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A CHARACTERIZATION OF TRIPLE SEMIGROUP OF TERNARY FUNCTIONS AND DEMORGAN TRIPLE SEMIGROUP OF TERNARY FUNCTIONS

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Abstract. We define algebras of triple semigroup and DeMorgan triple semigroup and by defining three Mann's compositions and one unary operation on the set of 3-place(ternary) functions over some set, we construct a DeMorgan triple semigroup of 3-place (ternary) functions and so find an abstract characterization of this algebras.

1. Introduction

Consideration of sets of functions and operations on functions play important roles in modern algebra and generally in mathematics. For example, group theory has become so important because of the theory of transformation groups. Also, the theory of transformation semigroups is the heart of semigroup theory.
While transformations of a set A are one-place functions, i.e., mappings of A into A, various parts of mathematics, beginning from calculus, have to consider multiplace functions, also called functions of many variables. Any mapping of a subset of A^n, i.e., of a subset of the nth Cartesian power of A, into A is called a partial n-place function. The set of all such

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functions is denoted by $F(A^n, A)$. The set of all full n-place functions on A, i.e., mappings defined for every $(a_1, ..., a_n) \in A^n$, is denoted by $T(A^n, A)$. Obviously, $T(A^n, A) \subseteq F(A^n, A)$. In many papers, n-place functions are called n-ary operations.
One has to consider various natural operations on sets of n-place functions, and among them the operation of superposition, i.e., of substituting fixed n-place functions $g_1, ..., g_n$ into another n-place function f, and thus forming a new n-place function $h = f[g_1, ..., g_n]$ in this way, is of paramount importance. In the middle of the 1940's, Menger observed that the superposition of n-place functions had some properties analogous to the associativity law.

On $F(A^n, A)$, we can define n binary compositions $\oplus_1, \oplus_2, ..., \oplus_n$ of two functions by putting

$$(f \oplus_i g)(a_1, ..., a_n) = f(a_1, ..., a_{i-1}, g(a_1, ..., a_n), a_{i+1}, ..., a_n),$$

for all $f, g \in F(A^n, A)$ and $(a_1, ..., a_n) \in A^n$. Since all compositions $\oplus_1, ..., \oplus_n$ are binary associative operations, algebras of the form $(\Phi, \oplus_1, ..., \oplus_n)$, where $\Phi \subseteq F(A^n, A)$, are called $(2, n)$-semigroups of n-place functions (cf. [14] and [16]). If $\Phi \subseteq T(A^n, A)$, then we say that $(\Phi, \oplus_1, ..., \oplus_n)$ is a $(2, n)$-semigroup of full n-place functions (or n-ary operations).
The study of such compositions of functions was initiated by Mann [8] for binary operations and continued by others (cf., for examples, [1], [12], [15] and [17]). Nowadays, such compositions are called Mann's compositions or Mann's superpositions. Mann's compositions of n-ary operations are described in [16]. Abstract algebras isomorphic to some sets of operations closed with respect to these compositions are described in [14]. The sets of partial functions closed with respect to these compositions and some additional operations are characterized in [5]. Also, the set of partial binary functions closed with respect to these compositions and one quasi-complementation operation is characterised in [12]. In this paper, we fined an abstract characterization of the set of 3-place functions (here we say ternary functions) closed with respect to Mann's compositions and one unary quasi-complementation operation. Results of this paper can be extended to n-place functions.

Definition 1.1. *An algebra $(D, +, ., *)$ is called triple semigroup if $+$, $.$ and $*$ are binary associative operations over the set D.*

Let Ω be a set. We denote all ternary functions from Ω^3 to Ω by T_Ω. In T_Ω, we define the binary operations below:

$$\begin{aligned}(A+B)(x,y,z) &= A(x,y,B(x,y,z)),\\ (A*B)(x,y,z) &= A(x,B(x,y,z),z),\\ (A\cdot B)(x,y,z) &= A(B(x,y,z),y,z).\end{aligned}$$

All these three operations are also associative (and Mann's compositions). The algebra $(T_\Omega,+,.,*)$ is called triple semigroup of ternary functions over the set Ω.

