

EXTENSIONS OF BAER AND QUASI-BAER MODULES

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Communicated by Freydoon Shahidi

ABSTRACT. We study the relationships between the Baer, quasi-Baer and p.q.-Baer property of an R -module M and the polynomial extensions of module M . As a consequence of our results, we obtain some results of [C.Y. Hong, N.K. Kim and T.K. Kwak, *J. Pure Appl. Algebra* **151** (2000) 215-226.] and [E. Hashemi and A. Moussavi, *Acta Math. Hungar.* **107** (2005) 207-224.].

1. Introduction

Throughout the paper, R will always denote an associative ring with identity and M_R will stand for a right R -module. Recall from [15] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [15], Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$ -regular rings. The class of Baer rings includes the von Neumann algebras. In [9], Clark defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with identity over an algebraically closed field is isomorphic to

MSC(2010): Primary: 16D80; Secondary: 16S36.

Keywords: (α, δ) -compatible modules, Reduced modules, Baer modules, quasi-Baer module, α -rigid rings, skew polynomial ring.

Received: 6 March 2008, Accepted: 15 October 2009.

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a twisted matrix units semigroup algebra. Every prime ring is a quasi-Baer ring. Another generalization of Baer rings is the p.p.-rings. A ring R is called *right (resp. left) p.p.* if right (resp. left) annihilator of an element of R is generated by an idempotent. Birkenmeier, et al. in [6] introduced the concept of principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal of R is generated by an idempotent.

In 1974, Armendariz considered the behavior of a polynomial ring over a Baer ring by obtaining the following result: Let R be a *reduced* ring (i.e., R has no nonzero nilpotent elements). Then, $R[x]$ is a Baer ring if and only if R is a Baer ring ([4], Theorem B). Armendariz provided an example to show that the reduced condition is not superfluous. In [6], Birkenmeier, et al. showed that the quasi-Baer condition is preserved by many polynomial extensions. Also, Birkenmeier, et al. [6] showed that a ring R is right p.q.-Baer if and only if $R[x]$ is right p.q.-Baer.

From now on, we always denote the Ore extension ring (or Ore polynomial ring) by $S := R[x; \alpha, \delta]$, where $\alpha : R \rightarrow R$ is an endomorphism and $\delta : R \rightarrow R$ is an α -derivation. Recall that an α -derivation δ is an additive operator on R with the property that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. The Ore extension S is then the ring consisting of all (left) polynomials of the form $\sum a_i x^i$ ($a_i \in R$), which are multiplied using the distributive law and the Ore commutation rule $xa = \alpha(a)x + \delta(a)$, for all $a \in R$. From this rule, an inductive argument can be made to calculate an expression for $x^j a$, for all $j \in \mathbb{N}$ and $a \in R$.

Notation [19]. Let δ be an α -derivation of R . For integers $j \geq i \geq 0$, write f_i^j for the sum of all “words” in α and δ in which there are i factors of α and $j - i$ factors of δ . For instance, $f_j^j = \alpha^j$, $f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$.

Using recursive formulas for the f_i^j and induction, as in [19], one can show with a routine computation that

$$(1.1) \quad x^j a = \sum_{i=0}^j f_i^j(a) x^i.$$

This formula uniquely determines a general product of (left) polynomials in S and will be used freely in what follows.

Given a right R -module M_R , we can make $M[x]$ into a right S -module by allowing polynomials from S to act on polynomials in $M[x]$ in the obvious way, and applying the above “twist” whenever necessary. The verification that this defines a valid S -module structure on $M[x]$ is almost identical to the verification that S is a ring, and it is straightforward.

For a nonempty subset X of M , put $\text{ann}_R(X) = \{a \in R \mid Xa = 0\}$. In [21], Lee and Zhou introduced the notions of Baer, quasi-Baer and p.p.-modules as follows: (1) M_R is called *Baer* if for any subset X of M , $\text{ann}_R(X) = eR$, where $e^2 = e \in R$. (2) M_R is called *quasi-Baer* if, for any submodule $X \subseteq M$, $\text{ann}_R(X) = eR$, where $e^2 = e \in R$. (3) M_R is called *p.p.* if for any element $m \in M$, $\text{ann}_R(m) = eR$, where $e^2 = e \in R$. Clearly, a ring R is Baer (resp. p.p. or quasi-Baer) if and only if R_R is Baer (resp. p.p. or quasi-Baer) module. If R is a Baer (resp. p.p. or quasi-Baer) ring, then for any right ideal I of R , I_R is Baer (resp. p.p. or quasi-Baer) module.

