ϕ-FACTORABLE OPERATORS AND WEYL-HEISENBERG FRAMES ON LCA GROUPS

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Abstract. We investigate ϕ-factorable operators and Weyl-Heisenberg frames with respect to a function-valued inner product, the so called ϕ-bracket product on $L^2(G)$, where $G$ is a locally compact abelian group and ϕ is a topological isomorphism on $G$. We introduce ϕ-factorable operators on $L^2(G)$ and extend the Riesz Representation Theorems for these operators. Finally, as an application of the ϕ-bracket product, we show that several well known theorems for Weyl-Heisenberg frames in $L^2(\mathbb{R})$ remain valid in $L^2(G)$, and they are unified within of group theory, in connection with the ϕ-bracket product.

1. Introduction

In [13], we have defined the ϕ-bracket product as a function-valued inner product on $L^2(G)$, where $G$ is a locally compact abelian (which will be abbreviated by “LCA”) group and ϕ is a topological isomorphism on $G$. The ϕ-bracket product, as a new inner product on $L^2(G)$, is applicable to extend many ideas and constructions from the theory of shift invariant spaces, factorable operators and Weyl-Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a more general and different
way. Whereas our work in [13] was devoted to basic properties of the $\varphi$-bracket product and $\varphi$-orthonormal bases, here we deal with characterizing $\varphi$-factorable operators on $L^2(G)$ and establishing Riesz Representation Theorems for such operators. We continue our investigation following the line of approach worked by Casazza and Lammers [5], but in a more general setting, using various tools in abstract harmonic analysis. In fact, our results generalize some of the results developed in [5] on $\mathbb{R}^n$, in which the authors want to be able to scale the lattice, and so they introduce a positive parameter $a$ and express their results relative to the lattice $a\mathbb{Z}$. Here, like in [13], we use a topological isomorphism which introduces an appropriate scale factor in the setting of LCA groups. $\varphi$-Factorable operators are useful and shed light to define and investigate $\varphi$-frames and $\varphi$-Riesz bases, which are worked out in a forthcoming paper. After investigating $\varphi$-Factorable operators, we then, as an application of the $\varphi$-bracket product, study Weyl-Heisenberg frames on LCA groups in connection with the $\varphi$-bracket product. Our results generalize some of the results appearing in the literature on the Weyl-Heisenberg frames. Such a unified approach is useful, since it determines the basic features of the Weyl-Heisenberg frames, and includes most of the special cases.

Here, we give some of the basics regarding LCA groups. For a comprehensive account of LCA groups, we refer to [8, 11]. Suppose $G$ is an LCA group with the Haar measure $dx$. A subgroup $L$ of $G$ is called a uniform lattice if it is discrete and co-compact (i.e., $G/L$ is compact). Let $\varphi$ be a topological isomorphism on $G$. If $L$ is a uniform lattice in $G$, then so is $\varphi(L)$. Indeed, obviously $\varphi(L)$ is discrete. Also, by [11, Theorem 5.34], $G/\varphi(L)$ is topologically isomorphic to $G/L$ and so it is compact. Here, we always assume that $G/\varphi(L)$ is normalized, i.e., $|G/\varphi(L)| = 1$. Denote by $\varphi(L)^\perp$, the annihilator of $\varphi(L)$ in $\hat{G}$, i.e., $\varphi(L)^\perp = \{\gamma \in \hat{G}; \ \gamma(\varphi(L)) = \{1\}\}$, which is a uniform lattice in $\hat{G}$ (see [12-16]).

Let $L$ be a uniform lattice in $G$. Choosing the counting measure on $L$, a relation between the Haar measures $dx$ on $G$ and $d\hat{x}$ on $G/\varphi(L)$ is given by the following special case of Weil’s formula [8]:

For $f \in L^1(G)$, we have $\sum_{k \in L} f(x\varphi(k^{-1})) \in L^1(G/\varphi(L))$ and

(1.1) $\int_G f(x)dx = \int_{G/\varphi(L)} \sum_{\varphi(k^{-1}) \in \varphi(L)} f(x\varphi(k^{-1}))d\hat{x},$

where, $\hat{x} = x\varphi(L)$. 

Let \( f, g \in L^2(G) \). The \( \varphi \)-bracket product of \( f, g \) is defined by
\[
[f, g]_\varphi(x) = \sum_{k \in L} f\overline{g}(x\varphi(k^{-1})),
\]
for all \( x \in G \). We define the \( \varphi \)-norm of \( f \) as \( \|f\|_\varphi(x) = ([f, f]_\varphi(x))^{1/2} \).

In the sequel, we recall some basic properties of the \( \varphi \)-bracket product, for the proofs of which and more details the reader is referred to [13].