Definition 1.2. *A left null element in any triple semigroup $(D,+,.,*)$ is an element $N\in D$ such that for all $X\in D$,*

$$N+X=N,\quad N\cdot X=N\quad \textit{and}\quad N*X=N.$$

The set of all left null elements of the triple semigroup D is denoted by n(D).

Now, we intend to find all left null elements of the triple semigroup $(T_\Omega,+,.,*)$. To this end, we denote by L, the set of all ternary functions $C_a\in T_\Omega$, where,

$$\forall x,y,z\in\Omega, C_a(x,y,z)=a.$$

It is easy to verify that for all $A\in T_\Omega$ and $a,b,c\in\Omega$,

$$C_a+A=C_a\cdot A=C_a*A=C_a, \tag{1.1}$$

and

$$\begin{aligned}(A+C_a)(x,y,z) &= A(x,y,a),\\ (A*C_a)(x,y,z) &= A(x,a,z),\\ (A\cdot C_a)(x,y.z) &= A(a,y,z).\end{aligned} \tag{1.2}$$

Also,

$$\begin{aligned}&[(A\cdot C_a)*C_b]+C_c=[(A*C_b)\cdot C_a]+C_c\\ &=[(A\cdot C_a)+C_c]*C_b=[(A+C_c)\cdot C_a]*C_b\\ &=[(A*C_b)+C_c]\cdot C_a=[(A+C_c)*C_b]\cdot C_a\\ &=C_{A(a,b,c)}.\end{aligned} \tag{1.3}$$

From (1.3), we conclude that for all $A,B\in T_\Omega$, if we have

$$[(A\cdot C_a)*C_b]+C_c=[(B\cdot C_a)*C_b]+C_c,$$

for all $a, b, c \in \Omega$, then $C_{A(a,b,c)} = C_{B(a,b,c)}$, and so $A(a, b, c) = B(a, b, c)$, for all a, b, c and thus $A = B$.
Moreover for all $A, B \in T_\Omega$ and $a \in \Omega$, we have

$$\begin{aligned}
&(A + B) \cdot C_a = (A \cdot C_a) + (B \cdot C_a),\\
&(A * B) \cdot C_a = (A \cdot C_a) * (B \cdot C_a),\\
&(A + B) * C_a = (A * C_a) + (B * C_a),\\
&(A \cdot B) * C_a = (A * C_a) \cdot (B * C_a),\\
&(A \cdot B) + C_a = (A + C_a) \cdot (B + C_a),\\
&(A * B) + C_a = (A + C_a) * (B + C_a).
\end{aligned} \tag{1.4}$$

Lemma 1.3. $n(D_\Omega) = L$.

Proof. From (1.1), it follows that $L \subseteq n(T_\Omega)$. Now, suppose $N \in n(T_\Omega)$. Let $a, b, c, \gamma \in \Omega$ and $N(a, b, c) = \gamma$. From (1.3), we get:

$$N = [(N \cdot C_a) * C_b] + C_c = C_{N(a,b,c)} = C_\gamma \in L.$$

Hence, $n(T_\Omega) = L$. □

2. Characterization of Triple Semigroup of Ternary Functions

Theorem 2.1. *In order for the triple semigroup $(D, +, ., *)$ to be isomorphic to the triple semigroup $(T_\Omega, +, ., *)$, it is necessary and sufficient that the following conditions hold:*

(I) *The set n(D) has the same cardinality as the set Ω.*

(II) *For all $A, B \in D$ and $N \in n(D)$,*
$(A+B) \cdot N = (A \cdot N) + (B \cdot N)$, $(A+B) * N = (A * N) + (B * N)$,
$(A \cdot B) + N = (A+N) \cdot (B+N)$, $(A * B) + N = (A+N) * (B+N)$,
$(A+B) * N = (A * N) + (B * N)$, $(A \cdot B) * N = (A * N) \cdot (B * N)$.