The module M_R is called *principally quasi-Baer* (or simply p.q.-Baer) if for any $m \in M$, $\text{ann}_R(mR) = eR$, where $e^2 = e \in R$. It is clear that R is a right p.q.-Baer ring if and only if R_R is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

Here, we impose (α, δ) -compatibility assumption on the module M_R and prove the following results, extending many results on rings to modules:

(1) The module M_R is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer), where $S = R[x; \alpha, \delta]$.

(2) If M_R is (α, δ) -Armendariz, then M_R is Baer (resp. p.p.) if and only if $M[x]_S$ is Baer (resp. p.p.).

Also, we give examples to show that (α, δ) -compatibility assumption on M_R in the preceding results is not superfluous. Among applications, we obtain some results of [12] and [10] as corollaries of our results.

2. Polynomials over Baer and Quasi-Baer Modules

Definition 2.1. (Annin [3]) *Given a module M_R , an endomorphism $\alpha : R \rightarrow R$, and an α -derivation $\delta : R \rightarrow R$, we say that M_R is α -compatible if for each $m \in M$, $r \in R$, we have $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, we say that M_R is δ -compatible if for each $m \in M$, $r \in R$, we have $mr = 0 \Rightarrow m\delta(r) = 0$. If M_R is both α -compatible and δ -compatible, we say that M_R is (α, δ) -compatible.*

Recall that an R -module N_R is called *prime* if $N \neq 0$ and $\text{ann}_R(N) = \text{ann}_R(N')$, for every nonzero submodule $N' \subseteq N$.

The following example shows that there exists an (α, δ) -compatible module M_R such that M_R and $M[x]_{R[x; \alpha, \delta]}$ are quasi-Baer.

Example 2.2. [3, Example 4.6] *Let R_0 be a domain of characteristic zero, and $R := R_0[t]$. Define $\alpha|_{R_0} = \text{Id}$ and $\alpha(t) = -t$. Now, for $a \in R_0$, set*

$$\delta(at^l) := \begin{cases} at^{l-1} & \text{if } l \text{ is odd} \\ 0 & \text{if } l \text{ is even.} \end{cases}$$

It is shown in [19] that δ is an α -derivation on R . Let $M_R := R_0 \oplus R_0 \oplus R_0 \oplus \cdots$, where $t \in R$ acts on M_R as follows: for $(m_0, m_1, m_2, \dots) \in M$, we set $(m_0, m_1, m_2, \dots)t := (0, m_0k_0, m_1k_1, m_2k_2, \dots)$, where the k_i ($i \in \mathbb{N}$) are fixed nonzero integers. We show that M_R is (α, δ) -compatible. For this, it suffices to show that $\text{ann}_R(m) = 0$, whenever $0 \neq m \in M$. Suppose that $(a_0, a_1, a_2, \dots)(b_r t^r + b_{r+1} t^{r+1} + \text{“higher terms”}) = 0$, where $a_i, b_i \in R_0$, for every $i \in \mathbb{N}$ and $b_r \neq 0$. First, applying t^r to (a_0, a_1, a_2, \dots) gives:

$$(0, 0, \dots, 0, a_0 k_0 k_1 \cdots k_{r-1}, a_1 k_1 k_2 \cdots k_r, \dots)(b_r + b_{r+1} t + \text{“higher terms”}) = 0.$$

Upon computing this expression, we deduce that $a_0 k_0 k_1 \cdots k_{r-1} b_r = 0$. Since the characteristic is zero, R is a domain, and $k_0 k_1 \cdots k_{r-1} b_r \neq 0$, we deduce that $a_0 = 0$. Now, we may proceed inductively to show that $a_i = 0$, for all i . From this calculation, we deduce at once that M_R is (α, δ) -compatible. Moreover, the calculation implies that M_R is prime, and $\text{ann}_R(N) = \{0\}$, for each nonzero submodule N of M . Therefore, M_R is quasi-Baer. Hence, $M[x]_{R[x; \alpha, \delta]}$ is quasi-Baer, by Theorem 2.11.

