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ON THE AVERAGE NUMBER OF SHARP CROSSINGS OF CERTAIN GAUSSIAN RANDOM POLYNOMIALS

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ABSTRACT. Let $Q_n(x) = \sum_{i=0}^{n} A_i x^i$ be a random algebraic polynomial where the coefficients $A_0, A_1, \cdots$ form a sequence of centered Gaussian random variables. Moreover, assume that the increments $\Delta_j = A_j - A_{j-1}$, $j = 0, 1, 2, \cdots$, are independent, assuming $A_{-1} = 0$. The coefficients can be considered as n consecutive observations of a Brownian motion. We obtain the asymptotic behaviour of the expected number of u-sharp crossings, $u > 0$, of polynomial $Q_n(x)$. We refer to u-sharp crossings as those zero up-crossings with slope greater than u, or those down-crossings with slope smaller than $-u$. We consider the cases where u is unbounded and increasing with n, say $u = o(n^{5/4})$, and $u = o(n^{3/2})$.

1. **Preliminaries**

The theory of the expected number of real zeros of random algebraic polynomials was addressed in the fundamental work of Kac [6]. The works of Logan and Shepp [7, 8], Ibragimov and Maslova [5], Wilkins [15], and Farahmand [3] and Sambandham [13, 14] are other fundamental contributions to the subject. For various aspects on random polynomials, see Bharucha-Reid and Sambandham [1], and Farahmand [4]. There

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has been recent interest in cases where the coefficients form certain random processes; see Rezakhah and Soltani [11, 12], and Rezakhah and Shemehsavar [9, 10]. Let

$$Q_n(x) = \sum_{i=0}^{n} A_i x^i, \ -\infty < x < \infty, \tag{1.1}$$

where, $A_0, A_1, \cdots$, are mean zero Gaussian random variables for which the increments $\Delta_i = A_i - A_{i-1}, \ i = 1, 2, \cdots$, are independent, and $A_{-1} = 0$. The sequence $A_0, A_1 \cdots$ may be considered as successive Brownian points, i.e., $A_j = W(t_j), \ j = 0, 1, \cdots$, where, $t_0 < t_1 < \cdots$, and $\{W(t), \ t \geq 0\}$ is the standard Brownian motion. In this physical interpretation, $\mathrm{Var}(\Delta_j)$ is the distance between successive times t_{j-1} *and* t_j. Thus, for $j = 1, 2, \cdots$, we have that $A_j = \Delta_0 + \Delta_1 + \cdots + \Delta_j$, where $\Delta_i \sim N(0, \sigma_i^2)$ are independent. Thus, $Q_n(x) = \sum_{k=0}^{n}(\sum_{j=k}^{n} x^j)\Delta_k = \sum_{k=0}^{n} a_k(x)\Delta_k$, and $Q_n'(x) = \sum_{k=0}^{n}(\sum_{j=k}^{n} jx^{j-1})\Delta_k = \sum_{k=0}^{n} b_k(x)\Delta_k$, where,

$$a_k(x) = \sum_{j=k}^{n} x^j, \qquad b_k(x) = \sum_{j=k}^{n} jx^{j-1}, \qquad k = 0, \cdots, n. \tag{1.2}$$

We say that $Q_n(x)$ has a zero up-crossing at t_0 if there exists $\varepsilon > 0$ such that $Q_n(x) \leq 0$ in $(t_0 - \varepsilon, t_0)$ and $Q_n(x) > 0$ in $(t_0, t_0 + \varepsilon)$. Similarly, $Q_n(x)$ is said to have a zero down crossing at t_0 if $Q_n(x) > 0$ in $(t_0 - \varepsilon, t_0)$ and $Q_n(x) \leq 0$ in $(t_0, t_0 + \varepsilon)$. We study the asymptotic behavior of the expected number of u-sharp zero crossings, those zero up-crossings with slope greater than $u > 0$, or those down-crossings with slope smaller than $-u$.