Let \( f, g \in L^2(G) \). Then, \( \|f, g\|_\varphi \leq \|f\|_\varphi \|g\|_\varphi \) (the Cauchy-Schwartz Inequality). Also, (1.1) implies \( \int_{G/\varphi(L)} [f, g]_\varphi(x) dx = <f, g>_{L^2(G)} \). For \( \gamma \in \hat{G} \), denote by \( M_\gamma \), the modulation operator on \( L^2(G) \), i.e., \( M_\gamma f(x) = \gamma(x)f(x) \), for all \( f \in L^2(G) \). Then, for \( f, g, \gamma \in \varphi(L)^\perp \), we have the following relation between the \( \varphi \)-bracket product and the usual inner product in \( L^2(G) \):
\[
[f, g]_\varphi(\gamma) = <f, M_\gamma g>_{L^2(G)}.
\]

We say \( g \in L^2(G) \) is \( \varphi \)-bounded if there exists \( M > 0 \) so that \( \|g\|_\varphi \leq M \) a.e.. For \( f, g \in L^2(G) \), the function \( [f, g]_\varphi \) need not generally be in \( L^2(G) \). But, we have the following result.

**Proposition 1.1.** If \( f, g, h \in L^2(G) \) and \( g \), \( h \) are \( \varphi \)-bounded, then \( [f, g]_\varphi h \in L^2(G) \).

A sequence \( (g_n)_{n \in \mathbb{N}} \subseteq L^2(G) \) is called \( \varphi \)-orthonormal if \( [g_n, g_m]_\varphi = 0 \), for all \( n \neq m \in \mathbb{N} \) and \( \|g_n\|_\varphi = 1 \), for all \( n \in \mathbb{N} \). Let \( f \in L^2(G) \) and \( (g_n)_{n \in \mathbb{N}} \) be a \( \varphi \)-orthonormal sequence in \( L^2(G) \). An extension of [5, Theorem 4.13] from \( \mathbb{R} \) to the setting of an LCA group gives Bessel’s Inequality for \( \varphi \)-bracket products as follows:
\[
\sum_{n \in \mathbb{N}} ||[f, g_n]_\varphi(x)||^2 \leq ||f||^2_\varphi(x), \text{ for a.e. } x \in G/\varphi(L).
\]

A \( \varphi \)-orthonormal sequence \( (g_n)_{n \in \mathbb{N}} \) is called a \( \varphi \)-orthonormal basis if \( [f, g_n]_\varphi = 0 \) a.e., for all \( n \in \mathbb{N} \), implies \( f = 0 \) a.e.. Let \( (g_n)_{n \in \mathbb{N}} \) be a \( \varphi \)-orthonormal sequence. It is not difficult to mimic the standard proofs for a usual orthonormal sequence in a Hilbert space to obtain equivalent conditions for \( (g_n)_{n \in \mathbb{N}} \subseteq L^2(G) \) to be a \( \varphi \)-orthonormal basis (see also [13]).

**Proposition 1.2.** If \( (g_n)_{n \in \mathbb{N}} \) is a \( \varphi \)-orthonormal sequence in \( L^2(G) \), then the following are equivalent.

1. \( (g_n)_{n \in \mathbb{N}} \) is a maximal \( \varphi \)-orthonormal sequence, i.e., \( (g_n)_{n \in \mathbb{N}} \) is not contained in any other \( \varphi \)-orthonormal set.
(2) \( (g_n)_{n \in \mathbb{N}} \) is a \( \varphi \)-orthonormal basis.
(3) For each \( f \in L^2(G) \), \( f = \sum_{n \in \mathbb{N}} [f, g_n] \varphi g_n \) a.e.
(4) \( \|f\|_2^2 = \sum_{n \in \mathbb{N}} |[f, g_n] \varphi|^2 \) a.e., for all \( f \in L^2(G) \) (the Parseval Identity).
(5) \( \{M_\gamma g_n\}_{n \in \mathbb{N}, \gamma \in \varphi(L)^\perp} \) is an orthonormal basis for \( L^2(G) \).

Thanks to Zorn’s Lemma and Proposition 1.2, \( L^2(G) \) admits a \( \varphi \)-orthonormal basis.

The rest of this paper is organized as follows. In Section 2, we introduce a \( \varphi \)-factorable operator on \( L^2(G) \), where \( G \) is an LCA group and establish the Riesz Representation Theorems for these operators.

Over the last ten years, there have been a lot of research on frame theory in general, and the Weyl-Heisenberg frame theory, in particular [2-4, 7, 18], most of which are on the Euclidean space. Our main goal in Section 3 is to represent the Weyl-Heisenberg frame identity and the frame operator of a Weyl-Heisenberg frame in terms of the \( \varphi \)-bracket product on an LCA group.

2. \( \varphi \)-factorable operators

Throughout this paper, we always assume that \( G \) is a second countable LCA group, \( \varphi \) is a topological isomorphism on \( G \) and the notation are as in Section 1.

A function \( h \in L^\infty(G) \) is said to be \( \varphi \)-periodic if \( h(x \varphi(k)) = h(x) \), for every \( k \in L, x \in G \).

**Definition 2.1.** We say an operator \( U : L^2(G) \to L^p(E) \), \( 1 \leq p \leq \infty \), is \( \varphi \)-factorable if \( U(hf) = hU(f) \), for all \( f \in L^2(G) \) and all \( \varphi \)-periodic \( h \in L^\infty(G) \), where \( E \) is a subgroup of \( G \) or \( G/\varphi(L) \).