Remark 2.3. (a) *If M_R is α -compatible (resp. δ -compatible), then so is any submodule of M_R .*

(b) *If M_R is α -compatible (resp. δ -compatible), then M_R is α^i -compatible (resp. δ^i -compatible), for all $i \geq 1$.*

Lemma 2.4. *Let M_R be an (α, δ) -compatible R -module. Let $m \in M$, and $a, b \in R$. Then, we have the followings:*

- (1) *If $ma = 0$, then $m\alpha^i(\delta^j(a)) = 0 = m\delta^j(\alpha^i(a))$, for any positive integers i, j .*
- (2) *If $mab = 0$, then $m\alpha^i(a)\delta^j(b) = 0 = m\delta^j(a)\alpha^i(b)$, for any positive integers i, j .*
- (3) *$\text{ann}_R(ma) = \text{ann}_R(m\alpha(a)) \subseteq \text{ann}_R(m\delta(a))$.*

Proof. (1) It follows from Remark 2.3.

(2) It is enough to show that $m\alpha(a)\delta(b) = 0 = m\delta(a)\alpha(a)$. Since M_R is δ -compatible, $mab = 0$ implies that $ma\delta(b) = 0$ and $m\delta(ab) = m\delta(a)b + m\alpha(a)\delta(b) = 0$. Since M_R is α -compatible, $mab = 0$ implies that $m\alpha(ab) = m\alpha(a)\alpha(b) = 0$, and so $m\alpha(a)b = 0$. Thus, $m\alpha(a)\delta(b) = 0$. Hence, $m\delta(a)b = 0$ and $m\delta(a)\alpha(a) = 0$.

(3) Observe that the α -compatibility of M_R yields $m\alpha(a)b = 0 \Leftrightarrow m\alpha(a)\alpha(b) = 0 \Leftrightarrow m\alpha(ab) = 0 \Leftrightarrow mab = 0$, for each $b \in R$. It remains only to show that $\text{ann}_R(ma) \subseteq \text{ann}_R(m\delta(a))$. Let $mab = 0$, for some $b \in R$. Using δ -compatibility, we get $0 = m\delta(ab) = m\alpha(a)\delta(b) + m\delta(a)b = 0$ and hence $m\delta(a)b = 0$, as desired. \square

Lemma 2.5. *Let M_R be an (α, δ) -compatible module, $m(x) = m_0 + \cdots + m_k x^k \in M[x]$ and $r \in R$. If $m(x)r = 0$, then $m_i r = 0$, for each i .*

Proof. An easy calculation using Eq. (1.1) shows that

$$0 = m(x)r = \sum_{i=0}^k \sum_{j=i}^k m_j f_i^j(r) x^i \text{ and so}$$

$$(2.1) \quad \sum_{j=i}^k m_j f_i^j(r) = 0 \text{ for each } i \leq k.$$

Starting with $i = k$, Eq. (2.1) yields $m_k \alpha^k(r) = 0$, and so α -compatibility of M_R yields $m_k r = 0$. Now, assume inductively that $m_j r = 0$, for each $j > i$. By (α, δ) -compatibility of M_R , for $j > i$ we have $m_j f_i^j(r) = 0$. Using Eq. (2.1) again, we deduce that $m_i \alpha^i(r) = 0$, and so $m_i r = 0$ as needed. \square

Following Anderson and Camillo [1], a module M_R is called *Armendariz* if whenever $m(x)f(x) = 0$, where $m(x) = \sum_{i=0}^s m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^t a_j x^j \in R[x]$, we have $m_i a_j = 0$, for all i, j .

Definition 2.6. *Given a module M_R , an endomorphism $\alpha : R \rightarrow R$, and an α -derivation $\delta : R \rightarrow R$, we say M_R is (α, δ) -quasi Armendariz (resp. (α, δ) -Armendariz), if whenever $m(x) = \sum_{i=0}^k m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^n b_j x^j \in R[x; \alpha, \delta]$ satisfy $m(x)R[x; \alpha, \delta]f(x) = 0$ (resp. $m(x)f(x) = 0$), we have $m_i x^i R b_j x^j = 0$ (resp. $m_i x^i a_j x^j = 0$), for all i, j .*

For a module M_R , put

$$\text{Ann}_R(\text{sub}(M)) = \{\text{ann}_R(N) \mid N \text{ is a submodule of } M\}.$$

Clearly, $A = \text{ann}_R(N)$ is an ideal of R for each submodule N of M .

