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LOWER BOUNDS OF COPSON TYPE FOR THE TRANSPOSE OF MATRICES ON WEIGHTED SEQUENCE SPACES

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ABSTRACT. Let $A = (a_{n,k})_{n,k\geq 0}$ be a non-negative matrix. Denote by $L_{w,p,q}(A)$, the supremum of those L, satisfying the following inequality:

$$\left(\sum_{n=0}^{\infty} w_n \left(\sum_{k=0}^{\infty} a_{n,k}x_k\right)^q\right)^{\frac{1}{q}} \geq L \left(\sum_{k=0}^{\infty} w_k x_k^p\right)^{\frac{1}{p}},$$

where, $x \geq 0$ and $x \in l_p(w)$ and also $w = (w_n)$ is a decreasing, non-negative sequence of real numbers. If $p = q$, then we use $L_{w,p}(A)$ inested of $L_{w,p,p}(A)$. Here, we focus on the evaluation of $L_{w,p}(A^t)$ for a lower triangular matrix A, where, $0 < p < 1$. In particular, we apply our results to summability matrices, weighted mean matrices, Nörlund matrices. Our results also generalize some results in Chen and Wang [C.-P. Chen and K.-Z. Wang, *J. Math. Anal. Appl.* **341** (2008) 1284-1294.], Foroutannia and Lashkaripour [D. Foroutannia and R. Lashkaripour, *Lobachevskii J. Math.* **27** (2007) 15-29.], and Lashkaripour and Foroutannia [R. Lashkaripour and D. Foroutannia, *J. Sci. Islam. Repub. Iran* **18** (2007) 49-56.].

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1. **Introduction**

Let $p \in \mathbb{R}\backslash\{0\}$ and let $l_p(w)$ denote the space of all real sequences $x = \{x_k\}_{k=0}^{\infty}$ such that $\|x\|_{w,p} := (\sum\limits_{k=0}^{\infty} w_k x_k^p)^{1/p} < \infty$, where, $w = (w_n)_{n=0}^{\infty}$ is a decreasing, non-negative sequence of real numbers with $\sum_{n=0}^{\infty} \frac{w_n}{n+1} = \infty$ with $w_0 = 1$.

We write $x \geq 0$ if $x_k \geq 0$, for all k. We also write $x \uparrow$ for the case that $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$. The symbol $x \downarrow$ is defined in a similar way. For $p, q \in \mathbb{R}\backslash\{0\}$, the lower bound involved here is the number $L_{w,p,q}(A)$, which is defined as the supremum of those L obeying the following inequality:

$$\left(\sum_{n=0}^{\infty} w_n \left(\sum_{k=0}^{\infty} a_{n,k} x_k\right)^q\right)^{\frac{1}{q}} \geq L \left(\sum_{k=0}^{\infty} w_k x_k^p\right)^{\frac{1}{p}}, \qquad (x \geq 0, x \in l_p(w)),$$

where, $A \geq 0$, that is, $A = (a_{n,k})_{n,k\geq 0}$ is a non-negative matrix. We have

$$L_{w,p,q}(A) \leq \|A\|_{w,p,q}.$$

In [3], the author obtained $L_{w,p}(C(1)^t) = p, (0 < p < 1)$, where, $(.)^t$ denotes the transpose of $(.)$ and $C(1) = (a_{n,k})_{n,k\geq 0}$ is the Cesaro matrix defined by

$$a_{n,k} = \begin{cases} \frac{1}{n+1} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This is an analogue of Copson's result [2, Eq. (1.1)] (see also [4], Theorem 344) for weighted sequence space $l_p(w)$ and has been generalized by Foroutannia [3]. He extended it in [3, Theorem 2.7.17 and Theorem 2.7.19] to those summability matrices A, whose rows are increasing or decreasing. Also, he gave upper bounds or lower bounds for $L_{w,p}(A)$, for such A. For the case of Hausdorff matrices, the related result with $0 < p < 1$ has been established in [3, Theorem 4.3.2], giving a Hardy-type formula for $L_{w,p}(H_\mu^t)$.

