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φ-FACTORABLE OPERATORS AND WEYL-HEISENBERG FRAMES ON LCA GROUPS

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ABSTRACT. We investigate φ-factorable operators and Weyl-Heisenberg frames with respect to a function-valued inner product, the so called φ-bracket product on $L^2(G)$, where G is a locally compact abelian group and φ is a topological isomorphism on G. We introduce φ-factorable operators on $L^2(G)$ and extend the Riesz Representation Theorems for these operators. Finally, as an application of the φ-bracket product, we show that several well known theorems for Weyl-Heisenberg frames in $L^2(\mathbb{R})$ remain valid in $L^2(G)$, and they are unified within of group theory, in connection with the φ-bracket product.

1. Introduction

In [13], we have defined the φ-bracket product as a function-valued inner product on $L^2(G)$, where G is a locally compact abelian (which will be abbreviated by "LCA") group and φ is a topological isomorphism on G. The φ-bracket product, as a new inner product on $L^2(G)$, is applicable to extend many ideas and constructions from the theory of shift invariant spaces, factorable operators and Weyl-Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a more general and different

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way. Whereas our work in [13] was devoted to basic properties of the φ-bracket product and φ-orthonormal bases, here we deal with characterizing φ-factorable operators on $L^2(G)$ and establishing Riesz Representation Theorems for such operators. We continue our investigation following the line of approach worked by Casazza and Lammers [5], but in a more general setting, using various tools in abstract harmonic analysis. In fact, our results generalize some of the results developed in [5] on $\mathbb{R}^n$, in which the authors want to be able to scale the lattice, and so they introduce a positive parameter a and express their results relative to the lattice $a\mathbb{Z}$. Here, like in [13], we use a topological isomorphism which introduces an appropriate scale factor in the setting of LCA groups. φ-Factorable operators are useful and shed light to define and investigate φ-frames and φ-Riesz bases, which are worked out in a forthcoming paper. After investigating φ-Factorable operators, we then, as an application of the φ-bracket product, study Weyl-Heisenberg frames on LCA groups in connection with the φ-bracket product. Our results generalize some of the results appearing in the literature on the Weyl-Heisenberg frames. Such a unified approach is useful, since it determines the basic features of the Weyl-Heisenberg frames, and includes most of the special cases.

Here, we give some of the basics regarding LCA groups. For a comprehensive account of LCA groups, we refer to [8, 11]. Suppose G is an LCA group with the Haar measure dx. A subgroup L of G is called a *uniform lattice* if it is discrete and co-compact (i.e., G/L is compact). Let φ be a topological isomorphism on G. If L is a uniform lattice in G, then so is $\varphi(L)$. Indeed, obviously $\varphi(L)$ is discrete. Also, by [11, Theorem 5.34], $G/\varphi(L)$ is topologically isomorphic to G/L and so it is compact. Here, we always assume that $G/\varphi(L)$ is normalized, i.e., $|G/\varphi(L)| = 1$. Denote by $\varphi(L)^\perp$, the annihilator of $\varphi(L)$ in $\hat{G}$, i.e., $\varphi(L)^\perp = \{\gamma \in \hat{G};\ \gamma(\varphi(L)) = \{1\}\}$, which is a uniform lattice in $\hat{G}$ (see [12-16]).

Let L be a uniform lattice in G. Choosing the counting measure on L, a relation between the Haar measures dx on G and $d\dot{x}$ on $G/\varphi(L)$ is given by the following special case of Weil's formula [8]:
For $f \in L^1(G)$, we have $\sum_{k\in L} f(x\varphi(k^{-1})) \in L^1(G/\varphi(L))$ and

$$\int_G f(x)dx = \int_{G/\varphi(L)} \sum_{\varphi(k^{-1})\in\varphi(L)} f(x\varphi(k^{-1}))d\dot{x}, \tag{1.1}$$

where, $\dot{x} = x\varphi(L)$.

Let $f, g \in L^2(G)$. The *φ-bracket product* of f, g is defined by

$$[f,g]_\varphi(\dot{x}) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1})), \tag{1.2}$$

for all $x \in G$. We define the φ-norm of f as $\|f\|_\varphi(\dot{x}) = ([f,f]_\varphi(\dot{x}))^{1/2}$. In the sequel, we recall some basic properties of the φ-bracket product, for the proofs of which and more details the reader is referred to [13]. Let $f,\ g \in L^2(G)$. Then, $|[f,g]_\varphi| \leq \|f\|_\varphi \|g\|_\varphi$ (the Cauchy-Schwartz Inequality). Also, (1.1) implies $\int_{G/\varphi(L)} [f,g]_\varphi(\dot{x})d\dot{x} =< f, g >_{L^2(G)}$. For $\gamma \in \hat{G}$, denote by M_γ, the modulation operator on $L^2(G)$, i.e., $M_\gamma f(x) = \gamma(x)f(x)$, for all $f \in L^2(G)$. Then, for $f, g \in L^2(G)$ and $\gamma \in \varphi(L)^\perp$, we have the following relation between the φ-bracket product and the usual inner product in $L^2(G)$:

$$\widehat{[f,g]_\varphi}(\gamma) =< f, M_\gamma g >_{L^2(G)} . \tag{1.3}$$

We say $g \in L^2(G)$ is *φ-bounded* if there exists $M > 0$ so that $\|g\|_\varphi \leq M$ *a.e.*. For $f, g \in L^2(G)$, the function $[f,g]_\varphi g$ need not generally be in $L^2(G)$. But, we have the following result.