Proposition 2.7. *Let M_R be an (α, δ) -compatible module and S be the skew polynomials ring $R[x; \alpha, \delta]$. Then, the following statements are equivalent:*

- (1) M_R is (α, δ) -quasi Armendariz.
- (2) $\psi : \text{Ann}_R(\text{sub}(M)) \rightarrow \text{Ann}_S(\text{sub}(M[x])); A \rightarrow AS$ is bijective.

Proof. (2) \Rightarrow (1). Let $m(x) = m_0 + m_1x + \dots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \dots + b_mx^m \in S$ satisfy $m(x)Sf(x) = 0$. Then, $f(x) \in \text{ann}_S(m(x)S) = AS$, where A is an ideal of R . Hence, $b_0, \dots, b_m \in A$, and so $m(x)Rb_j = 0$, for $j = 0, \dots, m$. By lemmas 2.4 and 2.5, $m_i x^i Rb_j x^j = 0$, for all i, j . Therefore, M_R is (α, δ) -quasi Armendariz.

(1) \Rightarrow (2). Let $A \in \text{Ann}_R(\text{sub}(M))$. Then, there exists a submodule N of M such that $A = \text{ann}_R(N)$, and hence $\text{ann}_S(N[x]) = AS$, by Lemmas 2.4 and 2.5. Thus, ψ is a well defined map. Assume that $B \in \text{Ann}_S(\text{sub}(M[x]))$. Then, there exists a submodule N of $M[x]$ such that $B = \text{ann}_S(N)$. Let B_1 denote the set of all coefficients of elements of B in R and N_1 denote the set of all coefficients of elements of N in M . We claim that $\text{ann}_R(N_1R) = B_1R$. Let $m(x) = m_0 + m_1x + \dots + m_kx^k \in N$ and $f(x) = b_0 + b_1x + \dots + b_mx^m \in B$. Then, $m(x)Sf(x) = 0$. Since M_R is (α, δ) -quasi Armendariz and (α, δ) -compatible, $m_i Rb_j = 0$, for all i, j . Thus, $(N_1R)(B_1R) = 0$, and so $B_1R \subseteq \text{ann}_R(N_1R)$. Since M_R is (α, δ) -compatible, $\text{ann}_R(N_1R) \subseteq B_1R$. Thus, $\text{ann}_R(N_1R) = B_1R$, and so $\text{ann}_S(N) = (B_1R)S$. \square

Following Tominaga [25], an ideal I of R is said to be *left s-unital* if for each $a \in I$ there is an $x \in I$ such that $xa = a$. If an ideal I of R is left s-unital, then, for any finite subset F of I , there exists an element $e \in I$ such that $ex = x$, for each $x \in F$. A submodule N of a right R -module M is called a *pure submodule* if $N \otimes_R L \rightarrow M \otimes_R L$ is a monomorphism for every left R -module L . By [25, Proposition 11.3.13], an ideal I is left s-unital if and only if R/I is flat as a right R -module if and only if I is pure as a right ideal of R .

Proposition 2.8. *Let M_R be an (α, δ) -compatible module and $S = R[x; \alpha, \delta]$. Then, the followings are equivalent:*

- (1) $\text{ann}_R(mR)$ is left s-unital for any element $m \in M$.
- (2) $\text{ann}_S(m(x)S)$ is left s-unital for any element $m(x) \in M[x]$. In this case, M_R is (α, δ) -quasi Armendariz.

Proof. (1) \Rightarrow (2). First, we prove that M_R is (α, δ) -quasi Armendariz. Suppose that $(m_0 + m_1x + \dots + m_kx^k)S(b_0 + b_1x + \dots + b_nx^n) = 0$, with $m_i \in M$ and $b_j \in R$. Then,