Obviously, the lower bound problems of Copson type for the weighted mean matrices, $(A_W^{WM}) = (a_{n,k})_{n,k\geq 0}$, and the Nörlund matrices, $(A_W^{NM}) = (b_{n,k})_{n,k\geq 0}$, or more generally for the summability matrices on weighted sequence spaces are still less satisfactory (cf. [1, problem 4.20]), where the weighted mean matrices and the Nörlund matrices are defined as:

$$a_{n,k} = \begin{cases} \frac{w'_n}{W'_n} & 0 \le k \le n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b_{n,k} = \begin{cases} \frac{w'_{n-k}}{W'_n} & 0 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$$

Here, $W'_n = \sum_{k=0}^{n} w'_k$, and $w' = (w'_n)$ is a non negative sequence with $w'_0 > 0$.

Here we are concerned with the problem of finding $L_{w,p}(A^t)$ and $L_{w,p^*}(A)$ (see Theorem 2.3), where, $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$ and A is a non-negative lower triangular matrix. Our result gives a lower estimate for these two values in terms of the constant M, defined by:

$$a_{n,k} \le M a_{n,j}, \ (0 \le k \le j \le n). \tag{1.1}$$

Here, $M \ge 1$. We shall assume that M is the smallest value appearing in (1.1). If (1.1) is not satisfied, then we set $M = \infty$. As a consequence, we prove that Theorem 2.3 generalizes some works of Lashkaripour and Foroutannia ([3], pp.53-54). Also, we obtain lower estimate and upper estimate for the weighted mean matrix and the Nörlund matrix in some cases.

2. **Main Result**

The purpose of this section is to establish general lower bounds for $L_{w,p}(A^t)$ and $L_{w,p^*}(A)$, where, $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$ and A is a non-negative lower triangular matrix. First, we generalize Lemma 5.2 of [2] to the weighted sequence space $l_p(w)$.

Lemma 2.1. *Suppose that* $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$ *and* $N \in \mathbb{N}$. *Let* $C_N^1 = (c_{n,k}(N))_{n,k \ge 0}$ *be the matrix with entries*

$$c_{n,k}(N) = \begin{cases} \frac{1}{n+N} & 0 \le k < n+N \\ 0 & k \ge n+N. \end{cases}$$

Then,

$$L_{w,p}\left((C_N^1)^t\right) = L_{w,p^*}\left(C_N^1\right) = p.$$

Moreover, for $r \in \mathbb{N}$ *and* $r > max\{N-2, \frac{1}{p}\}$, *there exists a sequence* $\{x_N^m\}_{m=0}^{\infty}$ *such that* $x_N^m = \left(0, ..., 0, x_{r-N+1}^m, ...\right) \ge 0$, $x_{r-N+1}^m \ge x_{r-N+2}^m \ge$

..., $\| x_N^m \|_{w,p} = 1$, *for all* m, *and also*

$$\lim_{m\to\infty} \|x_N^m\|_{w,1} = 0 \ , \lim_{m\to\infty} \left\| \left(C_N^1\right)^t x_N^m \right\|_{w,p} = p.$$

Proof. Applying Proposition 2.5 of [6], it suffices to prove the case $L_{w,p}\left((C_N^1)^t\right) = p$. For $x \geq 0$, we have

$$\left\| \left(C_N^1\right)^t x \right\|_{w,p} = \|C(1)^t x'\|_{w,p},$$

where, $x' = \{x'_k\}_{k=0}^{\infty}$ is defined by

$$x'_k = \begin{cases} 0 & 0 \leq k < N-1, \\ x_{k-N+1} & k \geq N-1. \end{cases} \tag{2.1}$$