Proposition 1.1. *If $f, g, h \in L^2(G)$ and g, h are φ-bounded, then $[f,g]_\varphi h \in L^2(G)$.*

A sequence $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is called *φ-orthonormal* if $[g_n, g_m]_\varphi = 0$, for all $n \neq m \in \mathbb{N}$ and $\|g_n\|_\varphi = 1$, for all $n \in \mathbb{N}$. Let $f \in L^2(G)$ and $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence in $L^2(G)$. An extension of [5, Theorem 4.13] from $\mathbb{R}$ to the setting of an LCA group gives Bessel's Inequality for φ-bracket products as follows:

$$\sum_{n\in\mathbb{N}} |[f,g_n]_\varphi(\dot{x})|^2 \leq \|f\|^2_\varphi(\dot{x}), \quad \textit{for a.e. } \dot{x} \in G/\varphi(L). \tag{1.4}$$

A φ-orthonormal sequence $(g_n)_{n\in\mathbb{N}}$ is called a *φ-orthonormal basis* if $[f, g_n]_\varphi = 0$ a.e., for all $n \in \mathbb{N}$, implies $f = 0$ a.e.. Let $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence. It is not difficult to mimic the standard proofs for a usual orthonormal sequence in a Hilbert space to obtain equivalent conditions for $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ to be a φ-orthonormal basis (see also [13]).

Proposition 1.2. *If $(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal sequence in $L^2(G)$, then the following are equivalent.*

(1) *$(g_n)_{n\in\mathbb{N}}$ is a maximal φ-orthonormal sequence, i.e., $(g_n)_{n\in\mathbb{N}}$ is not contained in any other φ-orthonormal set.*

(2) $(g_n)_{n\in\mathbb{N}}$ *is a* φ*-orthonormal basis.*
(3) *For each* $f \in L^2(G)$, $f = \sum_{n\in\mathbb{N}}[f, g_n]_\varphi g_n$ *a.e..*
(4) $\|f\|_\varphi^2 = \sum_{n\in\mathbb{N}} |[f, g_n]_\varphi|^2$ *a.e., for all* $f \in L^2(G)$ *(the Parseval Identity).*
(5) $\{M_\gamma g_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ *is an orthonormal basis for* $L^2(G)$.

Thanks to Zorn's Lemma and Proposition 1.2, $L^2(G)$ admits a φ-orthonormal basis.

The rest of this paper is organized as follows. In Section 2, we introduce a φ-factorable operator on $L^2(G)$, where G is an LCA group and establish the Riesz Representation Theorems for these operators.

Over the last ten years, there have been a lot of research on frame theory in general, and the Weyl-Heisenberg frame theory, in particular [2-4, 7, 18], most of which are on the Euclidean space. Our main goal in Section 3 is to represent the Weyl-Heisenberg frame identity and the frame operator of a Weyl-Heisenberg frame in terms of the φ-bracket product on an LCA group.

2. φ-factorable operators

Throughout this paper, we always assume that G is a second countable LCA group, φ is a topological isomorphism on G and the notation are as in Section 1.

A function $h \in L^\infty(G)$ is said to be φ-*periodic* if $h(x\varphi(k)) = h(x)$, for every $k \in L$, $x \in G$.

Definition 2.1. *We say an operator* $U : L^2(G) \to L^p(E)$, $1 \le p \le \infty$, *is* φ*-factorable if* $U(hf) = hU(f)$, *for all* $f \in L^2(G)$ *and all* φ*-periodic* $h \in L^\infty(G)$, *where* E *is a subgroup of* G *or* $G/\varphi(L)$.

A bounded operator U is φ-factorable if and only if it commutes with modulations. More precisely, we have the following result.