$$(2.2) \quad (m_0 + m_1x + \dots + m_kx^k)R(b_0 + b_1x + \dots + b_nx^n) = 0.$$

Since M_R is α -compatible, $m_kRb_n = 0$. Then, $b_n \in \text{ann}_R(m_kR)$, and so $m_kx^kRb_nx^n = 0$, by Lemma 2.4. Since $\text{ann}_R(m_kR)$ is left s -unital, there exists $e_k \in \text{ann}_R(m_kR)$ such that $e_kb_n = b_n$. Replacing R by Re_k in Eq. (2.2), and using Lemma 2.4, we obtain $(m_0 + m_1x + \dots + m_{k-1}x^{k-1})Re_k(b_0 + b_1x + \dots + b_nx^n) = 0$. Hence, $m_{k-1}Rb_n = 0$, since M_R is α -compatible. Then, $b_n \in \text{ann}_R(m_{k-1}R)$, and so $m_{k-1}x^{k-1}Rb_nx^n = 0$, by Lemma 2.4. Hence, $b_n \in \text{ann}_R(m_kR) \cap \text{ann}_R(m_{k-1}R)$. Since $\text{ann}_R(m_{k-1}R)$ is left s -unital, there exists $f \in \text{ann}_R(m_{k-1}R)$ such that $fb_n = b_n$. If we put $e_{k-1} = e_kf$, then $e_{k-1}b_n = b_n$ and $e_{k-1} \in \text{ann}_R(m_kR) \cap \text{ann}_R(m_{k-1}R)$. Next, replacing R by Re_{k-1} in Eq. (2.2), and using Lemma 2.4, we obtain $(m_0 + m_1x + \dots + m_{k-2}x^{k-2})Re_{k-1}(b_0 + b_1x + \dots + b_nx^n) = 0$. Hence, we have $b_n \in \text{ann}_R(m_{k-2}R)$, and so $m_{k-2}x^{k-2}Rb_nx^n = 0$, by Lemma 2.4. Continuing this process, we get $m_ix^iRb_nx^n = 0$, for $i = 0, \dots, k$. Using induction on $k + n$, we obtain $m_ix^iRb_jx^j = 0$, for all i, j . Therefore, M_R is (α, δ) -quasi Armendariz. Let $m(x) = m_0 + m_1x + \dots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \dots + b_mx^m \in \text{ann}_S(m(x)S)$. Then, $m_iRb_j = 0$, for all i, j . Since $\text{ann}_R(m_iR)$ is left s -unital, there exists $e_i \in \text{ann}_R(m_iR)$ such that $b_j = e_ib_j$, for $j = 0, 1, \dots, m$. Put $e = e_0e_1 \dots e_k$. Then, $b_j = eb_j$, for $j = 0, 1, \dots, m$, and so $ef(x) = f(x)$. Clearly, $e \in \text{ann}_S(m(x)S)$. Therefore, $\text{ann}_S(m(x)S)$ is left s -unital.

(2) \Rightarrow (1). Let $m \in M$. By using Lemma 2.4, $\text{ann}_R(mR) \subseteq \text{ann}_S(mS)$. Hence, for any $b \in \text{ann}_R(mR)$, there exists a polynomial $f(x) \in S$ such that $f(x)b = b$. Let a_0 be the constant term of $f(x)$. Then, $a_0b = b$, by (α, δ) -compatibility of M_R . Clearly, $a_0 \in \text{ann}_R(mR)$. Therefore, $\text{ann}_R(mR)$ is left s -unital. \square

By Proposition 2.8, if $\text{ann}_R(mR)$ is left s -unital for any element $m \in M$, then M_R is α -quasi Armendariz. But the converse is not true, in general. The following example shows that there exists an α -compatible ring R such that R_R is α -quasi Armendariz, but $\text{ann}_R(mR)$ is not left s -unital for some $m \in R$.

Example 2.9. [26, Example 2.4] For a given field F , let

$$S = \{(a_n)_{n=1}^{\infty} \in \prod F \mid a_n \text{ is eventually constant}\},$$

which is a subring of the countably infinite direct product $\prod F$. Then, S is a commutative ring. Let $R = S[[x]]$. Clearly S is a reduced ring. Suppose that $f(x) = a_0 + a_1x + \dots$ and $g(x) = b_0 + b_1x + \dots \in S[[x]]$ are such that $f(x)g(x) = 0$. Then, from [1, p. 2269], it follows that $a_i b_j = 0$, for all i, j . Thus, R is a reduced ring. Let α be the S -automorphism of R such that $\alpha(x) = -x$. Clearly, R_R is α -compatible. Hence R is α -quasi Armendariz, by [12, Proposition 6], and [10, Lemma 2.2]. We show that there exists $m \in R$ such that $\text{ann}_R(mR)$ is not left s -unital. Let $m = m_0 + m_1x + \dots$, where $m_0 = (0, 1, 0, 0, \dots)$, $m_1 = (0, 1, 0, 1, 0, 0, \dots)$, $m_2 = (0, 1, 0, 1, 0, 1, 0, 0, \dots)$, \dots . We show that $\text{ann}_R(mR)$ is not left s -unital. Suppose that $\text{ann}_R(mR)$ is left s -unital. Let $f = f_0 + f_1x + \dots \in R$, where