Cramer and Leadbetter [1967, p. 287] have shown that the expected number of total zeros of any Gaussian nonstationary process, say $Q_n(x)$, is calculated by the following formula,

$$EN(a, b) = \int_a^b dt \int_{-\infty}^{\infty} |y| p_t(0, y) dy,$$

where, $p_t(z, y)$ denotes the joint density of $Q_n(x)$ and its derivative $Q_n'(x)$, and

$$\begin{aligned} p_t(0, y) \ &= \ [2\pi\gamma\sigma(1 - \mu^2)^{1/2}]^{-1} \\ &\quad \times \exp\left\{ - \frac{\gamma^2 m^2 + 2\mu\gamma\sigma m(y - m' + \sigma^2(y - m')^2}{2\gamma^2\sigma^2(1 - \mu^2)} \right\}, \end{aligned}$$

in which, $m = E(Q_n(x))$, $m' = E(Q'_n(x))$, $\sigma^2 = \text{Var}(Q_n(x))$, $\gamma^2 = \text{Var}(Q'_n(x))$, and $\mu = \text{Cov}(Q_n(x), Q'_n(x))/(\gamma\sigma)$.

By a similar method as in Farahmand [4], we find that $ES_u(a,b)$, the expected number of u-sharp zero crossings of $Q_n(x)$ in any interval (a,b), satisfies

$$ES_u(a,b) = \int_a^b dt \int_{\{y:|y|>u\}} |y| p_t(0,y) dy = \int_a^b f_n(x) dx, \tag{1.3}$$

where,

$$f_n(x) = \tfrac{1}{\pi} g_{1,n}(x) \exp(g_{2,n}(x)), \tag{1.4}$$

and

$$g_{1,n}(x) = FA^{-2}, \qquad g_{2,n}(x) = -\tfrac{A^2u^2}{2F^2}, \tag{1.5}$$

in which,

$$A^2 = \text{Var}(Q_n(x)) = \sum_{k=0}^n a_k^2(x)\sigma_k^2, \quad B^2 = \text{Var}(Q'_n(x)) = \sum_{k=0}^n b_k^2(x)\sigma_k^2,$$

$$D = \text{Cov}(Q_n(x), Q'_n(x)) = \sum_{k=0}^n a_k(x) b_k(x)\sigma_k^2, \quad \text{and} \quad F^2 = A^2B^2 - D^2,$$

and $a_k(x)$ *and* $b_k(x)$ are defined by (1.2).

2. Asymptotic Behaviour of ES_u

In this section, we obtain the asymptotic behavior of the expected number of u-sharp zero crossings of $Q_n(x)$ given by (1.1). We prove the following theorem for the case that the increments $\Delta_1, \cdots, \Delta_n$ are independent and have the same distribution. Also, we assume that $\sigma_k^2 = 1$, for $k = 1, \cdots, n$.

Theorem 2.1. *Let $Q_n(x)$ be the random algebraic polynomial given by (1.1) for which $A_j = \Delta_1 + ... + \Delta_j$, where, Δ_k, $k = 1, ..., n$, are independent and $\Delta_j \sim N(0, \sigma_j^2)$. Then, the expected number of u-sharp zero crossings of $Q_n(x)$ satisfies:*

(i) for $u = o(n^{5/4})$,

$$\begin{aligned} ES_u(-\infty,\infty) &= \frac{1}{\pi}\log(2n+1) + \frac{1}{\pi}(1.920134502) \\ &+ \frac{1}{\pi\sqrt{2n}}\left(-\pi + 2\arctan(\frac{1}{2\sqrt{2n}})\right) + \frac{C_1}{n\pi} \\ &+ \frac{u^2}{n^3\pi}\left(19.05803659 - \frac{8}{3}\ln(n^3+1)\right) \\ &+ \frac{u^4}{n^6\pi}\left(-34989.96324 + \frac{32}{3}\ln(n^6+1)\right) + o(n^{-1}), \end{aligned}$$

where, $C_1 = -0.7190843756$ *for* n *even and* $C1 = 1.716159410$ *for* n *odd.*
(ii) for $u = o(n^{3/2})$,

$$ES_u(-\infty,\infty) = \frac{1}{\pi}\log(2n+1) + \frac{1}{\pi}(1.920134502) + o(1).$$