A bounded operator \( U \) is \( \varphi \)-factorable if and only if it commutes with modulations. More precisely, we have the following result.

**Lemma 2.2.** Let \( U \) be a bounded operator from \( L^2(G) \) to \( L^2(E) \), where \( E \) is a subgroup of \( G \) or \( G/\varphi(L) \). \( U \) is \( \varphi \)-factorable if and only if

\begin{equation}
(2.1) \quad U(M_\gamma g) = M_\gamma U(g), \text{ for all } g \in L^2(G), \; \gamma \in \varphi(L)^\perp.
\end{equation}

**Proof.** If \( U \) is \( \varphi \)-factorable and \( \gamma \in \varphi(L)^\perp(\subseteq G/\varphi(L)) \), then since \( \gamma \) is \( \varphi \)-periodic, (2.1) obviously holds. Conversely, assume (2.1). Then, \( U \) is \( \varphi \)-factorable using the facts that \( \varphi(L)^\perp(= G/\varphi(L)) \) is an orthonormal basis for \( L^2(G/\varphi(L)) \) and \( L^\infty(G/\varphi(L)) \subseteq L^2(G/\varphi(L)) \). Note that there
is a one-to-one correspondence between \( L^\infty(G/\varphi(L)) \) and the set of all \( \varphi \)-periodic \( h \in L^\infty(G) \). □

Our main goal in this section is to characterize \( \varphi \)-factorable operators \( U : L^2(G) \to L^p(G/\varphi(L)) \), for \( p = 1 \) and \( p = 2 \).

Clearly, the operator \( U \), defined by \( U(f) = [f, g]_\varphi \), for \( f \in L^2(G) \), is \( \varphi \)-factorable. We will also show that every \( \varphi \)-factorable operator \( U : L^2(G) \to L^1(G/\varphi(L)) \) is of this form. First, we establish a lemma in which we show that two \( \varphi \)-factorable operators are equal on \( L^2(G) \) if and only if their integrals over \( G/\varphi(L) \) are the same.

**Lemma 2.3.** Let \( U_1, U_2 : L^2(G) \to L^1(G/\varphi(L)) \) be two \( \varphi \)-factorable operators. Then, \( U_1 = U_2 \) if and only if
\[
\int_{G/\varphi(L)} U_1(f)(\hat{x}) d\hat{x} = \int_{G/\varphi(L)} U_2(f)(\hat{x}) d\hat{x},
\]
for every \( f \in L^2(G) \).

**Proof.** The necessity is obvious. For the converse, by [8, Theorem 4.33], it is enough to show that \( \hat{U}_1(f) = \hat{U}_2(f) \), for all \( f \in L^2(G) \). Let \( \xi \in \varphi(L)^\perp \) and \( \hat{f} \in L^2(G) \). Since \( \xi \) as a function in \( L^\infty(G) \) is \( \varphi \)-periodic, we obtain:
\[
\begin{align*}
U_1(f)(\xi) &= \int_{G/\varphi(L)} U_1(f)(\hat{x}) \overline{\xi(\hat{x})} d\hat{x} \\
&= \int_{G/\varphi(L)} U_1(\xi^{-1}.f)(\hat{x}) d\hat{x} \\
&= \int_{G/\varphi(L)} U_2(\xi^{-1}.f)(\hat{x}) d\hat{x} \\
&= U_2(f)(\xi).
\end{align*}
\]
Hence, \( U_1 = U_2 \). □

Now, we have the following Riesz Representation Theorem which generalizes [5, Theorem 4.5.5] and characterizes all \( \varphi \)-factorable operators from \( L^2(G) \) to \( L^1(G/\varphi(L)) \).

**Theorem 2.4.** A bounded operator \( U : L^2(G) \to L^1(G/\varphi(L)) \) is \( \varphi \)-factorable if and only if there exists \( g \in L^2(G) \) such that \( U(f) = [f, g]_\varphi \) a.e., for all \( f \in L^2(G) \). Moreover, \( \|U\| = \|g\| \).

**Proof.** Let \( U : L^2(G) \to L^1(G/\varphi(L)) \) be a bounded \( \varphi \)-factorable operator. Define the linear functional \( \psi : L^2(G) \to \mathbb{C} \) by
\[
\psi(f) = \int_{G/\varphi(L)} U(f)(\hat{x}) d\hat{x}.
\]
By the standard Riesz Representation Theorem [9, Theorem 5.25], there exists \( g \in L^2(G) \) such that \( \psi(f) = \langle f, g \rangle_{L^2(G)} \) for all \( f \in L^2(G) \).

Thus, \( \int_{G/\varphi(L)} U(f) \hat{\psi} d\hat{x} = \psi(f) = \langle f, g \rangle_{L^2(G)} = \int_{G/\varphi(L)} [f, g]_{\varphi}(\hat{x}) d\hat{x} \).