$$f_0 = (1, 0, 0, 0, \dots), f_1 = (1, 0, 1, 0, 0, 0, \dots), f_2 = (1, 0, 1, 0, 1, 0, 0, 0, \dots), \dots$$

Then, $mf = 0$, and so $mRf = 0$, since R is reduced. Hence, $f \in \text{ann}_R(mR)$. Thus, there exists $h \in \text{ann}_R(mR)$ such that $hf = f$. Suppose that $h = h_0 + h_1x + \dots$. Now, $mh = 0$ and from [1, p. 2269], it follows that $m_i h_j = 0$, for all i, j , and so there exists $n_j \in \mathbb{N}$ such that h_j has the form $(b_1^j, 0, b_3^j, 0, \dots, b_{2n_j+1}^j, 0, 0, 0, \dots)$, where $b_k^j \in F$, $j = 0, 1, 2, \dots$. From $(h - 1)f = 0$, it follows that $(h_0 - 1)f_i = 0$ and $h_j f_i = 0$, for all i and $j \geq 1$, and so there exists $m_j \in \mathbb{N}$ such that h_j has the form $(0, b_2^j, 0, b_4^j, 0, \dots, b_{2m_j}^j, 0, 0, 0, \dots)$, where $b_k^j \in F$, $j = 1, 2, \dots$. Thus, $h_1 = h_2 = \dots = 0$, and so $h = h_0$. This contradicts with $h_0 f_i = f_i$, $i = 0, 1, \dots$. Thus, $\text{ann}_R(mR)$ is not left s -unital.

Clearly, if M_R is quasi-Baer, then $\text{ann}_R(mR)$ is left s -unital for each $m \in M$. But the converse is not true, in general. The following example shows that there exists a ring R such that $\text{ann}_R(mR)$ is left s -unital for each $m \in R$, but R is not quasi-Baer. Recall that a ring R is called a *right Bezout* ring if every finitely generated right ideal of R is principal. Recall that the weak global dimension of a ring R is defined as $\sup\{fd(A) \mid A \text{ is a right } R\text{-module}\}$. Note that the weak global dimension ≤ 1 if and only if every right ideal of R is flat.

Example 2.10. [26, Example 2.5] Let \mathbb{Z} be the ring of integers and let

$$S = \left(\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} \right) / \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} \right).$$

Then, S is clearly a Boolean ring and by [8, p. 64], the weak global dimension of $S[[x]]$ is one and $S[[x]]$ is not semihereditary. Let $R = S[[x]]$. Then, every principal ideal of R is flat, and so $R/\text{ann}_R(aR) = R/\text{ann}_R(a) \cong aR$ is flat. Thus, $\text{ann}_R(aR)$ is pure as a right ideal of R , for every $a \in R$. In [8, Theorem 43], it was shown that the power series ring $A[[x]]$ over a von Neumann regular ring A is semihereditary if and only if $A[[x]]$ is a Bezout ring, in which all principal ideals are projective. On the other hand, by [8, Theorem 42], $S[[x]]$ is a Bezout ring since the weak global dimension of $S[[x]]$ is one. Thus, R is not p.q.-Baer.

Since quasi-Baer (p.q.-Baer) modules satisfy the hypotheses of Proposition 2.8, by using Proposition 2.7 we have the following results.

Theorem 2.11. *Let M_R be an (α, δ) -compatible module. Then, M_R is quasi-Baer (resp. p.q.-Baer) if and only if $M[x]_S$ is quasi-Baer (resp. p.q.-Baer); in this case, M_R is an (α, δ) -quasi Armendariz module.*

The following examples show that the α -compatibility condition on M_R in Theorem 2.11 is not superfluous.