Lemma 2.2. *Let* U *be a bounded operator from* $L^2(G)$ *to* $L^2(E)$, *where* E *is a subgroup of* G *or* $G/\varphi(L)$. U *is* φ*-factorable if and only if*

$$U(M_\gamma g) = M_\gamma U(g), \text{ for all } g \in L^2(G),\ \gamma \in \varphi(L)^\perp. \tag{2.1}$$

Proof. If U is φ-factorable and $\gamma \in \varphi(L)^\perp (\subseteq \hat{G} \subseteq L^\infty(G))$, then since γ is φ-periodic, (2.1) obviously holds. Conversely, assume (2.1). Then, U is φ-factorable using the facts that $\varphi(L)^\perp (= \widehat{G/\varphi(L)})$ is an orthonormal basis for $L^2(G/\varphi(L))$ and $L^\infty(G/\varphi(L)) \subseteq L^2(G/\varphi(L))$. Note that there

is a one-to-one correspondence between $L^\infty(G/\varphi(L))$ and the set of all φ-periodic $h \in L^\infty(G)$. □

Our main goal in this section is to characterize φ-factorable operators $U : L^2(G) \to L^p(G/\varphi(L))$, for $p = 1$ and $p = 2$.

Clearly, the operator U, defined by $U(f) = [f, g]_\varphi$, for $f \in L^2(G)$, is φ-factorable. We will also show that every φ-factorable operator $U : L^2(G) \to L^1(G/\varphi(L))$ is of this form. First, we establish a lemma in which we show that two φ-factorable operators are equal on $L^2(G)$ if and only if their integrals over $G/\varphi(L)$ are the same.

Lemma 2.3. *Let $U_1, U_2 : L^2(G) \to L^1(G/\varphi(L))$ be two φ-factorable operators. Then, $U_1 = U_2$ if and only if*

$$\int_{G/\varphi(L)} U_1(f)(\dot{x})d\dot{x} = \int_{G/\varphi(L)} U_2(f)(\dot{x})d\dot{x},$$

for every $f \in L^2(G)$.

Proof. The necessity is obvious. For the converse, by [8, Theorem 4.33], it is enough to show that $\widehat{U_1(f)} = \widehat{U_2(f)}$, for all $f \in L^2(G)$. Let $\xi \in \varphi(L)^\perp$ and $f \in L^2(G)$. Since ξ as a function in $L^\infty(G)$ is φ-periodic, we obtain:

$$\begin{aligned}
\widehat{U_1(f)}(\xi) &= \textstyle\int_{G/\varphi(L)} U_1(f)(\dot{x})\bar{\xi}(\dot{x})d\dot{x} \\
&= \textstyle\int_{G/\varphi(L)} U_1(\xi^{-1}.f)(\dot{x})d\dot{x} \\
&= \textstyle\int_{G/\varphi(L)} U_2(\xi^{-1}.f)(\dot{x})d\dot{x} \\
&= \widehat{U_2(f)}(\xi).
\end{aligned}$$

Hence, $U_1 = U_2$. □

Now, we have the following Riesz Representation Theorem which generalizes [5, Theorem 4.5.5] and characterizes all φ-factorable operators from $L^2(G)$ to $L^1(G/\varphi(L))$.

Theorem 2.4. *A bounded operator $U : L^2(G) \to L^1(G/\varphi(L))$ is φ-factorable if and only if there exists $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e., for all $f \in L^2(G)$. Moreover, $\|U\| = \|g\|$.*

Proof. Let $U : L^2(G) \to L^1(G/\varphi(L))$ be a bounded φ-factorable operator. Define the linear functional $\psi : L^2(G) \to \mathbb{C}$ by

$$\psi(f) = \int_{G/\varphi(L)} U(f)(\dot{x})d\dot{x}.$$

By the standard Riesz Representation Theorem [9, Theorem 5.25], there exists $g \in L^2(G)$ such that $\psi(f) =< f, g >_{L^2(G)}$, for all $f \in L^2(G)$. Thus, $\int_{G/\varphi(L)} U(f)(\dot{x})d\dot{x} = \psi(f) =< f, g >_{L^2(G)} = \int_{G/\varphi(L)}[f, g]_\varphi(\dot{x})d\dot{x}$. By Lemma 2.3, $U(f) = [f, g]_\varphi$ a.e., for all $f \in L^2(G)$. Moreover, for any $f \in L^2(G)$,

$$\begin{aligned} \|U(f)\|_1 &= \int_{G/\varphi(L)} |[f, g]_\varphi(\dot{x})|d\dot{x} \\ &\leq \int_{G/\varphi(L)} \|f\|_\varphi(\dot{x})\|g\|_\varphi(\dot{x})d\dot{x} \\ &\leq (\int_{G/\varphi(L)} \|f\|^2_\varphi(\dot{x})d\dot{x})^{1/2}(\int_{G/\varphi(L)} \|g\|^2_\varphi(\dot{x})d\dot{x})^{1/2} \\ &= \|f\|_2\|g\|_2. \end{aligned}$$

So, $\|U\| \leq \|g\|_2$. Also, $\|Ug\|_1 = \int_{G/\varphi(L)} |[g, g]_\varphi(\dot{x})|d\dot{x} = \|g\|^2_2$. Therefore, $\|U\| = \|g\|_2$. □

The following theorem, which generalizes [5, Theorem 4.5.8], characterizes φ-factorable operators from $L^2(G)$ to $L^2(G/\varphi(L))$.