By Lemma 2.3, \( U(f) = [f, g]_{\varphi} \) a.e., for all \( f \in L^2(G) \). Moreover, for any \( f \in L^2(G) \),

\[
\|U(f)\|_1 = \int_{G/\varphi(L)} \|f\|_{\varphi}(\hat{x}) \|g\|_{\varphi}(\hat{x}) d\hat{x} \\
\leq \int_{G/\varphi(L)} \|f\|_{\varphi}(\hat{x}) \|g\|_{\varphi}(\hat{x}) d\hat{x} \\
\leq (\int_{G/\varphi(L)} \|f\|_{\varphi}(\hat{x})^2 d\hat{x})^{1/2} (\int_{G/\varphi(L)} \|g\|_{\varphi}(\hat{x})^2 d\hat{x})^{1/2} \\
= \|f\|_2 \|g\|_2.
\]

So, \( \|U\| \leq \|g\|_2 \). Also, \( \|Ug\|_1 = \int_{G/\varphi(L)} \|g\|_{\varphi}(\hat{x}) d\hat{x} = \|g\|_2^2 \). Therefore, \( \|U\| = \|g\|_2 \).

The following theorem, which generalizes [5, Theorem 4.5.8], characterizes \( \varphi \)-factorable operators from \( L^2(G) \) to \( L^2(G/\varphi(L)) \).

**Theorem 2.5.** A bounded operator \( U : L^2(G) \to L^2(G/\varphi(L)) \) is \( \varphi \)-factorable if and only if there exists a \( \varphi \)-bounded \( g \in L^2(G) \) such that \( U(f) = [f, g]_{\varphi} \) a.e., for all \( f \in L^2(G) \). Moreover,

\[
\|U\|^2 = \text{ess sup}_{x \in G/\varphi(L)} \|g\|_{\varphi}^2(\hat{x}).
\]

**Proof.** Let \( U(f) = [f, g]_{\varphi} \) a.e., for some \( \varphi \)-bounded \( g \in L^2(G) \). Then, obviously \( U \) is \( \varphi \)-factorable and by the Cauchy-Shwartz Inequality, we have

\[
\|U(f)\|_{L^2(G/\varphi(L))}^2 = \int_{G/\varphi(L)} |U(f)(\hat{x})|^2 d\hat{x} \\
= \int_{G/\varphi(L)} |[f, g]_{\varphi}(\hat{x})|^2 d\hat{x} \\
\leq \int_{G/\varphi(L)} \|f\|_{\varphi}^2(\hat{x}) \|g\|_{\varphi}^2(\hat{x}) d\hat{x} \\
\leq \text{ess sup}_{x \in G/\varphi(L)} \|g\|_{\varphi}^2(\hat{x}) \|f\|_{L^2(G)}^2.
\]

Letting \( f = g \) above, we get \( \|U\|^2 = \text{ess sup}_{x \in G/\varphi(L)} \|g\|_{\varphi}^2(\hat{x}) \).

For the converse, let \( U \) be a \( \varphi \)-factorable operator from \( L^2(G) \) to \( L^2(G/\varphi(L)) \). Since \( G/\varphi(L) \) is compact, \( L^2(G/\varphi(L)) \subseteq L^1(G/\varphi(L)) \) and so by Theorem 2.4, there exists \( g \in L^2(G) \) such that \( U(f) = [f, g]_{\varphi} \) a.e., for all \( f \in L^2(G) \). But, also \( g \) is \( \varphi \)-bounded. To show this observe that \( |U(g)(\hat{x})| \leq \|U\| \|g\|_{\varphi}(\hat{x}) \) for a.e. \( \hat{x} \in G/\varphi(L) \). In fact, for every \( \varphi \)-periodic \( h \in L^\infty(G) \), we have
\[ \int_{G/\varphi(L)} |h(\dot{x})|^2 |U(g)(\dot{x})|^2 d\dot{x} = \int_{G/\varphi(L)} |U(hg)(\dot{x})|^2 d\dot{x} = \|U(hg)\|_{L^2(G/\varphi(L))}^2 \leq \|U\|^2 \int_{G} |hg(x)|^2 dx = \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \sum_{n \in \varphi(L)} |g(x\varphi(k^{-1}))|^2 d\dot{x} = \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \|g\|^2 \varphi(\dot{x}) d\dot{x}, \]

that is, \(|U(g)(\dot{x})| \leq \|U\| \|g\| \varphi(\dot{x})\) for a.e. \(\dot{x} \in G/\varphi(L)\). So, we get \(\|g\|^2 \varphi(\dot{x}) = |U(g)(\dot{x})| \leq \|U\| \|g\| \varphi(\dot{x})\) for a.e. \(\dot{x} \in G/\varphi(L)\). Hence, \(\|g\| \varphi(\dot{x}) \leq \|U\|\) a.e. That is, \(g\) is \(\varphi\)-bounded. \(\square\)

Next, we show that every bounded \(\varphi\)-factorable operator on \(L^2(G)\) is adjointable.

