: $\mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{Y})$ is a surjective linear map preserving the spectral radius. Then, $\phi = c\ \varphi$, where c is a complex number with $|\, c \,| = 1$, and φ is in the form (1) or (2) of Theorem 2.3.*

There are other generalizations. For complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let φ: $\mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{Y})$ be a bijective unital linear transformation preserving invertibility in both directions (i.e., $\varphi\ (T)$ is invertible if and only if T is.) Then, obviously, φ preserves the spectrum. Theorem 2.3 shows that the converse is also true. So, Theorem 2.3 can be viewed as a characterization of bijective unital linear maps preserving invertibility in both directions, and hence a generalization of Theorem 2.2(1). One advantage of looking at the problem from this viewpoint is that it makes sense for *real* Banach spaces as well. In [54], Sourour proved, among other things, a generalization of this version of Theorem 2.3 by showing that it is enough to assume that the map φ preserves invertibility *only* in one direction, namely $\varphi\ (T)$ is invertible if T is. (In [10], Brešar and Šemrl gave a simpler proof of Sourour's result.)

A *Jordan isomorphism* from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a bijective linear map φ satisfying $\varphi\ (\mathrm{T}^2)=(\varphi\ (\mathrm{T}))^2$, for every T in $\mathcal{A}$. Every isomorphism or antiisomorphism of $\mathcal{A}$ to $\mathcal{B}$ is a Jordan isomorphism, but the converse is not true in general. Every Jordan isomorphism between unital algebras preserves invertibility (see for example, [54], Proposition 1.3.) Another conclusion of Theorem 2.3 and Sourour's result [54] is: *Jordan isomorphisms are the only bijective unital linear maps between* $\mathcal{B}(\mathcal{X})$ *and* $\mathcal{B}(\mathcal{Y})$ *that preserve invertibility.* This provides a positive answer, in a particular case, to an earlier question of I. Kaplansky as seen below.

Question (Kaplansky [33]) *Let* φ: $\mathcal{A} \longrightarrow \mathcal{B}$ *be a unital invertibility preserving linear map, where* $\mathcal{A}$ *and* $\mathcal{B}$ *are unital complex Banach algebras. Is* φ *a Jordan homomorphism*?

The original question of Kaplansky was about additive invertibility preserving maps on rings. He was motivated by the Gleason-Kahane-Żelazko Theorem (see [21], [32], [56], and [1], [12], [53] for its generalizations) and the above results of Dieudonné, Marcus and Purves. (The Gleason-Kahane-Żelazko Theorem says: *Every unital invertibility preserving linear functional on a unital complex Banach algebra is necessarily multiplicative.*)

The above question of Kaplansky is very general and the answer to it is negative in its generality. The first example below shows that if φ is not surjective, then it might not be a Jordan isomorphism.

Example 1. [54]. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H})$ be the linear map defined by

$$\varphi(A) = \begin{bmatrix} A & f(A) \\ 0 & A \end{bmatrix},$$

where f is any nonzero linear functional on $\mathcal{B}(\mathcal{H})$ satisfying $f(\ I\ ) = 0$. Then, φ is a unital invertibility preserving linear map but not a Jordan isomorphism.

The next example shows that if the Banach algebras involved are *not* semi-simple, then the answer to the Kaplansky question may be negative.

Example 2. [9]. Let $\mathcal{A} = \mathcal{B} = \mathcal{T}_n$ be the algebra of all $n \times n$ upper triangular matrices, and let φ: $\mathcal{T}_n \longrightarrow \mathcal{T}_n$ be any unital linear map which keeps the diagonal elements fix. Then, φ preserves invertibility, but in general it is not a Jordan homomorphism. Here, the radical of $\mathcal{T}_n$ is the set of all strictly upper triangular matrices, and hence $\mathcal{T}_n$ is not semi-simple.