Theorem 2.5. *A bounded operator $U : L^2(G) \to L^2(G/\varphi(L))$ is φ-factorable if and only if there exists a φ-bounded $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e., for all $f \in L^2(G)$. Moreover,*

$$\|U\|^2 = ess\ sup_{\dot{x}\in G/\varphi(L)}\|g\|^2_\varphi(\dot{x}).$$

Proof. Let $U(f) = [f, g]_\varphi$ *a.e.*, for some φ-bounded $g \in L^2(G)$. Then, obviously U is φ-factorable and by the Cauchy-Shwartz Inequality, we have

$$\begin{aligned} \|U(f)\|^2_{L^2(G/\varphi(L))} &= \int_{G/\varphi(L)} |U(f)(\dot{x})|^2d\dot{x} \\ &= \int_{G/\varphi(L)} |[f, g]_\varphi(\dot{x})|^2d\dot{x} \\ &\leq \int_{G/\varphi(L)} \|f\|^2_\varphi(\dot{x})\|g\|^2_\varphi(\dot{x})d\dot{x} \\ &\leq ess\ sup_{\dot{x}\in G/\varphi(L)}\|g\|^2_\varphi(\dot{x})\|f\|^2_{L^2(G)}. \end{aligned} \tag{2.2}$$

Letting $f = g$ above, we get $\|U\| = ess\ sup_{\dot{x}\in G/\varphi(L)}\|g\|_\varphi(\dot{x})$.

For the converse, let U be a φ-factorable operator from $L^2(G)$ to $L^2(G/\varphi(L))$. Since $G/\varphi(L)$ is compact, $L^2(G/\varphi(L)) \subseteq L^1(G/\varphi(L))$ and so by Theorem 2.4, there exists $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ *a.e.*, for all $f \in L^2(G)$. But, also g is φ-bounded. To show this observe that $|U(g)(\dot{x})| \leq \|U\|\|g\|_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. In fact, for every φ-periodic $h \in L^\infty(G)$, we have

$$\begin{aligned}
\int_{G/\varphi(L)} |h(\dot{x})|^2 |U(g)(\dot{x})|^2 d\dot{x} &= \int_{G/\varphi(L)} |U(hg)(\dot{x})|^2 d\dot{x} \\
&= \|U(hg)\|^2_{L^2(G/\varphi(L))} \\
&\leq \|U\|^2 \int_G |hg(x)|^2 dx \\
&= \|U\|^2 \int_{G/\varphi(L)} \sum_{\varphi(k)\in\varphi(L)} |hg(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \sum_{\varphi(k)\in\varphi(L)} |g(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \|g\|^2_\varphi(\dot{x}) d\dot{x},
\end{aligned}$$

that is, $|U(g)(\dot{x})| \leq \|U\| \|g\|_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. So, we get $\|g\|^2_\varphi(\dot{x}) = |U(g)(\dot{x})| \leq \|U\| \|g\|_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. Hence, $\|g\|_\varphi(\dot{x}) \leq \|U\|$ *a.e.* That is, g is φ-bounded. □

Next, we show that every bounded φ-factorable operator on $L^2(G)$ is adjointable.

Proposition 2.6. *Let $U : L^2(G) \to L^2(G)$ be a bounded φ-factorable operator and U^* be its adjoint. Then, U^* is φ-factorable. Moreover,*

$$[U(f), g]_\varphi = [f, U^*(g)]_\varphi, \quad \text{a.e., for all } f, g \in L^2(G). \tag{2.3}$$

Proof. Clearly U^* is φ-factorable. Indeed, for $f, g \in L^2(G)$ and φ-periodic $h \in L^\infty(G)$, we have

$$\begin{aligned}
< U^*(hf), g >_{L^2(G)} &= < hf, U(g) >_{L^2(G)} \\
&= < f, \bar{h}U(g) >_{L^2(G)} \\
&= < f, U(\bar{h}g) >_{L^2(G)} \\
&= < U^*(f), \bar{h}g >_{L^2(G)} \\
&= < hU^*(f), g >_{L^2(G)} .
\end{aligned}$$

Moreover, given $f, g \in L^2(G)$, we have

$$\begin{aligned}
\int_{G/\varphi(L)} [U(f), g]_\varphi(\dot{x}) d\dot{x} &= < U(f), g >_{L^2(G)} \\
&= < f, U^*(g) >_{L^2(G)} \\
&= \int_{G/\varphi(L)} [f, U^*(g)]_\varphi(\dot{x}) d\dot{x},
\end{aligned}$$

which implies (2.3). □

Example 2.7. *Let $G = \mathbb{R}^n$, for $n \in \mathbb{N}$. Then, $L = \mathbb{Z}^n$ is a uniform lattice in G. Let A be an invertible $n \times n$ real matrix. Define $\varphi : G \to G$ by $\varphi(x) = Ax$, for $x \in \mathbb{R}^n$. Then, for $g \in L^2(G)$, the operator U given by $U(f) = [f, g]_\varphi$, where $[f, g]_\varphi(x) = \sum_{k\in\mathbb{Z}^n} f\bar{g}(x - Ak)$, is a φ-factorable operator from $L^2(G)$ to $L^1(G/\varphi(L))$ $(= L^1(\mathbb{T}^n))$.*