Proof. Due to the behavior of $Q_n(x)$, the asymptotic behavior is treated separately on the intervals $1 < x < \infty$, $-\infty < x < -1$, $0 < x < 1$ and $-1 < x < 0$. For $1 < x < \infty$, using (1.3), the change of variable $x = 1 + \frac{t}{n}$ and the equality $\left(1 + \frac{t}{n}\right)^n = e^t\left(1 - \frac{t^2}{n}\right) + O\left(\frac{1}{n^2}\right)$, we find that

$$ES_u(1,\infty) = \frac{1}{n}\int_0^\infty f_n\big(1 + \frac{t}{n}\big)dt,$$

where, $f_n(\cdot)$ is defined by (1.4). Using (1.5), and by some tedious manipulations, we have that

$$g_{2,n}\left(1 + \frac{t}{n}\right) = o(n^{-2}),$$

and

$$n^{-1}g_{1,n}\left(1 + \tfrac{t}{n}\right) = \left(R_1(t) + \tfrac{S_1(t)}{n} + O\left(\tfrac{1}{n^2}\right)\right), \tag{2.1}$$

where,

$$R_1(t) = \frac{\sqrt{(4t-15)e^{4t} + (24t+32)e^{3t} - e^{2t}(8t^3+12t^2+36t+18) + 8e^t t + 1}}{2t(-1 - 3e^{2t} + 4e^t + 2te^{2t})}.$$

Also, $S_1(t) = S_{11}(t)/S_{12}(t)$, in which,

$$\begin{aligned} S_{11}(t) &= -0.25\big((4t^2-6t-27)e^{6t}+(156-84t+116t^2-24t^3)e^{5t} \\ &+ (8t^4+8t^3-32t^2+42t-153)e^{2t}+(28-4t-4t^2)e^{t} \\ &+ (16t^5-72t^4+96t^3-212t^2+220t-331)e^{4t} \\ &+ (328-168t+128t^2-104t^3)e^{3t}-1\big), \end{aligned}$$

and

$$\begin{aligned} S_{12}(t) &= \left(2\,e^{2\,t}t-1+4\,e^{t}-3\,e^{2\,t}\right)^2 \\ &\times(1-8t^3e^{2t}-12e^{2t}t^2+8e^{t}t-18e^{2t}-36e^{2t}t-15e^{4t}+32e^{3t}+24e^{3t}t+4te^{4t})^{1/2}. \end{aligned}$$

One can easily verify that as $\lim_{t\to 0} R_1(t)=0$, and as $t\to\infty$,

$$R_1(t)=\frac{1}{2t^{3/2}}+O(t^{-2}), \qquad S_1(t)=-\frac{1}{8t^{1/2}}+O(t^{-3/2}).$$

As (2.1) can not be integrated term by term, we use the equality

$$\frac{I_{[t>1]}}{8n\sqrt{t}}=\frac{I_{[t>1]}}{8n\sqrt{t}+t\sqrt{t}}+O\left(\frac{1}{n^2}\right), \tag{2.2}$$

where,

$$I_{[t>1]}=\begin{cases} 1 & \text{if } \ t\geq 1 \\ 0 & \text{if } \ t<1 \end{cases}.$$

This is by the fact that $\left[\frac{I_{[t>1]}}{8n\sqrt{t}}-\frac{I_{[t>1]}}{8n\sqrt{t}+t\sqrt{t}}\right]n^2\to\sqrt{t}/64$, as $n\to\infty$, which is constant for fixed t.