**Proposition 2.6.** Let \(U : L^2(G) \to L^2(G)\) be a bounded \(\varphi\)-factorable operator and \(U^*\) be its adjoint. Then, \(U^*\) is \(\varphi\)-factorable. Moreover,

\[
\|U(f), g\| \varphi = \|f, U^*(g)\| \varphi, \text{ a.e., for all } f, g \in L^2(G). \tag{2.3}
\]

**Proof.** Clearly \(U^*\) is \(\varphi\)-factorable. Indeed, for \(f, g \in L^2(G)\) and \(\varphi\)-periodic \(h \in L^\infty(G)\), we have

\[
\langle U^*(h), g \rangle_{L^2(G)} = \langle hf, U(g) \rangle_{L^2(G)} = \langle f, hU(g) \rangle_{L^2(G)} = \langle f, U(hg) \rangle_{L^2(G)} = \langle U^*(f), hg \rangle_{L^2(G)} = \langle hU^*(f), g \rangle_{L^2(G)}.
\]

Moreover, given \(f, g \in L^2(G)\), we have

\[
\int_{G/\varphi(L)} |U(f), g| \varphi(\dot{x}) d\dot{x} = \langle U(f), g \rangle_{L^2(G)} = \langle f, U^*(g) \rangle_{L^2(G)} = \int_{G/\varphi(L)} |f, U^*(g)| \varphi(\dot{x}) d\dot{x},
\]

which implies (2.3). \(\square\)

**Example 2.7.** Let \(G = \mathbb{R}^n\), for \(n \in \mathbb{N}\). Then, \(L = \mathbb{Z}^n\) is a uniform lattice in \(G\). Let \(A\) be an invertible \(n \times n\) real matrix. Define \(\varphi : G \to G\) by \(\varphi(x) = Ax\), for \(x \in \mathbb{R}^n\). Then, for \(g \in L^2(G)\), the operator \(U\) given by \(U(f) = [f, g] \varphi\), where \([f, g] \varphi(x) = \sum_{k \in \mathbb{Z}^n} fg(x - Ak)\), is a \(\varphi\)-factorable operator from \(L^2(G)\) to \(L^1(G/\varphi(L))\) (\(= L^1(T^n)\)).

**Example 2.8.** Fix a prime \(p\). Let \(\Delta_p\) denote the group of \(p\)-adic integers, as defined in [11, Definition 10.2]. Consider the LCA group \(G = \mathbb{R} \times \Delta_p\) and let \(L\) be the subgroup \(\{ (n, nu) \}_{n \in \mathbb{Z}}\) of \(\mathbb{R} \times \Delta_p\), where \(u = (1, 0, 0, ...)\). Then, \(L\) is a uniform lattice in \(\mathbb{R} \times \Delta_p\) (obviously,
L is discrete and by [11, Theorem 10.13], $\mathbb{R} \times \Delta_p/L$ is compact). Let $a := (1/p, 0, 0, ... \in \Delta_p$. Then, the mapping $\varphi : \mathbb{R} \times \Delta_p \to \mathbb{R} \times \Delta_p$, defined for $(x, v) \in \mathbb{R} \times \Delta_p$, by $\varphi(x, v) = (2x, av)$, is a topological isomorphism on $\mathbb{R} \times \Delta_p$. For $g \in L^2(\mathbb{R} \times \Delta_p)$, the operator $U$, given by $U(f)(x, v) = \sum_{k \in \mathbb{Z}} \hat{f}(x - 2k, v - ku)$, is a $\varphi$-factorable operator from $L^2(\mathbb{R} \times \Delta_p)$ to $L^1(\mathbb{R} \times \Delta_p/L)$.

The next section is devoted to an application of the $\varphi$-bracket product to the Weyl-Heisenberg systems.

3. Applications to Weyl-Heisenberg frames

In this section, we investigate the Weyl-Heisenberg frames with regard to the $\varphi$-bracket product. For general references on the Weyl-Heisenberg frames on $\mathbb{R}$, we refer to the survey articles [2, 3].

Suppose $L_1$ and $L_2$ are two uniform lattices in $G$, $g \in L^2(G)$ and $T_{\varphi(k)}g$ is the translation of $g$ by $\varphi(k)$. We call $(M_{\gamma}T_{\varphi(k)}g)_{\gamma \in \varphi(L_2), k \in L_1}$ a Weyl-Heisenberg system (Gabor’s system). If this system is a frame in $L^2(G)$, we call it a Weyl-Heisenberg frame. In this case, the frame operator associated with it is defined to be

$$S(f) = \sum_{\gamma \in \varphi(L_2)} \sum_{k \in L_1} <f, M_{\gamma}T_{\varphi(k)}g>^2 = \sum_{l \in L_2} \int_{G/\varphi(L_1)} [T_{\varphi(l^{-1})}f, f]_{\varphi,L_1}(\hat{x})g, T_{\varphi(l^{-1})}g]_{\varphi,L_1}(\hat{x}) d\hat{x},$$

where, $[f, g]_{\varphi,L_i}(\hat{x}) = \sum_{k \in L_i} \hat{f}(x\varphi(k^{-1}))$, $i = 1, 2$. 