Example 2.8. *Fix a prime p. Let Δ_p denote the group of p-adic integers, as defined in* [11, Definition 10.2]. *Consider the LCA group $G = \mathbb{R} \times \Delta_p$ and let L be the subgroup $\{(n, n\boldsymbol{u})\}_{n\in\mathbb{Z}}$ of $\mathbb{R} \times \Delta_p$, where $\boldsymbol{u} = (1, 0, 0, ...)$. Then, L is a uniform lattice in $\mathbb{R} \times \Delta_p$ (obviously,*

L is discrete and by [11, Theorem 10.13]*, $\mathbb{R}\times\Delta_p/L$ is compact). Let $\boldsymbol{a}:=(1/p,0,0,...)\in\Delta_p$. Then, the mapping $\varphi:\mathbb{R}\times\Delta_p\to\mathbb{R}\times\Delta_p$, defined for $(x,\boldsymbol{v})\in\mathbb{R}\times\Delta_p$, by $\varphi(x,\boldsymbol{v})=(2x,\boldsymbol{av})$, is a topological isomorphism on $\mathbb{R}\times\Delta_p$. For $g\in L^2(\mathbb{R}\times\Delta_p)$, the operator U, given by $U(f)(x,\boldsymbol{v})=\sum_{k\in\mathbb{Z}}f\bar{g}(x-2k,\boldsymbol{v}-k\boldsymbol{au})$, is a φ-factorable operator from $L^2(\mathbb{R}\times\Delta_p)$ to $L^1(\mathbb{R}\times\Delta_p/L)$.*

The next section is devoted to an application of the φ-bracket product to the Weyl-Heisenberg systems.

3. **Applications to Weyl-Heisenberg frames**

In this section, we investigate the Weyl-Heisenberg frames with regard to the φ-bracket product. For general references on the Weyl-Heisenberg frames on $\mathbb{R}$, we refer to the survey articles [2, 3].

Suppose L_1 and L_2 are two uniform lattices in G, $g\in L^2(G)$ and $T_{\varphi(k)}g$ is the translation of g by $\varphi(k)$. We call $(M_\gamma T_{\varphi(k)}g)_{\gamma\in\varphi(L_2)^\perp,k\in L_1}$, *a Weyl-Heisenberg system (Gabor's system).* If this system is a frame in $L^2(G)$, we call it a *Weyl-Heisenberg frame.* In this case, the frame operator associated with it is defined to be

$$S(f)=\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1}<f,M_\gamma T_{\varphi(k)}g>M_\gamma T_{\varphi(k)}g.$$

We would like to consider the Weyl-Heisenberg frame Identity and the frame operator of a Weyl-Heisenberg frame in terms of the φ-bracket product. The following proposition is an extension of the Weyl-Heisenberg frame Identity ([5, Theorem 4.6.2]) with regards to the φ-bracket product; see also [6].

Proposition 3.1. *Let L_1 and L_2 be two uniform lattices in G. Let $g\in L^2(G)$ be φ-bounded. Then, for every $f\in L^2(G)$ which is bounded and compactly supported, we have*

$$\sum_{k\in L_1}\sum_{\gamma\in\varphi(L_2)^\perp}|<f,M_\gamma T_{\varphi(k)}g>|^2= \tag{3.1}$$

$$\sum_{l\in L_2}\int_{G/\varphi(L_1)}[T_{\varphi(l^{-1})}f,f]_{\varphi,L_1}(\dot{x})[g,T_{\varphi(l^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x},$$

where, $[f,g]_{\varphi,L_i}(\dot{x})=\sum_{k\in L_i}f\overline{g}(x\varphi(k^{-1}))$, $i=1,2$.

Proof. For $k \in L_1$, using the Plancherel Theorem, we have

$\sum_{\gamma\in\varphi(L_2)^\perp} | < f, M_\gamma T_{\varphi(k)} g > |^2$

$= \sum_{\gamma\in\varphi(L_2)^\perp} | \int_G f(x)\overline{M_\gamma T_{\varphi(k)} g(x)}dx|^2$

$= \sum_{\gamma\in\varphi(L_2)^\perp} | \int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))\overline{\gamma}(x)d\dot{x}|^2$

$= \sum_{\gamma\in\varphi(L_2)^\perp} |\hat{F}_k(\gamma)|^2$

$= \|\hat{F}_k\|^2_{L^2(\widehat{G/\varphi(L_2)})}$

$= \|F_k\|^2_{L^2(G/\varphi(L_2))}$,

where, $F_k(x) = \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))$. So, we get