Example 2.12. [3, Example 2.7] *Let F be any field of characteristic zero, and set $R := F[t]$. Let α be the F -automorphism of R such that $\alpha(t) = t + 1$, and set $S := R[x; \alpha]$. Consider the right R -module $M_R := \frac{F[t]}{(t^2)}$ and the right S -module $P_S := M[x]_S$. Using “ $\bar{}$ ” to mean “modulo (t^2) ”, note that since $\bar{t} \cdot t = \bar{0}$ but $\bar{t} \cdot (t + 1) \neq \bar{0}$, the α -compatibility condition fails here. We show that P_S is prime. It suffices to show that, for any nonzero submodule $P'_S \subseteq P_S$, we have $\text{ann}_S(P') = 0$. Choose any $0 \neq p' \in P'$. We may write*

$$p' = \overline{g_k(t)}x^k + \overline{g_{k+1}(t)}x^{k+1} + \cdots \in P,$$

where $\overline{g_k(t)} \neq \bar{0}$ in M_R . It suffices to show that $\text{ann}(p'S_S) = 0$. Suppose there exists $s \in S$ with $(p'S)s = 0$. Write $s = f_0(t) + f_1(t)x + \cdots \in S$ with $f_j(t) \in R$, for each j . Now, for each $i \geq 0$, we have

$$\begin{aligned} \bar{0} &= (\overline{g_k(t)}x^{k+i} + \text{“higher terms”})(f_0(t) + \text{“higher terms”}) \\ &= \overline{g_k(t)}f_0(t + k + i)x^{k+i} + \text{“higher terms.”} \end{aligned}$$

Hence, we have $\overline{g_k(t)}f_0(t + k + i) = \bar{0}$ in M_R . So, for each $i \geq 0$, we have $g_k(t)f_0(t + k + i) \in (t^2)$ in R . But $\overline{g_k(t)} \neq \bar{0}$ implies that $g_k(t) \notin (t^2)$. From this, we conclude that t divides $f_0(t + k + i)$, for

each $i \geq 0$. Putting $t = 0$, we have that $f_0(k + i) = 0$, for each $i \geq 0$. Since F has characteristic zero, we conclude that $f_0(t) = 0$. Now, we may go back and repeat this argument for f_1, f_2, \dots , in turn, eventually concluding that $s = 0$. Thus, as desired, we have $\text{ann}_S(p'S) = 0$. Hence, P_S is prime with $\text{ann}_S(P) = 0$. Thus, $M[x]_S$ is quasi-Baer. Since $\text{ann}_R(M) = (t^2)$ and (t^2) does not have any idempotents, M_R is not quasi-Baer.

Example 2.13. Let R_0 denote any domain and let $R := R_0[t]$. Let $\alpha : R \rightarrow R$ be defined by $\alpha(t) = 0$ and $\alpha|_{R_0} = \text{Id}$. Next, let $M := R$ and $S = R[x; \alpha]$. Observe that α -compatibility evidently fails in this case. Since R is a domain, it is quasi-Baer. Now, consider the S -submodule $Q = xS$. Then, $\text{ann}_S(Q) = tS$ and tS does not have any idempotents. Hence, $M[x]_S$ is not quasi-Baer.

The following example shows that δ -compatibility condition on R_R in Theorem 2.11 is not superfluous.

Example 2.14. [4, Example 11] There is a ring R and a derivation δ of R such that $R[x; \delta]$ is a Baer (hence a quasi-Baer) ring, but R is not quasi-Baer. In fact, let $R = \mathbb{Z}_2[t]/(t^2)$ with the derivation δ such that $\delta(\bar{t}) = 1$, where $\bar{t} = t + (t^2)$ in R and $\mathbb{Z}_2[t]$ is the polynomial ring over the field \mathbb{Z}_2 of two elements. Consider the Ore extension $R[x; \delta]$. If we set $e_{11} = \bar{t}x$, $e_{12} = \bar{t}$, $e_{21} = \bar{t}x^2 + x$, and $e_{22} = 1 + \bar{t}x$ in $R[x; \delta]$, then they form a system of matrix units in $R[x; \delta]$. Now, the centralizer of these matrix units in $R[x; \delta]$ is $\mathbb{Z}_2[x^2]$. Therefore, $R[x; \delta] \cong M_2(\mathbb{Z}_2[x^2]) \cong M_2(\mathbb{Z}_2)[y]$, where $M_2(\mathbb{Z}_2)[y]$ is the polynomial ring over $M_2(\mathbb{Z}_2)$. So, $R[x; \delta]$ is a Baer ring, but R is not quasi-Baer.