(III) *For all $A, B \in D$, if we have*

$$[(A \cdot N_1) * N_2] + N_3 = [(B \cdot N_1) * N_2] + N_3$$

for all $N_1, N_2, N_3 \in n(D)$, then A=B.

(IV) *If $(D', +, ., *))$ is a supertriple semigroup of the triple semigroup $(D, +, ., *)$, satisfying conditions* (II) *and* (III)*, where $n(D') = n(D)$, then $D = D'$.*

Proof. First, note that condition (II) implies that for all $A \in D$ and $N_1, N_2, N_3 \in n(D)$,

$$\begin{aligned} &(A \cdot N_1) + N_2 = (A + N_2) \cdot N_1, \\ &(A \cdot N_1) * N_2 = (A * N_2) \cdot N_1, \\ &(A + N_1) * N_2 = (A * N_2) + N_1, \end{aligned} \tag{2.1}$$

and also from (2.1), we get:

$$\begin{aligned} &[(A \cdot N_1) * N_2] + N_3 = [(A * N_2) \cdot N_1] + N_3 \\ &= [(A * N_2) + N_3] \cdot N_1 = [(A + N_3) * N_2] \cdot N_1 \\ &= [(A + N_3) \cdot N_1] * N_2 = [(A \cdot N_1) + N_3] * N_2 \in n(D). \end{aligned} \tag{2.2}$$

To prove the last part of the above relations, we have for each $B \in D$,

$$[[(A \cdot N_1) * N_2] + N_3] + B = [(A \cdot N_1) * N_2] + (N_3 + B) = [(A \cdot N_1) * N_2] + N_3,$$

and

$$\begin{aligned} [[(A \cdot N_1) * N_2] + N_3] \cdot B &= [[(A * N_2) + N_3] \cdot N_1] \cdot B && ;\ \text{by (2.2)} \\ &= [(A * N_2) + N_3] \cdot (N_1 \cdot B) \\ &= [(A * N_2) + N_3] \cdot N_1 \\ &= [(A \cdot N_1) * N_2] + N_3. && ;\ \text{by (2.2)} \end{aligned}$$

Also,

$$\begin{aligned} [[(A \cdot N_1) * N_2] + N_3] * B &= [[(A \cdot N_1) + N_3] * N_2] * B && ;\ \text{by (2.2)} \\ &= [(A \cdot N_1) + N_3] * (N_2 * B) \\ &= [(A \cdot N_1) + N_3] * N_2 \\ &= [(A \cdot N_1) * N_2] + N_3. && ;\ \text{by (2.2)} \end{aligned}$$

So, all terms of the equalities (2.2) belong to $n(D)$. Now, let us prove the necessity. Suppose $D \approx T_\Omega$ and we must prove the conditions (I) $-$ (IV) hold in D. It is enough to prove that these conditions hold in T_Ω. By Lemma 1.3, condition (I) holds in T_Ω. By (1.4), condition (II) and by (1.3), condition (III) holds in T_Ω.

Let $(D',+,.,*)$ be a supertriple semigroup of T_Ω, and D' satisfies conditions (II) and (III) and $n(D') = n(T_\Omega) = L$. We will show $T_\Omega = D'$.