Since Kaplansky raised his question, there have been many interesting partial results, but the question is still open even for C*-algebras and it is known as *Harris - Kadison Conjecture* ([22],[23]). Let us briefly mention some of these results. Aupetit and du Mouton [4] extended Theorem 2.3 above to a Banach algebra whose socle is an essential ideal. In [6], Brešar, Fošner and Šemrl generalized the results of [4] and [54].

Theorem 2.6. [6]. *Let $\mathcal{A}$ be a primitive complex Banach algebra with nonzero socle, and let $\mathcal{B}$ be a semisimple complex Banach algebra. If φ: $\mathcal{A} \longrightarrow \mathcal{B}$ is a unital bijective linear map that preserves invertibility, then φ is either an isomorphism or an antiisomorphism.*

Finally, we point out the article [47] ([11]), which proves that an inverti-bility and *-preserving positive linear map on von Neumann algebras (respectively, C*-algebras) is a Jordan*-homomorphism.

In view of the above, it seems quite natural that Aupetit and several others working on the question made the following conjecture.

Conjecture 1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital semisimple complex Banach algebras and φ: $\mathcal{A} \longrightarrow \mathcal{B}$ is a unital bijective linear map preserving invertibility. Then, φ is a Jordan isomorphism.*

For the case of Banach algebras with "enough" idempotents, it is our belief that the following characterization of idempotents by Aupetit [3] could be a key in solving the conjecture.

Proposition 2.7. [3] *An element a of a semisimple complex Banach algebra $\mathcal{A}$ is idempotent if and only if $\sigma(a) \subset \{0, 1\}$ and there exist real numbers r and $C > 0$ such that*

$$\sigma(x) \subset \sigma(a) + C \left\| x - a \right\|,$$

for all x in $\mathcal{A}$ with $\|x - a\| < r$.

Aupetit used this characterization in a relatively recent breakthrough solution of Kaplansky's problem for general von Neumann algebras.

Theorem 2.8. [3] *A bijective unital linear map between two von Neumann algebras that preserves invertibility is a Jordan isomorphism.*

It should be pointed out that Cui and Hou [14] showed that the *bijectivity* condition of the map in Theorem 2.8 could be replaced with the less restrictive condition of *surjectivity*.

It would be interesting to consider extension of Theorem 2.8 to the case of surjective unital *additive* invertibility preserving maps between von Neumann algebras. It is known that this is true [43] for the *additive* maps between algebras of *all* bounded linear operators on Hilbert spaces. This gives rise to the following conjecture.

Conjecture 2. *A surjective unital additive map between two von Neumann algebras that preserves invertibility is a Jordan isomorphism.*

There have been several relatively recent results about additive versions of linear preserver problems (see for example, [13],[15],[18],[25], [26], [34]-[36], and [43],[44]). It is our strong feeling that some of the techniques employed in these papers can be used to prove this conjecture.

In most of the above problems and results, the underlying Banach algebras or spaces are over the *complex* field. A natural question is: *How much of the existing results can be extended to real Banach algebras?* The answer is: *Not much is known.* However, in the case of $\mathcal{B}(\mathcal{X})$ it is tempting to make the following conjecture.

Conjecture 3. *Let φ: $\mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be a bijective unital invertibility preserving linear map, where $\mathcal{X}$ is a real Banach space. Then, φ is either an automorphism or an antiautomorphism.*

Let us mention that the finite dimensional version of this conjecture is true even for additive maps (see for example, [18], Corollary 1.3).

Finally, it would be interesting to consider the following generalization of Dieudonné's result to infinite dimensions. Note that if the map φ is invertible, instead of being surjective, then the inverse of φ preserves invertibility and the statement is true by Sourour's result [54].

Conjecture 4. Let φ: $\mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a surjective linear map preserving singularity, where $\mathcal{H}$ is a complex Hilbert space. Then, there are invertible operators P and Q such that

$$\varphi(T) = PTQ, \text{ for all } T; \text{ or } \quad \varphi(T)=PT^tQ, \text{ for all } T,$$

where T^t is the transpose of T with respect to a fixed orthonormal basis of $\mathcal{H}$.

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