Corollary 2.15. [7, Corollary 2.8] Let R be a ring. Then, R is quasi-Baer (resp. right p.q.-Baer) if and only if $R[x]$ is quasi-Baer (resp. right p.q.-Baer).

Corollary 2.16. [10, Corollary 2.8] Let R be an (α, δ) -compatible ring. Then, R is quasi-Baer (resp. right p.q.-Baer) if and only if $R[x; \alpha, \delta]$ is quasi-Baer (resp. right p.q.-Baer).

According to Lee-Zhou [21], a module M_R is called *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is clear that R is a reduced ring if and only if R_R is reduced. If M_R is reduced, then M_R is p.p. if and only if M_R is p.q.-Baer.

Lemma 2.17. *The followings are equivalent for a module M_R .*

- (1) M_R is reduced and (α, δ) -compatible.
- (2) The following conditions hold: for any $m \in M$ and $a \in R$,
 - (a) $ma = 0$ implies $mRa = 0 = mR\alpha(a)$.
 - (b) $m\alpha(a) = 0$ implies $ma = 0$.
 - (c) $ma = 0$ implies $m\delta(a) = 0$.
 - (d) $ma^2 = 0$ implies $ma = 0$.

Proof. The proof is straightforward. \square

Lemma 2.18. *Let M_R be a reduced (α, δ) -compatible module. Then, M_R is (α, δ) -Armendariz.*

Proof. Let $m(x) = m_0 + \cdots + m_k x^k \in M[x]$, and $f(x) = a_0 + \cdots + a_n x^n \in R[x; \alpha, \delta]$ such that $m(x)f(x) = 0$. Hence, $m_k R a_n = 0$, by Lemmas 2.4 and 2.17. Thus, the coefficient of x^{k+n-1} in equation $m(x)f(x) = 0$ is $m_k \alpha^k(a_{n-1}) + m_{k-1} \alpha^{k-1}(a_n) = 0$. Multiplying this equation by a_n from the right-hand side, we obtain $m_{k-1} \alpha^{k-1}(a_n) a_n = 0$. Hence, $m_{k-1} a_n^2 = 0$, and so $m_{k-1} a_n = 0$, by Lemma 2.17. Therefore, $m_k a_{n-1} = 0$, and so $m_k x^k a_{n-1} x^{n-1} = m_{k-1} x^{k-1} a_n x^n = 0$, by Lemma 2.4. Continuing this process, we can prove $m_i x^i a_j x^j = 0$, for each i, j . \square

For a module M_R , put $\text{Ann}_R(2^M) = \{\text{ann}_R(N) \mid N \text{ is a subset of } M\}$. In a similar way as in the proof of Proposition 2.7, we can prove the following.

Proposition 2.19. *Let M_R be an (α, δ) -compatible module and S be the skew polynomial ring $R[x; \alpha, \delta]$. Then, the following statements are equivalent.*

- (1) M_R is (α, δ) -Armendariz.
- (2) $\psi : \text{Ann}_R(2^M) \rightarrow \text{Ann}_S(2^{M[x]}); A \rightarrow AS$ is bijective.

Theorem 2.20. *Let M_R be an (α, δ) -compatible module and $S = R[x; \alpha, \delta]$. If M_R is (α, δ) -Armendariz, then M_R is Baer (resp. p.p.) if and only if $M[x]_S$ is Baer (resp. p.p.).*

Proof. It follows from Lemma 2.18 and Proposition 2.19. \square

According to Krempa [18], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$. A ring R is said to be α -*rigid* if there exists a rigid endomorphism α of R .

Corollary 2.21. [12, Theorem 14] *Let R be an α -rigid ring. Then, R is Baer (resp. p.p.) if and only if $R[x; \alpha, \delta]$ is Baer (resp. p.p.).*

Proof. Since α -rigid rings are reduced and (α, δ) -compatible, the proof follows from Lemma 2.18 and Theorem 2.20. \square

Corollary 2.22. [4, Theorem B] *Let R be a reduced ring. Then, R is Baer (resp. p.p.) if and only if $R[x]$ is Baer (resp. p.p.).*

Acknowledgments

The author thanks the referee for his/her helpful suggestions. This research was supported by Shahrood University of Technology.

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