Since D' satisfies condition (II), so by (2.2) for each $X \in D'$ and $a, b, c \in \Omega$, there is $\gamma \in \Omega$ such that

$$[(X \cdot C_a) * C_b] + C_c = C_\gamma. \tag{2.3}$$

In view of (2.3), for each $X \in D'$, we assign an element $\varphi X \in T_\Omega$ by:

$$\forall a, b, c, \gamma \in \Omega : \quad \varphi X(a, b, c) = \gamma \Longleftrightarrow [(X \cdot C_a) * C_b] + C_c = C_\gamma. \tag{2.4}$$

Since $\varphi X \in T_\Omega$, by using (1.3) we get from (2.4),

$$\begin{aligned}
\varphi X(a, b, c) = \gamma \quad &\Longrightarrow C_{\varphi X(a,b,c)} = C_\gamma \\
&\Longrightarrow [(\varphi X \cdot C_a) * C_b] + C_c = C_\gamma && ;\text{by}(1.3) \\
&\Longrightarrow [(\varphi X \cdot C_a) * C_b] + C_c = [(X \cdot C_a) * C_b] + C_c && ;\text{by}(2.4) \\
&\Longrightarrow \varphi X = X && ;\text{by(III)} \\
&\Longrightarrow X \in T_\Omega \\
&\Longrightarrow D' = T_\Omega.
\end{aligned}$$

Conversely, now suppose the triple semigroup $(D, +, ., *)$ satisfies conditions (I) $-$ (IV). We must prove $D \approx T_\Omega$. By condition (I), each element of $n(D)$ can be denoted by $N_a (a \in \Omega)$. Also, by (2.2), we have for all $X \in D$ and $a, b, c \in \Omega$, there is $\gamma \in \Omega$ such that

$$[(X \cdot N_a) * N_b] + N_c = N_\gamma.$$

Then, to each $X \in D$ we can assign the following ternary function $\varphi X \in T_\Omega$:

$$\varphi X(a, b, c) = \gamma \Longleftrightarrow [(X \cdot N_a) * N_b] + N_c = N_\gamma.$$

By condition (III), it follows that φ is a one to one mapping from D to T_Ω. We prove this mapping is an isomorphism. For all $X, Y \in D$ and $a, b, c, \gamma \in \Omega$, we have

$$\begin{aligned}
&\varphi(X \cdot Y)(a, b, c) = \gamma \Longleftrightarrow [[(X \cdot Y) \cdot N_a] * N_b] + N_c = N_\gamma \\
&\Longleftrightarrow [[X \cdot (Y \cdot N_a)] * N_b] + N_c = N_\gamma \\
&\Longleftrightarrow [(X * N_b) \cdot [(Y \cdot N_a) * N_b]] + N_c = N_\gamma && ;\ \text{by}(2.1) \\
&\Longleftrightarrow [(X * N_b) + N_c] \cdot [[(Y \cdot N_a) * N_b] + N_c = N_\gamma, && ;\ \text{by}(2.1)
\end{aligned}$$

but by (2.2), $[[(Y \cdot N_a) * N_b] + N_c] \in n(D)$, and so there is $\lambda \in \Omega$ such that $[[(Y \cdot N_a) * N_b] + N_c]{=}N_\lambda$ and then $\varphi Y(a, b, c) = \lambda$. Hence, from

the above, we get:

$$\begin{aligned}
&\Longleftrightarrow [(X * N_b) + N_c] \cdot N_\lambda = N_\gamma \\
&\Longleftrightarrow [(X \cdot N_\lambda) * N_b] + N_c = N\gamma \qquad ;\ \text{by}(2.2) \\
&\Longleftrightarrow \varphi X(\lambda, b, c) = \gamma \\
&\Longleftrightarrow \varphi X\left(\varphi Y(a, b, c), b, c\right) = \gamma \\
&\Longleftrightarrow \left(\varphi X \cdot \varphi Y\right)(a, b, c) = \gamma.
\end{aligned}$$

Since a, b and c were arbitrary, we conclude that $\varphi X \cdot \varphi Y = \varphi(X \cdot Y)$. Similarly, we can see

$$\varphi(X + Y) = \varphi X + \varphi Y \qquad \text{and} \qquad \varphi(X * Y) = \varphi X * \varphi Y.$$