$\sum_{k\in L_1} \sum_{\gamma\in\varphi(L_2)^\perp} | < f, M_\gamma T_{\varphi(k)} g > |^2$

$= \sum_{k\in L_1} \int_{G/\varphi(L_2)} | \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))|^2 d\dot{x}$

$= \sum_{k\in L_1} \int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} \overline{f}(x\varphi(l))g(x\varphi(lk^{-1}))$

$\sum_{\varphi(m)\in\varphi(L_2)} f(x\varphi(m))\overline{g}(x\varphi(mk^{-1}))d\dot{x} \qquad (put\ m = nl)$

$= \sum_{k\in L_1} \int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} \overline{f}(x\varphi(l))g(x\varphi(lk^{-1}))$

$\sum_{\varphi(n)\in\varphi(L_2)} f(x\varphi(nl))\overline{g}(x\varphi(nlk^{-1}))d\dot{x}$

$= \sum_{k\in L_1} \int_G \bar{f}(x)g(x\varphi(k^{-1})) \sum_{\varphi(n)\in\varphi(L_2)} f(x\varphi(n))\overline{g}(x\varphi(nk^{-1}))dx$

$= \sum_{n\in L_2} \int_G \bar{f}(x)f(x\varphi(n)) \sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1}))dx$

$= \sum_{n\in L_2} \int_G \bar{f}(x)f(x\varphi(n))[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(x)dx$

$= \sum_{n\in L_2} \int_{G/\varphi(L_1)} \sum_{\varphi(l)\in\varphi(L_1)} \overline{f}(x\varphi(l))T_{\varphi(n^{-1})}f(x\varphi(l))[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x}$

$= \sum_{n\in L_2} \int_{G/\varphi(L_1)} [T_{\varphi(n^{-1})}f, f]_{\varphi,L_1}(\dot{x})[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x}.$

□

As a consequence of Proposition 3.1, we have the following corollary.

Corollary 3.2. *Let L_1 and L_2 be two uniform lattices in G. Let $g \in L^2(G)$ such that*

(3.2)

$B := sup_{\dot{x}\in G/\varphi(L_1)} \sum_{k_2\in L_2} |[g, T_{\varphi(k_2)}g]_{\varphi,L_1}(\dot{x})| < \infty$, *and*

$A := inf_{\dot{x}\in G/\varphi(L_1)} [\|g\|^2_{\varphi,L_1}(\dot{x}) - \sum_{1_G\neq k_2\in L_2} |[g, T_{\varphi(k_2)}g]_{\varphi,L_1}(\dot{x})|] > 0.$

Then, $(M_\gamma T_{\varphi(k)} g)_{k\in L_1,\gamma\in\varphi(L_2)^\perp}$ is a Weyl-Heisenberg frame with bounds A and B.

Proof. Put $H_n(x) = \sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1}))$. Then,

$$\sum_{0\neq k_2\in L_2} |T_{\varphi(k_2)} H_{k_2}(x)| = \sum_{0\neq k_2\in L_2} |H_{k_2}(x)|.$$

Using Proposition 3.1, we have

$\sum_{k\in L_1}\sum_{\gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)} g>|^2$
$|\sum_{0\neq n\in L_2}\int_G \bar{f}(x) f(x\varphi(n))\sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1}))dx|$
$\leq \sum_{0\neq n\in L_2}\int_G |f(x)|\sqrt{|H_n(x)|}\ |T_{\varphi(n^{-1})}f(x)|\sqrt{|H_n(x)|}dx$
$\leq \sum_{0\neq n\in L_2}(\int_G |f(x)|^2|H_n(x)|dx)^{1/2}(\int_G |T_{\varphi(n^{-1})}f(x)|^2|H_n(x)|dx)^{1/2}$
$\leq (\sum_{0\neq n\in L_2}\int_G |f(x)|^2|H_n(x)|dx)^{1/2}(\sum_{0\neq n\in L_2}\int_G |T_{\varphi(n^{-1})}f(x)|^2|H_n(x)|dx)^{1/2}$
$\leq (\int_G |f(x)|^2\sum_{0\neq n\in L_2}|H_n(x)|dx)^{1/2}(\int_G |f(x)|^2\sum_{0\neq n\in L_2}|T_{\varphi(n)}H_n(x)|dx)^{1/2}$
$= \int_G |f(x)|^2\sum_{0\neq n\in L_2}|H_n(x)|dx.$

Thus, by (3.2) we, get the desired inequalities:

$$A\|f\|_2^2 \leq \sum_{k\in L_1\gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)} g>|^2 \leq B\|f\|_2^2.$$

□

It is useful to note also that the Weyl-Heisenberg system has the following property.