Proposition 3.1. Let $L_1$ and $L_2$ be two uniform lattices in $G$. Let $g \in L^2(G)$ be $\varphi$-bounded. Then, for every $f \in L^2(G)$ which is bounded and compactly supported, we have

$$(3.1) \quad \sum_{k \in L_1} \sum_{\gamma \in \varphi(L_2)} |<f, M_{\gamma}T_{\varphi(k)}g>|^2 = \sum_{l \in L_2} \int_{G/\varphi(L_1)} [T_{\varphi(l^{-1})}f, f]_{\varphi,L_1}(\hat{x})g, T_{\varphi(l^{-1})}g]_{\varphi,L_1}(\hat{x}) d\hat{x},$$
\[
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\]

**Proof.** For \( k \in L_1 \), using the Plancherel Theorem, we have
\[
\sum_{\gamma \in \varphi(L_2)^*} | \langle f, M_{\gamma} T_{\varphi(k)} g \rangle |^2
\]
\[
= \sum_{\gamma \in \varphi(L_2)^*} \left| \int_G f(x) M_{\gamma} T_{\varphi(k)} g(x) \, dx \right|^2
\]
\[
= \sum_{\gamma \in \varphi(L_2)^*} \left| \int_{G/\varphi(L_2)} \sum_{\phi(l) \in \varphi(L_2)} f(x \varphi(l)) \overline{g}(x \varphi(lk^{-1})) \gamma(x) \, dx \right|^2
\]
\[
= \sum_{\gamma \in \varphi(L_2)^*} \left| \hat{F}_k(\gamma) \right|^2
\]
\[
= \| \hat{F}_k \|_{L_2(G/\varphi(L_2))}^2
\]
\[
= \| F_k \|_{L^2(G/\varphi(L_2))}^2
\]
where, \( F_k(x) = \sum_{\phi(l) \in \varphi(L_2)} f(x \varphi(l)) \overline{g}(x \varphi(lk^{-1})) \). So, we get
\[
\sum_{k \in L_1} \sum_{\gamma \in \varphi(L_2)^*} | \langle f, M_{\gamma} T_{\varphi(k)} g \rangle |^2
\]
\[
= \sum_{k \in L_1} \int_{G/\varphi(L_2)} \left| \sum_{\phi(l) \in \varphi(L_2)} f(x \varphi(l)) \overline{g}(x \varphi(lk^{-1})) \right|^2 \, dx
\]
\[
= \sum_{k \in L_1} \int_{G/\varphi(L_2)} \sum_{\phi(l) \in \varphi(L_2)} \overline{f}(x \varphi(l)) g(x \varphi(lk^{-1})) \sum_{\phi(m) \in \varphi(L_2)} f(x \varphi(m)) \overline{g}(x \varphi(mk^{-1})) \, dx
\]
\[
= \sum_{k \in L_1} \int_{G/\varphi(L_2)} \sum_{\phi(l) \in \varphi(L_2)} \overline{f}(x \varphi(l)) g(x \varphi(lk^{-1})) \sum_{\phi(nl) \in \varphi(L_2)} f(x \varphi(nl)) \overline{g}(x \varphi(nlk^{-1})) \, dx
\]
\[
= \sum_{n \in L_2} \int_{G/\varphi(L_1)} \int_{G/\varphi(L_1)} \int_{G/\varphi(L_1)} f(x) T_{\varphi(n^{-1})} f(x) \overline{g}(x) \, dx \, dx \, dx
\]
\[
= \sum_{n \in L_2} \int_{G/\varphi(L_1)} \int_{G/\varphi(L_1)} \int_{G/\varphi(L_1)} f(x) T_{\varphi(n^{-1})} f(x) \overline{g}(x) \, dx \, dx \, dx
\]
\[
= \sum_{n \in L_2} \left| \left| \sum_{k \in L_2} [g, T_{\varphi(k)} g]_{\varphi(L_1)}(\hat{x}) \right| \right|^2
\]
\[
= \sum_{n \in L_2} \left| \left| \sum_{k \in L_2} [g, T_{\varphi(k)} g]_{\varphi(L_1)}(\hat{x}) \right| \right|^2
\]
\[
= \sum_{n \in L_2} \left| \left| \sum_{k \in L_2} [g, T_{\varphi(k)} g]_{\varphi(L_1)}(\hat{x}) \right| \right|^2
\]

As a consequence of Proposition 3.1, we have the following corollary.