Now, we show that φ is onto. Since $D \approx \varphi(D) \subseteq T_\Omega$, by replacing in the triple semigroup T_Ω each element of the form φX by its preimage X , we get a triple semigroup $(D', +, ., *)$, which is isomorphic to T_Ω and such that D is a subtriple semigroup of D'. It is clear that $n(D') \subseteq n(D)$. Now, suppose $N_\gamma \in n(D)$. Then, for all $a, b, c \in \Omega$, we have

$$\begin{aligned}
[(N_\gamma \cdot N_a) * N_b] + N_c = N_\gamma \quad &\Longrightarrow \varphi N_\gamma(a, b, c) = \gamma \\
&\Longrightarrow \varphi N_\gamma(a, b, c) = C_\gamma(a, b, c).
\end{aligned}$$

Thus, $\varphi N_\gamma = C_\gamma \in n(T_\Omega) \approx n(D')$ and $N_\gamma \in n(D')$. Thus, $n(D) = n(D)')$. Since T_Ω satisfies conditions (II) and (III), D' satisfies them too, and from (IV) we have $T_\Omega \approx D' = D$. Then, φ is an isomorphism. So, we have $D \approx T_\Omega$ □

Definition 2.2. *An ideal of any triple semigroup $(T, +, ., *)$ is a non-empty subset $L \subseteq T$ such that for all $X \in T$ and $A \in L$, all of the elements below are in L:*

$$A \cdot X \quad , \quad X \cdot A \quad , \quad A + X \quad , \quad X + A \quad , \quad A * X \quad , \quad X * A.$$

Definition 2.3. *An equivalence relation E on the triple semigroup $(T, +, ., *)$ is called a congruence of algebra T, if for all $X \in T$ and $(A, B) \in E$ all of the ordered pairs below are in E:*

$$\begin{array}{lll}
(A * X, B * X), & (X * A, X * B), & (A + X, B + X), \\
(X + A, X + B), & (A \cdot X, B \cdot X), & (X \cdot A, X \cdot B).
\end{array}$$

An equivalence relation E on the algebra T is called **trivial**, if $(A, B) \in E$ implies $A = B$.

In triple semigroup $(T_\Omega, +, ., *)$, we denote the projection functions by I_1 , I_2 and I_3:

$$I_1(x, y, z) = x \quad , \quad I_2(x, y, z) = y \quad , \quad I_3(x, y, z) = z.$$

For each $A \in T_\Omega$, we have the followings:

$$\begin{aligned} &I_1 \cdot A = A \cdot I_1 = A, I_2 * A = A * I_2 = A, I_3 + A = A + I_3 = A, \\ &I_1 + A = I_1 * A = I_1, I_2 \cdot A = I_2 + A = I_2, I_3 * A = I_3 \cdot A = I_3. \end{aligned} \tag{2.5}$$

Theorem 2.4. *The triple semigroup $(T_\Omega, +, ., *)$ has no ideals distinct from T_Ω.*

Proof. Let D be an ideal of T_Ω and suppose that A be an arbitrary element of T_Ω and $B \in D$. We have

$$\begin{aligned} B \in D \quad &\Longrightarrow I_1 + B \in D \\ &\Longrightarrow A = A \cdot I_1 = A \cdot (I_1 + B) \in D \\ &\Longrightarrow A \in D \\ &\Longrightarrow D = T_\Omega. \end{aligned}$$

□

Theorem 2.5. *The algebra $(T_\Omega, +, ., *)$ has no non-trivial congruences.*

Proof. Let E be a congruence of T_Ω distinct from equality. Then, there exist $A, B \in T_\Omega$ and $a, b, c, \gamma, \delta \in \Omega$ such that $A(a, b, c) = \gamma$, $B(a, b, c) = \delta$, $\gamma \neq \delta$ and $(A, B) \in E$. By (3), we have

$$[(A \cdot C_a) * C_b] + C_c = C_\gamma \quad \textit{and} \quad [(B \cdot C_a) * C_b] + C_c = C_\delta.$$