Proposition 3.3. *Let L_1 and L_2 be two uniform lattices in G. If $f, g \in L^2(G)$ and g is φ-bounded, then*

(3.3)
$$\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |< f, M_\gamma T_{\varphi(k)} g>|^2 = \sum_{k\in L_1} \|[f, T_{\varphi(k)}g]_{\varphi,L_2}\|^2_{L^2(G/\varphi(L_2))}.$$

Proof. Using the Plancherel Theorem we have the following calculations which proves (3.3):

$\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |< f, M_\gamma T_{\varphi(k)} g>|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\int_G f(x)\overline{T_{\varphi(k)}g}(x)\overline{\gamma}(x)dx|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\int_{G/\varphi(L_2)}\sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{T_{\varphi(k)}g}(x\varphi(l))\overline{\gamma}(x)d\dot{x}|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\int_{G/\varphi(L_2)}[f, T_{\varphi(k)}g]_{\varphi,L_2}(\dot{x})\overline{\gamma}(\dot{x})d\dot{x}|^2$
$= \sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1} |\widehat{[f, T_{\varphi(k)}g]}_{\varphi,L_2}(\gamma)|^2$
$= \sum_{k\in L_1} \|\widehat{[f, T_{\varphi(k)}g]}_{\varphi,L_2}\|^2_{L^2(\widehat{G/\varphi(L_2)})}$
$= \sum_{k\in L_1} \|[f, T_{\varphi(k)}g]_{\varphi,L_2}\|^2_{L^2(G/\varphi(L_2))}.$

□

In the sequel, we will identify the frame operator of a Weyl-Heisenberg frame. For this, we need a couple of lemmas.

Lemma 3.4. *Suppose $g \in L^2(G)$ is φ-bounded and φ-periodic. Let L be a uniform lattice in G. Then,*

$$\sum_{\gamma\in\varphi(L)^\perp} < f, M_\gamma g > M_\gamma g = [f,g]_\varphi g \quad a.e. \quad \text{for all } f \in L^2(G), \tag{3.4}$$

where the series converges in $L^2(G)$. In particular, if $\|g\|_\varphi = 1$ a.e., and P is the orthogonal projection onto $\overline{span}\{M_\gamma g\}_{\gamma\in\varphi(L)^\perp}$, then $Pf = [f,g]_\varphi g$ a.e..

Proof. Let $f \in L^2(G)$. By (1.3), we have $\sum_{\gamma\in\varphi(L)^\perp} < f, M_\gamma g > \gamma(\dot{x}) = \sum_{\gamma\in\varphi(L)^\perp} \widehat{[f,g]_\varphi}(\gamma)\gamma(\dot{x}) = [f,g]_\varphi(\dot{x})$, for a.e. $\dot{x} \in G/\varphi(L)$. Hence, (3.4) holds, where the convergence of the series in $L^2(G)$ follows from Proposition 1.1. In particular, if $\|g\|_\varphi = 1$, then $(M_\gamma g)_{\gamma\in\varphi(L)^\perp}$ is an orthonormal basis for $\overline{span}\{M_\gamma g\}_{\gamma\in\varphi(L)^\perp}$. So, $Pf = \sum_{\gamma\in\varphi(L)^\perp} < f, M_\gamma g > M_\gamma g = [f,g]_\varphi g$ a.e.. □

Lemma 3.5. *Let L_1 and L_2 be two uniform lattices in G, $g \in L^\infty(G/\varphi(L_1))$ and $(M_\gamma T_{\varphi(k)} g)_{\gamma\in\varphi(L_1)^\perp, k\in L_2}$ be a Bessel sequence with bound B in $L^2(G)$. Then, $\|g\|^2_{\varphi,L_2} \le B$.*

Proof. Let $f \in L^2(G)$ be φ-periodic and $k \in L_2$. Then, $f \cdot T_{\varphi(k)}\overline{g} \in L^2(G/\varphi(L_1))$. Since $\varphi(L_1)^\perp$ is an orthonormal basis for $L^2(G/\varphi(L_1))$, we have

$$\begin{aligned} \sum_{\gamma\in\varphi(L_1)^\perp} | < f\cdot T_{\varphi(k)}\overline{g}, M_\gamma > |^2 &= \|f\cdot T_{\varphi(k)}\overline{g}\|^2_{L^2(G/\varphi(L_1))} \\ &= \int_{G/\varphi(L_1)} |f(x)|^2 |g(x\varphi(k^{-1}))|^2 d\dot{x}. \end{aligned}$$

So,

$$\begin{aligned} \sum_{\gamma\in\varphi(L_1)^\perp, k\in L_2} | < f, M_\gamma T_{\varphi(k)} g > |^2 &= \sum_{\gamma\in\varphi(L_1)^\perp, k\in L_2} | < f\cdot T_{\varphi(k)}\overline{g}, M_\gamma > |^2 \\ &= \int_{G/\varphi(L_1)} |f(x)|^2 \sum_{k\in L_2} |g(x\varphi(k^{-1}))|^2 d\dot{x} \\ &= \int_{G/\varphi(L_1)} |f(x)|^2 \|g\|^2_{\varphi,L_2}(x) d\dot{x}. \end{aligned} \tag{3.5}$$