Thus, by (2.2), we have that

$$\frac{1}{n}f_n\left(1+\frac{t}{n}\right)=-\frac{I_{[t>1]}}{\pi(8n\sqrt{t}+t\sqrt{t})}+\frac{R_1(t)}{\pi}+\frac{1}{\pi}\left(\frac{S_1(t)}{n}+\frac{I_{[t>1]}}{8n\sqrt{t}}\right)+O\left(\frac{1}{n^2}\right).$$

One should note that the term $O(n^{-2})$ in the above equation is the result of some added term of the same order which comes from equations (2.1) and (2.2), which has a complicated structure. On the other hand, as the total number of real zeros of the polynomial $Q_n(x)$ on $(1,\infty)$ is less than or equal to n, so the left hand side of the above equation is integrable and finite. Also, the integral of the other terms on the right hand side is also finite and integrable, and so the integral of the term $O(n^{-2})$ is finite. Thus, this expression is term by term integrable, and provides that

$$\begin{aligned} ES_u(1,\infty) &= \frac{1}{n}\int_0^\infty f_n\left(1+\frac{t}{n}\right)dt=\frac{1}{2\pi\sqrt{2n}}\left(-\pi+2\arctan\left(\frac{1}{2\sqrt{2n}}\right)\right) \\ &+\frac{1}{\pi}\int_0^\infty R_1(t)dt+\frac{1}{\pi n}\int_0^\infty\left(S_1(t)+\frac{I_{[t>1]}}{8\sqrt{t}}\right)dt+O\left(\frac{1}{n^2}\right), \end{aligned}$$

where, $\int_0^\infty R_1(t)dt = 0.7348742023$, and $\int_0^\infty \left(S_1(t) + I_{[t>1]}(8\sqrt{t})^{-1}\right) dt = -0.2496371198$. For $-\infty < x < -1$, using $x = -1-\frac{t}{n}$, we have $ES_u(-\infty,-1) = \frac{1}{n}\int_0^\infty f_n\left(-1-\frac{t}{n}\right) dt$, where, $f_n(\cdot)$ is defined by (1.4), and by (1.5),

$$n^{-1}g_{1,n}\left(-1-\tfrac{t}{n}\right) = \left(R_2(t) + \tfrac{S_2(t)}{n} + O\left(\tfrac{1}{n^2}\right)\right), \tag{2.3}$$

where,

$$R_2(t) = 1/2\sqrt{\frac{-2\,e^{2\,t} + e^{4\,t} + 1 - 12\,e^{2\,t}t^2 - 8\,t^3e^{2\,t} + 4\,te^{4\,t} - 4\,te^{2\,t}}{t^2\left(e^{2\,t} - 1 + 2\,te^{2\,t}\right)^2}},$$

for n even, $S_2(t) = \frac{S_{21}(t)+S_{22}(t)}{4S_{23}(t)}$, and for n odd, $S_2(t) = \frac{S_{21}(t)-S_{22}(t)}{4S_{23}(t)}$, in which,

$$\begin{aligned}
S_{21}(t) &= 1 + \left(-8\,t^4 + 30\,t - 8\,t^3 + 48\,t^2 - 3\right)e^{2\,t}\\
&+ \left(3 - 12t + 52t^2 + 96t^3 + 40t^4 - 16\,t^5\right)e^{4\,t} - \left(18t + 4t^2 + 1\right)e^{6\,t},\\
S_{22}(t) &= \left(8\,t + 32\,t^3 + 40\,t^2\right)e^{3\,t} + \left(-8\,t^2 - 12\,t\right)e^{5\,t} + 4\,e^t t,\\
S_{23}(t) &= \left(e^{4\,t}(4t+1) - 2e^{2t}(1 + 2t + 6t^2 + 4t^3) + 1\right)^{1/2}\left(e^{2\,t}(2t+1) - 1\right)^2.
\end{aligned}$$