Hence, $(C_\gamma, C_\delta) \in E$. Now, let X be an element of T_Ω. Our claim is that $(X, I_2) \in E$. To this end, first assume that Ω is a finite set: $\Omega = \{x_1, x_2, ..., x_n\}$. Since $\gamma, \delta \in \Omega$, so $n \geq 2$. Let $a, b \in \Omega$ and $a \neq b$. Denote by X_a and D_a, the following ternary functions over Ω:

$$X_a(x, y, z) = \begin{cases} X(x, y, z) & \text{if } x = a \\ y & \text{if } x \neq a \end{cases}$$

$$D_a(x, y, z) = \begin{cases} a & \text{if } x = a \text{ and } y = \delta \\ b & \text{if } x \neq a \text{ or } y \neq \delta. \end{cases}$$

By using (1.2), for all $\lambda \in \Omega$, we get:

$$[X_a \cdot (D_a * C_\lambda)](x,y,z) = X_a((D_a * C_\lambda)(x,y,z),y,z) = X_a(D_a(x,\lambda,z),y,z).$$

Hence,

$$\begin{aligned}[X_a \cdot (D_a * C_\gamma)](x,y,z) &= X_a((D_a(x,\gamma,z),y,z) \\ &= X_a(b,y,z) = y = I_2(x,y,z),\end{aligned}$$

and so, $X_a \cdot (D_a * C_\gamma) = I_2$. Also,

$$\begin{aligned}X_a \cdot (D_a * C_\delta)(x,y,z) &= X_a(D_a(x,\delta,z),y,z) \\ &= \begin{cases} X_a(b,y,z) & \text{if } x \neq a \\ X_a(x,y,z) & \text{if } x = a \end{cases} \\ &= \begin{cases} y & \text{if } x \neq a \\ X_a(x,y,z) & \text{if } x = a \end{cases} \\ &= \begin{cases} X_a(x,y,z) & \text{if } x \neq a \\ X_a(x,y,z) & \text{if } x = a \end{cases} \\ &= X_a(x,y,z).\end{aligned}$$

Hence, $X_a \cdot (D_a * C_\delta) = X_a$, and thus we get:

$$\begin{aligned}(C_\gamma, C_\delta) \in E &\Longrightarrow (X_a \cdot (D_a * C_\gamma), X_a \cdot (D_a * C_\delta)) \in E \\ &\Longrightarrow (I_2, X_a) \in E.\end{aligned}$$

It is easy to verify that $X = X_{x_1} * X_{x_2} * ... * X_{x_n}$. Also, by (2.5), $I_2 = I_2 * I_2 * ... * I_2$. Since $(X_{x_i}, I_2) \in E$, for $i = 1, 2, ..., n$, we get $(X, I_2) \in E$.

Now, consider that the set Ω is infinite. Then, the sets Ω and Ω^3 have the same power and there is a one to one mapping F from Ω^3 onto Ω. Hence, for each $y \in \Omega$, there exist elements $r, s, t \in \Omega$ such that $F(r,s,t) = y$. So, the function $K : \Omega^3 \to \Omega$ can be defined as follows:

$$K(x,y,z) = K(x, F(r,s,t), z) = \begin{cases} X(r,s,t) & \text{if } x = \delta \\ s & \text{if } x \neq \delta. \end{cases}$$

We have

$$\begin{aligned}[(K \cdot C_\gamma) * F](x,y,z) &= K \cdot C_\gamma(x, F(x,y,z), z) \\ &= K(\gamma, F(x,y,z), z) = y = I_2(x,y,z),\end{aligned}$$

and

$$[(K \cdot C_\delta) * F](x,y,z) = K \cdot C_\delta(x, F(x,y,z), z) = X(x,y,z).$$