On the other hand,

$$\sum_{\gamma\in\varphi(L_1)^\perp, k\in L_2} | < f, M_\gamma T_{\varphi(k)} g > |^2 \le B\|f\|^2_{L^2(G/\varphi(L_1))}. \tag{3.6}$$

Hence, (3.5) and (3.6) imply that $\|g\|^2_{\varphi,L_2} \le B$, a.e.. □

The frame operator of a Weyl-Heisenberg frame is given by the following theorem, which is a generalization of [5, Theorem 4.6.8].

Theorem 3.6. *Let L_1 and L_2 be two uniform lattices in G and $g \in L^\infty(G/\varphi(L_1))$. Suppose $(M_\gamma T_{\varphi(k)} g)_{\gamma\in\varphi(L_1),k\in L_2}$ is a Weyl-Heisenberg frame with the frame operator S. Then, S has the form*

$$S(f) = \sum_{k\in L_2} [f, T_{\varphi(k)}g]_{\varphi,L_1} T_{\varphi(k)} g, \tag{3.7}$$

where the series converges unconditionally in $L^2(G)$.

Proof. By Lemma 3.5, $T_{\varphi(k)}g$ is φ-bounded, and so we can use Lemma 3.4 to obtain:

$$\begin{aligned} S(f) &= \sum_{\gamma\in\varphi(L_1)^\perp,k\in L_2} < f, M_\gamma T_{\varphi(k)} g > M_\gamma T_{\varphi(k)} g \\ &= \sum_{k\in L_2} [f, T_{\varphi(k)}g]_{\varphi,L_1} T_{\varphi(k)} g. \end{aligned}$$

□

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References

1. C. de Boor, R.A. DeVore and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$ *J. Funct. Anal.* **119** (1994) 37-78.
2. P.G. Casazza, The art of frame theory, *Taiwanese J. Math.* **4** (2000) 129-201.
3. P.G. Casazza, Modern tools for Weyl-Heisenberg (Gabor) frame theory, *Adv. Imaging Electron Phys.* **115** (2001) 1-127.
4. P.G. Casazza and O. Christensen, Weyl-Heisenberg frames for subspaces of $L^2(\mathbb{R})$, *Proc. Amer. Math. Soc.* **129** (2001) 145-154.
5. P.G. Casazza and M.C. Lammers, Bracket Products for Weyl-Heisenberg Frames, *Advances in Gabor Analysis*, 71-98, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, 2003.
6. O. Christensen, An introduction to frames and Riesz bases, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Inc., Boston, MA, 2003.
7. H.G. Feichtinger and T. Strohmer, (editors), Gabor Analysis and Algorithms. Theory and Applications, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, MA, 1998.
8. G. B. Folland, A Course in Abstract Harmonic Analysis, Studies in Advanced Mathematic, CRC Press, Boca Raton, FL, 1995.
9. G. B. Folland, *Real Analysis. Modern Techniques and their Applications*, Pure and Applied Mathematics (New York), A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1984.

10. E. Hernández and G. Weiss, A First Course on Wavelets, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.
11. E. Hewitt, K.A. Ross, Abstract Harmonic Analysis, Vol. I: Structure of topological groups, Integration theory, group representations, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
12. R. A. Kamyabi Gol and R. Raisi Tousi, A range function approach to shift-invariant spaces on locally compact abelian groups, *Int. J. Wavelets. Multiresolut. Inf. Process.* **8** (2010) 49-59.
13. R. A. Kamyabi Gol and R. Raisi Tousi, Bracket products on locally compact abelian groups, *J. Sci. Islam. Repub. Iran* **19** (2008) 153-157.
14. R. A. Kamyabi Gol and R. Raisi Tousi, The structure of shift invariant spaces on a locally compact abelian group, *J. Math. Anal. Appl.* **340** (2008) 219-225.
15. E. Kaniuth and G. Kutyniok, Zeros of the Zak transform on locally compact abelian groups, *Proc. Amer. Math. Soc.* **126** (1998) 3561-3569.
16. G. Kutyniok and D. Labate, The theory of reproducing systems on locally compact abelian groups, *Colloq. Math.* **106** (2006) 197-220.
17. A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$, *Canad. J. Math.* **47** (1995) 1051-1094.
18. A. Safapour and R. A. Kamyabi Gol, A necessary condition for Weyl-Heisenberg frames, *Bull. Iranian Math. Soc.* **30** (2004) 67-79.
19. R. Young, An Introduction to Nonharmonic Fourier Series, Pure and Applied Mathematics, 93, Academic Press, Inc., New York, London, 1980.

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