Also,

$$g_{2,n}\left(-1-\tfrac{t}{n}\right) = \left(\tfrac{u^2}{n^3}\right)G_{2,1}(t) + o\left(\tfrac{1}{n}\right), \tag{2.4}$$

where,

$$G_{2,1}(t) = \frac{16\left(2\,e^{2\,t}t + e^{2\,t} - 1\right)t^3}{2\,e^{2\,t} - e^{4\,t} + 12\,e^{2\,t}t^2 - 1 + 8\,e^{2\,t}t^3 - 4\,e^{4\,t}t + 4\,e^{2\,t}t}.$$

It can be seen that as $t \to \infty$,

$$R_2(t) = \frac{1}{2t^{3/2}} + O(t^{-2}), \qquad S_2(t) = \frac{-1}{8t^{1/2}} + O(t^{-3/2}), \qquad G_{2,1}(t) = o(e^{-t}).$$

Now, by using the equality (2.2), we have that

$$\begin{aligned}
\frac{1}{n}f_n(-1-\frac{t}{n}) &= -\frac{I_{[t>1]}}{\pi(8n\sqrt{t} + t\sqrt{t})} + \frac{1}{\pi}\Bigg(R_2(t) + \frac{1}{n}(S_2(t) + \frac{I_{[t>1]}}{8\sqrt{t}})\\
&\quad + \frac{u^2}{n^3}R_2(t)G_{2,1}(t) + \frac{u^4}{2n^6}R_2(t)G_{2,1}^2(t)\Bigg) + o\left(n^{-1}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
ES_u(-\infty,-1) &= \frac{1}{n}\int_0^\infty f_n(-1-\frac{t}{n}) = \frac{1}{2\pi\sqrt{2n}}\left(-\pi + 2\arctan(\frac{1}{2\sqrt{2n}})\right)\\
&+ \frac{1}{\pi}\int_0^\infty R_2(t)dt + \frac{1}{n\pi}\int_0^\infty\left(S_2(t) + \frac{I_{[t>1]}}{8\sqrt{t}}\right)dt\\
&+ \frac{u^2}{n^3\pi}\int_0^\infty R_2(t)G_{2,1}(t)dt + \frac{u^4}{n^6\pi}\int_0^\infty \frac{1}{2}R_2(t)G_{2,1}^2(t)dt + o\left(n^{-1}\right),
\end{aligned}$$

where, $\int_0^\infty R_2(t)dt = 1.095640061$, and

$$\int_0^\infty R_2(t)G_{2,1}(t)dt = -2.418589510, \qquad \int_0^\infty \frac{1}{2}R_2(t)G_{2,1}^2(t)dt = 7.057233216,$$

and for n odd, $\int_0^\infty \left(S_2(t) + I_{[t>1]}(8\sqrt{t})^{-1}\right) dt = -0.0322863105$, and for n even, $\int_0^\infty \left(S_2(t) + I_{[t>1]}(8\sqrt{t})^{-1}\right) dt = -0.4677136959$. For $0 < x < 1$, let $x = 1 - \frac{t}{n+t}$. Then, $ES_u(0,1) = \int_0^\infty \left(\frac{n}{(n+t)^2}\right) f_n\left(1 - \frac{t}{n+t}\right) dt$, where, $f_n(\cdot)$ is defined by (1.4), and by (1.5),

$$g_{2,n}\left(1 - \tfrac{t}{n+t}\right) = o(n^{-2}), \tag{2.5}$$

and

$$\begin{aligned} \frac{n}{(n+t)^2}\, g_{1,n}\left(1 - \frac{t}{n+t}\right) &= \left(1 - \frac{2t}{n} + O\left(\frac{1}{n^2}\right)\right)\left(R_3(t) + \frac{S_3(t)}{n} + O\left(\frac{1}{n^2}\right)\right) \\ &= \left(R_3(t) + \frac{S_3(t) - 2tR_3(t)}{n}\right) + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{2.6}$$

where, we observe that $R_3(t) \equiv R_1(-t)$ and $S_3(t) = S_{31}(t)/S_{32}(t)$, in which,