**Corollary 3.2.** Let \( L_1 \) and \( L_2 \) be two uniform lattices in \( G \). Let \( g \in L^2(G) \) such that
\[
B := \sup_{x \in \varphi(L_1)} \sum_{k_2 \in L_2} \left| [g, T_{\varphi(k_2)} g]_{\varphi(L_1)}(\hat{x}) \right| < \infty, \quad \text{and}
\]
\[
A := \inf_{\hat{x} \in \varphi(L_1)} \| [g, T_{\varphi(k_2)} g]_{\varphi(L_1)}(\hat{x}) \|^2_{\varphi(L_1)} - \sum_{1 \neq k_2 \in L_2} \| [g, T_{\varphi(k_2)} g]_{\varphi(L_1)}(\hat{x}) \|^2_{\varphi(L_1)} > 0.
\]
Then, \((M_{\gamma} T_{\varphi(k)} g)_{k \in L_1, \gamma \in \varphi(L_2)^*}\) is a Weyl-Heisenberg frame with bounds \( A \) and \( B \).
Proof. Put $H_n(x) = \sum_{k \in L_1} g(x \varphi(k^{-1})) \overline{\varphi}(x \varphi(nk^{-1}))$. Then,

$$
\sum_{0 \neq k_2 \in L_2} |T_{\varphi(k_2)} H_{k_2}(x)| = \sum_{0 \neq k_2 \in L_2} |H_{k_2}(x)|.
$$

Using Proposition 3.1, we have

$$
\sum_{k \in L_1} \sum_{\gamma \in \varphi(L_2)^{+}} |<f,M_{\gamma} T_{\varphi(k)} g>|^2 \\
\leq \sum_{0 \neq n \in L_2} \int_G |f(x)|\sqrt{|H_n(x)|} |T_{\varphi(n^{-1})} f(x)|\sqrt{|H_n(x)|} dx \\
\leq \sum_{0 \neq n \in L_2} \left( \int_G |f(x)|^2 |H_n(x)| dx \right)^{1/2} \left( \int_G |T_{\varphi(n^{-1})} f(x)|^2 |H_n(x)| dx \right)^{1/2} \\
\leq \left( \sum_{0 \neq n \in L_2} \int_G |f(x)|^2 |H_n(x)| dx \right)^{1/2} \left( \sum_{0 \neq n \in L_2} \int_G |T_{\varphi(n^{-1})} f(x)|^2 |H_n(x)| dx \right)^{1/2} \\
\leq \left( \int_G |f(x)|^2 \sum_{0 \neq n \in L_2} |H_n(x)| dx \right)^{1/2} \left( \sum_{0 \neq n \in L_2} |T_{\varphi(n)} H_n(x)| dx \right)^{1/2} \\
= \int_G |f(x)|^2 \sum_{0 \neq n \in L_2} |H_n(x)| dx.
$$

Thus, by (3.2) we, get the desired inequalities:

$$
A\|f\|_2^2 \leq \sum_{k \in L_1} \sum_{\gamma \in \varphi(L_2)^{+}} |<f,M_{\gamma} T_{\varphi(k)} g>|^2 \leq B\|f\|_2^2.
$$

\[\square\]

It is useful to note also that the Weyl-Heisenberg system has the following property.

**Proposition 3.3.** Let $L_1$ and $L_2$ be two uniform lattices in $G$. If $f, g \in L^2(G)$ and $g$ is $\varphi$-bounded, then

$$
\sum_{\gamma \in \varphi(L_2)^{+}} \sum_{k \in L_1} |<f,M_{\gamma} T_{\varphi(k)} g>|^2 = \sum_{k \in L_1} \|f, T_{\varphi(k)} g\|_{L^2(G/\varphi(L_2))}^2
$$

**Proof.** Using the Plancherel Theorem we have the following calculations which proves (3.3):

$$
\sum_{\gamma \in \varphi(L_2)^{+}} \sum_{k \in L_1} |<f,M_{\gamma} T_{\varphi(k)} g>|^2 \\
= \sum_{\gamma \in \varphi(L_2)^{+}} \sum_{k \in L_1} |\int_G f(x) T_{\varphi(k)\gamma}(x) \overline{\gamma}(x) dx|^2 \\
= \sum_{\gamma \in \varphi(L_2)^{+}} \sum_{k \in L_1} |\int_{G/\varphi(L_2)} f(x \varphi(l)) T_{\varphi(k)\gamma}(x \varphi(l)) \overline{\gamma}(x \varphi(l)) dx|^2 \\
= \sum_{\gamma \in \varphi(L_2)^{+}} \sum_{k \in L_1} |\int_{G/\varphi(L_2)} f \overline{T_{\varphi(k)\gamma}}(x \varphi(l)) \overline{\gamma}(x \varphi(l)) dx|^2 \\
= \sum_{\gamma \in \varphi(L_2)^{+}} \sum_{k \in L_1} \|f, T_{\varphi(k)\gamma}\|_{L^2(G/\varphi(L_2))}^2 \\
= \sum_{k \in L_1} \|f, T_{\varphi(k)\gamma}\|_{L^2(G/\varphi(L_2))}^2 \\
= \sum_{k \in L_1} \|f, T_{\varphi(k)\gamma}\|_{L^2(G/\varphi(L_2))}^2.
$$

\[\square\]
In the sequel, we will identify the frame operator of a Weyl-Heisenberg frame. For this, we need a couple of lemmas.