Then, $(K \cdot C_\gamma) * F = I_2$ and $(K \cdot C_\delta) * F = X$, and since $(C_\delta, C_\gamma) \in E$, we have $(X, I_2) \in E$. Thus, in both cases for Ω it was proved that if $X \in T_\Omega$, then $(X, I_2) \in E$. Now, let $X, Y \in T_\Omega$. We have

$$\begin{aligned}(I_2, Y) \in E,\ (X, I_2) \in E &\Longrightarrow (X * I_2, X * Y) \in E, (X * Y, I_2 * Y) \in E \\ &\Longrightarrow (X, X * Y) \in E, \quad (X * Y, Y) \in E \\ &\Longrightarrow (X, Y) \in E.\end{aligned}$$

Hence, E is trivial. □

3. Characterization of DeMorgan Triple Semigroup of Ternary Functions

Definition 3.1. *A DeMorgan triple semigroup is an algebra* $(D, +, ., *, -, 0, 1, e)$ *such that* $(D, +, ., *)$ *is a triple semigroup and the quasi- complementation operation* $-\,; D \to D$ *and constants 0, 1, e satisfy the followings, for all* $x, y \in D$,

(1) $x + 0 = 0 + x = x$
(2) $x \cdot 1 = 1 \cdot x = x$
(3) $x * e = e * x = x$
(4) $\bar{\bar{x}} = x$
(5) $\overline{x + y} = \bar{x} \cdot \bar{y}$
(6) $\overline{x \cdot y} = \bar{x} + \bar{y}$
(7) $\overline{x * y} = \bar{x} * \bar{y}$.

It is easy to see that

$$\bar{1} = 0 \quad , \quad \bar{0} = 1 \quad \text{and} \quad \bar{e} = e.$$

Let Ω be a set. For each ternary function $A \in T_\Omega$, we define the quasi-complementation operation $^-$ as follows:

$$\bar{A}(x, y, z) = A(z, y, x).$$

For projection functions I_1, I_2, I_3 in T_Ω, we have

$$\begin{aligned}\bar{I_1}(x, y, z) &= I_1(z, y, x) = z = I_3(x, y, z) \\ \bar{I_2}(x, y, z) &= I_2(z, y, x) = y = I_2(x, y, z) \\ \bar{I_3}(x, y, z) &= I_3(z, y, x) = x = I_1(x, y, z).\end{aligned}$$

Hence,

$$\bar{I_1} = I_3 \quad , \quad \bar{I_3} = I_1 \quad \text{and} \quad \bar{I_2} = I_2,$$

and for all $A \in T_\Omega$ in (2.5) we had

$$I_1 \cdot A = A \cdot I_1 = A, \quad I_2 * A = A * I_2 = A, \quad I_3 + A = A + I_3 = A.$$

For all $A, B \in T_\Omega$ and $x, y, z \in \Omega$, we have

$$\begin{aligned} \overline{A+B}(x,y,z) = (A+B)(z,y,x) &= A(z,y,B(z,y,x)) \\ \bar{A} \cdot \bar{B}(x,y,z) = \bar{A}(\bar{B}(x,y,z),y,z) &= \bar{A}(B(z,y,x),y,z) \\ &= A(z,y,B(z,y,x)). \end{aligned}$$

So, we have $\overline{A+B} = \bar{A} \cdot \bar{B}$. Similarly we can see

$$\overline{A \cdot B} = \bar{A} + \bar{B} \quad \text{and} \quad \overline{A * B} = \bar{A} * \bar{B}.$$

From our discussion given above, the following theorem is concluded.

Theorem 3.2. *The algebra* $(T_\Omega, +, ., *, -, I_3, I_1, I_2)$ *is a DeMorgan triple semigroup.*

For all $a \in \Omega$, we have

$$\bar{C_a}(x,y,z) = C_a(z,y,x) = a = C_a(x,y,z) \Longrightarrow \bar{C_a} = C_a. \tag{3.1}$$

In the next theorem, we present a characterization of the DeMorgan triple semigroup $(T_\Omega, +, ., *, -, I_3, I_1, I_2)$.