$$\begin{aligned} S_{31}(t) &= \left(-7t^2 - \frac{69}{2}t - \frac{63}{4}\right)e^{-6t} + \left(6t^3 + 35t - 55t^2 + 39\right)e^{-5t} \\ &+ \left(49t - 4t^5 + 22t^4 + 91t^2 - \frac{63}{4} - 12t^3\right)e^{-4t} \\ &- (6t^3 + 30 + 44t^2 + 66t)e^{-3t} + \left(-9 - t - t^2\right)e^{-t} + 3/4 \\ &+ \left(\frac{35}{2}t + 2t^4 - 6t^3 + 16t^2 + \frac{123}{4}\right)e^{-2t}, \end{aligned}$$

and, $S_{32}(t) \equiv S_{12}(-t)$. Now, as $t \to \infty$, $R_3(t) = \frac{1}{2t} + O(t^{-1/2}e^{-t/2})$, and $S_3(t) = \frac{3}{4} + O(t^2 e^{-t})$. Since the relation (2.6) is not term by term integrable, we use the equality

$$\tfrac{I_{[t>1]}}{2t} - \tfrac{I_{[t>1]}}{4n+2t} = \tfrac{I_{[t>1]}}{2t} - \tfrac{I_{[t>1]}}{4n} + O\left(\tfrac{1}{n^2}\right), \tag{2.7}$$

which is valid by a similar reason as stated for the equality (2.2). Thus, by the relations (1.4), (2.5), (2.6) and (2.7) we have that

$$\begin{aligned} \frac{n}{(n+t)^2} f_n(1 - \frac{t}{n+t}) &= \frac{1}{\pi}\left(R_3(t) - \frac{I_{[t>1]}}{2t}\right) + \frac{1}{\pi}\left(\frac{I_{[t>1]}}{2t} - \frac{I_{[t>1]}}{4n+2t}\right) \\ &+ \frac{1}{n\pi}\left(S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}}{4}\right) + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Using these calculations, one can easily verify that

$$
\begin{aligned}
ES_u(0,1) &= \int_0^\infty \frac{n}{(n+t)^2} f_n(1-\frac{t}{n+t})dt \\
&= \frac{1}{\pi}\int_0^\infty \left(R_3(t) - \frac{I_{[t>1]}}{2t}\right) dt + \frac{1}{2\pi}(\log(4n+2)-\log(2)) \\
&+\frac{1}{n\pi}\int_0^\infty \left(S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}}{4}\right) dt + O\left(\frac{1}{n^2}\right), \qquad (2.8)
\end{aligned}
$$

where,

$$
\begin{aligned}
\int_0^\infty \left(R_3(t) - \frac{I_{[t>1]}}{2t}\right) dt &= -0.2897712456 \\
\int_0^\infty \left(S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}}{4}\right) dt &= 0.498174649.
\end{aligned}
$$

This is valid by a similar reason as stated for $ES_u(1,\infty)$.

For $-1 < x < 0$, let $x = -1 + \frac{t}{n+t}$. Then,

$$
ES_u(-1,0) = \int_0^\infty \left(\frac{n}{(n+t)^2}\right) f_n\left(-1+\frac{t}{n+t}\right) dt.
$$

Using (1.4) and (1.5) we have that

$$
\left(\tfrac{n}{(n+t)^2}\right) g_{1,n}(t) = \left(\tfrac{n^2}{(n+t)^2}\right)\left(R_4(t) + \tfrac{S_4(t)}{n} + O\left(\tfrac{1}{n^2}\right)\right), \qquad (2.9)
$$

in which, $R_4(t) \equiv R_2(-t)$. Here, for n even, $S_4(t) = \frac{S_{41}(t)+S_{42}(t)}{4S_{43}(t)}$ and for n odd, $S_4(t) = \frac{S_{41}(t)-S_{42}(t)}{4S_{43}(t)}$, where,

$$
\begin{aligned}
S_{41}(t) &= 8\left(-9/4\,t + 6\,t^2 - 3\,t^3 - \frac{9}{8} + t^4\right)e^{-2\,t} \\
&+8\left(15\,t^4 - 3/2\,t - 22\,t^3 + \frac{9}{8} + 19/2\,t^2 - 2\,t^5\right)e^{-4\,t} \\
&+8\left(\frac{15}{4}\,t - 7/2\,t^2 - 3/8\right)e^{-6\,t} + 3,
\end{aligned}
$$