**Lemma 3.4.** Suppose \( g \in L^2(G) \) is \( \varphi \)-bounded and \( \varphi \)-periodic. Let \( L \) be a uniform lattice in \( G \). Then,

\[
\sum_{\gamma \in \varphi(L)} \langle f, M_\gamma g \rangle > M_\gamma g = [f, g]_\varphi g \quad \text{a.e.} \quad \text{for all} \quad f \in L^2(G),
\]

where the series converges in \( L^2(G) \). In particular, if \( \|g\|_\varphi = 1 \) a.e., and \( P \) is the orthogonal projection onto \( \text{span}\{M_\gamma g\}_{\gamma \in \varphi(L)} \), then \( Pf = [f, g]_\varphi g \) a.e..

**Proof.** Let \( f \in L^2(G) \). By (1.3), we have

\[
\sum_{\gamma \in \varphi(L)} \langle f, M_\gamma g \rangle = \int_{G/\varphi(L)} [\hat{f}, \hat{g}] \varphi(x) \varphi(\hat{x}) \, dx = [f, g]_\varphi(g), \quad \text{for a.e.} \quad \hat{x} \in G/\varphi(L).
\]

Hence, (3.4) holds, where the convergence of the series in \( L^2(G) \) follows from Proposition 1.1. In particular, if \( \|g\|_\varphi = 1 \), then \( \langle M_\gamma g, \gamma \rangle \) is an orthonormal basis for \( \text{span}\{M_\gamma g\}_{\gamma \in \varphi(L)} \). So, \( Pf = \sum_{\gamma \in \varphi(L)} \langle f, M_\gamma g \rangle = [f, g]_\varphi g \) a.e.. \( \Box \)

**Lemma 3.5.** Let \( L_1 \) and \( L_2 \) be two uniform lattices in \( G \), \( g \in L^\infty(G/\varphi(L_1)) \) and \( (M_\gamma T_{\varphi(k)} g)_{\gamma \in \varphi(L_1), k \in L_2} \) be a Bessel sequence with bound \( B \) in \( L^2(G) \). Then, \( \|g\|_{\varphi,L_2}^2 \leq B \).

**Proof.** Let \( f \in L^2(G) \) be \( \varphi \)-periodic and \( k \in L_2 \). Then, \( f \cdot T_{\varphi(k)} g \in L^2(G/\varphi(L_1)) \). Since \( \varphi(L_1) \) is an orthonormal basis for \( L^2(G/\varphi(L_1)) \), we have

\[
\sum_{\gamma \in \varphi(L_1)} |\langle f, T_{\varphi(k)} g \rangle| = \frac{\|f \cdot T_{\varphi(k)} g\|_{L^2(G/\varphi(L_1))}^2}{\|g\|_{L^2(G/\varphi(L_1))}^2} = \int_{G/\varphi(L_1)} |f(x)|^2 |g(x\varphi^{-1}(k))|^2 \, dx.
\]

So,

\[
\sum_{\gamma \in \varphi(L_1)} |\langle f, M_\gamma T_{\varphi(k)} g \rangle|^2 = \sum_{\gamma \in \varphi(L_1), k \in L_2} |\langle f, M_\gamma T_{\varphi(k)} g \rangle|^2 = \int_{G/\varphi(L_1)} |f(x)|^2 \sum_{k \in L_2} |g(x\varphi^{-1}(k))|^2 \, dx = \int_{G/\varphi(L_1)} |f(x)|^2 \|g\|_{\varphi,L_2}^2 \, dx.
\]

On the other hand,

\[
\sum_{\gamma \in \varphi(L_1), k \in L_2} \langle f, M_\gamma T_{\varphi(k)} g \rangle = \sum_{\gamma \in \varphi(L_1)} |\langle f, M_\gamma T_{\varphi(k)} g \rangle|^2 \leq B \|f\|_{L^2(G/\varphi(L_1))}^2.
\]

Hence, (3.5) and (3.6) imply that \( \|g\|_{\varphi,L_2}^2 \leq B \), a.e.. \( \Box \)

The frame operator of a Weyl-Heisenberg frame is given by the following theorem, which is a generalization of [5, Theorem 4.6.8].
Theorem 3.6. Let $L_1$ and $L_2$ be two uniform lattices in $G$ and $g \in L^\infty(G/\varphi(L_1))$. Suppose $(M_\gamma T_{\varphi(k)g})_{\gamma \in \varphi(L_1), k \in L_2}$ is a Weyl-Heisenberg frame with the frame operator $S$. Then, $S$ has the form

\begin{equation}
S(f) = \sum_{k \in L_2} \langle f, T_{\varphi(k)g} \rangle_{\varphi, L_1} T_{\varphi(k)g},
\end{equation}

where the series converges unconditionally in $L^2(G)$.

Proof. By Lemma 3.5, $T_{\varphi(k)g}$ is $\varphi$-bounded, and so we can use Lemma 3.4 to obtain:

\begin{align*}
S(f) &= \sum_{\gamma \in \varphi(L_1), k \in L_2} \langle f, M_\gamma T_{\varphi(k)g} \rangle_{L_2} M_\gamma T_{\varphi(k)g} \\
&= \sum_{k \in L_2} \langle f, T_{\varphi(k)g} \rangle_{\varphi, L_1} T_{\varphi(k)g}.
\end{align*}

□

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