Theorem 3.3. *The DeMorgan triple semigroup* $(D, +, ., *, -, 0, 1, e)$ *is isomorphic to the DeMorgan triple semigroup* $(T_\Omega, +, ., *, -, I_3, I_1, I_2)$*, if and only if the following conditions hold:*

(I) *The set n(D) has the same cardinality as the set* Ω.

(II) *For all* $A, B \in D$ *and* $N \in n(D)$,
$(A+B) \cdot N = (A \cdot N) + (B \cdot N)$, $(A+B) * N = (A * N) + (B * N)$,
$(A \cdot B) + N = (A+N) \cdot (B+N)$, $(A * B) + N = (A+N) * (B+N)$,
$(A+B) * N = (A * N) + (B * N)$, $(A \cdot B) * N = (A * N) \cdot (B * N)$.

(III) *For all* $A, B \in D$*, if we have*

$$[(A \cdot N_1) * N_2] + N_3 = [(B \cdot N_1) * N_2] + N_3,$$

for all $N_1, N_2, N_3 \in n(D)$*, then* $A = B$.

(IV) *If* $(D', +, ., *))$ *is a supertriple semigroup of the triple semigroup* $(D, +, ., *)$ *satisfying conditions* (II) *and* (III)*, where* $n(D') = n(D)$*, then* $D = D'$.

(V) *For each* $N \in n(D), \bar{N} = N$.

Proof. The proof is similar to the proof of Theorem 2.1. For the necessary part, we assume $D \approx T_\Omega$ and prove that D satisfies the conditions (I) – (V). To do this, it is enough to prove that these conditions hold in T_Ω. As we saw in the proof of Theorem 2.1, the conditions (I) – (IV) hold in T_Ω and by (3.1), the condition (V) holds in T_Ω.

For the sufficient part, we assume the conditions (I) – (V) hold in D and it must be proved $D \approx T_\Omega$. By condition (I), each element of $n(D)$ can be denoted by the form N_a, where $a \in \Omega$. Also, by (2.2), for all $X \in D$ and $a, b, c \in \Omega$, there is $\gamma \in \Omega$ such that

$$[(X \cdot N_a) * N_b] + N_c = N_\gamma.$$

Hence, to each $X \in D$, we can assign a ternary function $\varphi X \in T_\Omega$ by

$$\varphi X(a, b, c) = \gamma \Longleftrightarrow [(X \cdot N_a) * N_b] + N_c = N_\gamma.$$

As we saw in the proof of Theorem 2.1, the mapping $\varphi : D \to T_\Omega$ is one to one and onto. Also, for all $X, Y \in D$,

$$\varphi(X \cdot Y) = \varphi X \cdot \varphi Y, \quad \varphi(X + Y) = \varphi X + \varphi Y \quad \text{and} \quad \varphi(X * Y) = \varphi X * \varphi Y.$$

Moreover, we have

$$\begin{aligned}
\varphi\bar{X}(a, b, c) = \gamma \quad &\Longleftrightarrow [(\bar{X} \cdot N_a) * N_b] + N_c = N_\gamma \\
&\Longleftrightarrow \overline{[(\bar{X} \cdot N_a) * N_b] + N_c} = \bar{N}_\gamma \\
&\Longleftrightarrow [(X + \bar{N}_a) * \bar{N}_b] \cdot \bar{N}_c = \bar{N}_\gamma \\
&\Longleftrightarrow [(X + N_a) * N_b] \cdot N_c = N_\gamma \qquad ;\ \text{by (V)} \\
&\Longleftrightarrow [(X \cdot N_c) * N_b] + N_a = N_\gamma \qquad ;\ \text{by(2.2)} \\
&\Longleftrightarrow \varphi X(c, b, a) = \gamma \\
&\Longleftrightarrow \overline{\varphi X}(a, b, c) = \gamma = \varphi(\bar{X})(a, b, c) \\
&\Longleftrightarrow \varphi\bar{X} = \overline{\varphi X}.
\end{aligned}$$

Hence, the mapping $\varphi : D \to T_\Omega$ is an isomorphism.

□

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