$S_{42}(t) \equiv S_{22}(-t)$, and $S_{43}(t) \equiv S_{23}(-t)$. Finally, we have

$$
g_{2,n}\left(-1+\tfrac{t}{n+t}\right) = \left(\tfrac{u^2}{n^3}\right) G_{2,1}(t) + o(n^{-1}), \qquad (2.10)
$$

where,

$$
G_{2,1}(t) = -\frac{16\left(-e^{-2\,t} + 1 + 2\,e^{-2\,t}t\right)t^3}{\left(-4\,e^{-4\,t}t + e^{-4\,t} - 12\,e^{-2\,t}t^2 + 4\,e^{-2\,t}t + 1 + 8\,e^{-2\,t}t^3 - 2\,e^{-2\,t}\right)}.
$$

As $t \to \infty$, we have

$$R_4(t) = \frac{1}{2t} + O(t^{1/2}e^{-t}), \quad S_4(t) = \frac{3}{4} + O(te^{-t}), \quad G_{2,1}(t) = -16t^3 + O(t^4e^{-2t}).$$

Therefore,

$$\begin{aligned}\frac{n^2}{(n+t)^2} f_n(-1+\frac{t}{n+t}) &= \frac{1}{\pi}\left(1-\frac{2t}{n}+O(\frac{1}{n^2})\right)\left(R_4(t)+\frac{S_4(t)}{n}+O(\frac{1}{n^2})\right)\\ &\quad\times\left(1+\frac{u^2}{n^3}G_{2,1}(t)+\frac{u^4}{2n^6}G_{2,1}^2(t)+o(n^{-1})\right)\\ &= \frac{1}{\pi}\left\{R_4(t)+\frac{S_4(t)-2tR_4(t)}{n}+\frac{u^2}{n^3}R_4(t)G_{2,1}(t)\right.\\ &\qquad \left.+\frac{u^4}{2n^6}R_4(t)G_{2,1}^2(t)\right\}+o(n^{-1}).\end{aligned}$$

Since this is not term by term integrable, following this we use (2.7) and the following equalities:

$$\frac{8u^2t^2}{n^3} = \frac{8u^2t^2}{n^3+exp(t^3)} + o(n^{-2})\,, \qquad \frac{64u^4t^5}{n^6} = \frac{64u^4t^5}{n^6+exp(t^6)} + o(n^{-2}).$$

Thus, we have that

$$\begin{aligned}ES_u(-1,0) &= \int_0^\infty \frac{n^2}{(n+t)^2} f_n(-1+\frac{t}{n+t})\\ &= \frac{1}{2\pi}(\log(2n+1)) + \frac{1}{\pi}\left\{\int_0^\infty (R_4(t)-\frac{I_{[t>1]}}{2t})dt(2))dt\right.\\ &+ \frac{1}{n}\int_0^\infty (S_4(t)-2tR_4(t)+\frac{I_{[t>1]}}{4})dt\\ &+ \frac{u^2}{n^3}\int_0^\infty (R_4(t)G_{2,1}(t)+8t^2)dt - u^2\int_0^\infty \frac{8t^2}{n^3+exp(t^3)}dt\\ &+ \left.\frac{u^4}{n^6}\int_0^\infty (\frac{1}{2}R_4(t)G_{2,1}^2(t)-64t^5)dt + u^4\int_0^\infty \frac{64t^5}{n^6+exp(t^6)}dt\right\}\\ &+ o(n^{-1}),\end{aligned}$$

where,

$$\int_0^\infty (R_4(t)-\frac{I_{[t>1]}}{2t})dt = 0.3793914851,$$

$$\begin{aligned}\int_0^\infty (R_4(t)G_{2,1}(t)+8t^2)dt &= 21.47662610,\\ \int_0^\infty (\frac{1}{2}R_4(t)G_{2,1}^2(t)-64t^5)dt &= -34997.02047,\end{aligned}$$

and we have for n even, $\int_0^\infty (S_4(t) - 2tR_4(t) + \frac{I_{[t>1]}}{4})dt = -0.4999081999$, and for n odd, $\int_0^\infty (S_4(t) - 2tR_4(t) + \frac{I_{[t>1]}}{4})dt = 1.499908200$. Also, we have that

$$\int_0^\infty \frac{8t^2}{n^3 + exp(t^3)}dt = \frac{8\log(n^3+1)}{3n^3}, \qquad \int_0^\infty \frac{64t^5}{n^6 + exp(t^6)}dt = \frac{32\log(n^6+1)}{3n^6}.$$

So, we arrive at the first assertion of the theorem.

Now, for the proof of the second part of the theorem, that is, for the case $k = o(n^{3/2})$, we study the asymptotic behavior of $ES_u(a, b)$ for different intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ separately.

For $1 < x < \infty$, using the change of variable $x = 1 + \frac{t}{n}$, and by (1.4), (1.5), and (2.1), we find that

$$\frac{1}{n}\int_0^\infty f_n(1 + \frac{t}{n})dt = \frac{1}{\pi}\int_0^\infty R_1(t)dt + o(1),$$

where, $\int_0^\infty R_1(t)dt = 0.734874192$.

For $-\infty < x < -1$, using the change of variable $x = -1 - \frac{t}{n}$, and by the fact that $u^2/n^3 = o(1)$, as $n \to \infty$, we have that

$$\exp\{u^2 G_{2,1}(-1 - t/n)/n^3\} = 1 + o(1).$$

Therefore, the relations (2.4) implies that $\exp\{g_2(-1-t/n)\} = 1+o(1)$. Thus, by the relations (1.4), (1.5), (2.3), (2.4), we have that

$$\frac{1}{n}\int_0^\infty f_n(-1 - \frac{t}{n})dt = \frac{1}{\pi}\int_0^\infty R_2(t)dt + o(1),$$

where, $\int_0^\infty R_2(t)dt = 1.095640061$.

For $0 < x < 1$, using the change of variable $x = 1 - \frac{t}{n+t}$, and relations (1.4),(1.5),(2.5),(2.6), and by using the equality (2.7), we have that

$$\begin{aligned}\int_0^\infty \frac{n}{(n+t)^2} f_n(-1 - \frac{t}{n+t})dt &= \frac{1}{\pi}\int_0^\infty (R_3(t) - \frac{I[t>1]}{2t})dt \\ &+ \frac{1}{2\pi}\log(2n+1) + o(1),\end{aligned}$$

where, $\int_0^\infty (R_3(t) - \frac{I[t>1]}{2t})dt = -0.28977126$.

For $-1 < x < 0$, using the change of variable $x = -1 + \frac{t}{n+t}$, (2.9), and by the same reasoning as above, the case $-\infty < x < -1$, we have that $\exp\{g_2(-1 + \frac{t}{n+t})\} = 1 + o(1)$. Thus, by using the relations (2.9),(2.10), and by using the equality (2.7), we have that

$$\begin{aligned}\int_0^\infty \frac{n}{(n+t)^2} f_n(-1 + \frac{t}{n+t})dt &= \frac{1}{\pi}\int_0^\infty (R_4(t) - \frac{I[t>1]}{2t})dt \\ &+ \frac{1}{2\pi}\log(2n+1) + o(1),\end{aligned}$$

where, $\int_0^{\infty}(R_4(t) - \frac{I[t>1]}{2t})dt = 0.3793914850$. This complete the proof of the theorem. □

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