This implies that $L_{w,p}\left((C_N^1)^t\right) \geq L_{w,p}\left(C(1)^t\right) = p$. For the rest of the proof, it suffices to prove the existence of $\{x_N^m\}_{m=0}^{\infty}$, for $r \in \mathbb{N}$, with $r > max\{N-2, \frac{1}{p}\}$. Choose a sequence, say $\{\rho_m\}_{m=0}^{\infty}$, such that $\rho_0 \leq r$ and $\rho_m \downarrow \frac{1}{p}$. Define $x_N^m = \{x_k^m\}_{k=0}^{\infty}$ by

$$x_k^m = \begin{cases} 0 & 0 \leq k < r-N+1, \\ (\phi(\rho_m))^{-1} \ \binom{k+N-1-\rho_m}{k+N-1-r} \Big/ \binom{k+N-1}{r} & k \geq r-N+1, \end{cases}$$

where,

$$\phi(t) = \left(\sum_{k=r-N+1}^{\infty} w_k \left\{ \binom{k+N-1-t}{k+N-1-r} \Big/ \binom{k+N-1}{r} \right\}^p \right)^{\frac{1}{p}}.$$

We have $x_N^m = \left(0, ..., 0, x_{r-N+1}^m, ...\right) \geq 0$, $x_k^m \downarrow$, for all $k \geq r-N+1$, and $\|x_N^m\|_{w,p} = 1$, for all m. Applying ([7, Vol.I], p.77, Eq. (1.15)), we have

$$\binom{k+N-1-\rho_m}{k+N-1-r} \Big/ \binom{k+N-1}{r} \sim \frac{\Gamma(r+1)}{\Gamma(r-\rho_m+1)} (k+N-1-r)^{-\rho_m}, \quad as\ k \to \infty.$$

Since $\rho_m \downarrow, \frac{1}{p}$ and $\frac{1}{p} > 1$, it follows from the monotone convergence theorem that $\lim_{m\to\infty} \phi(\rho_m) = \infty$. Moreover, there exists a constant C such that

$$\limsup_{m\to\infty} \sum_{k=r-N+1}^{\infty} w_k \left\{ \binom{k+N-1-\rho_m}{k+N-1-r} \Big/ \binom{k+N-1}{r} \right\} \leq C \sum_{n=1}^{\infty} w_n n^{-1/p} < \infty.$$

So, $\lim_{m\to\infty} \|x_m^N\|_{w,1} = 0$. We know that $C(1)$ is the same as the Hausdorff matrix H_μ with $d\mu(\theta) = d\theta$. By modifying the argument given in ([3], pp. 80-81), we can prove that

$$\|(C_N^1)^t x_N^m\|_{w,p} = \|C(1)^t (x_N^m)'\|_{w,p} \to p, \quad as \quad m \to \infty,$$

where, $(x_N^m)'$ is obtained from x_N^m by means of (2-1). This completes the proof of the lemma □

In the following lemma, we extend Lemma 2.1 from matrix C_N^1 to general matrix C_N^l, with $l \in \mathbb{N}$.

Lemma 2.2. *Suppose that* $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$ *and* $l, N \in \mathbb{N}$. *Let* $C_N^l = (c_{n,k}^l)_{n,k\geq 0}$ *be the matrix with*

$$c_{n,k}^l = \begin{cases} \frac{1}{n+N} & 0 \leq k < n + N - l + 1 \\ 0 & k \geq n + N - l + 1. \end{cases}$$

Then,

$$L_{w,p}\left((C_N^l)^t\right) = L_{w,p^*}\left(C_N^l\right) \leq p.$$

Moreover, the following two assertions hold:

(i) *For* $l \leq N$ *and* $x \geq 0$ *with* $x \downarrow$, *we have*

$$\left\|(C_N^l)^t x\right\|_{w,p}^p \leq \left\|C(1)^t x'\right\|_{w,p}^p \leq \left\|(C_N^l)^t x\right\|_{w,p}^p + \frac{l^p(l+1)}{N^p} \|x\|_{w,p}^p, \tag{2.2}$$

where, $x' = \{x'_k\}_{k=0}^\infty$ *is defined by* (2.1).

(ii) *There exists a sequence* $\{x_N\}_{N=0}^\infty$ *such that* $x_N \geq 0$, $x_N \downarrow$, $\|x_N\|_{w,p} = 1$, *and also*

$$\lim_{N\to\infty} \|x_N\|_{w,1} = 0, \lim_{N\to\infty} \left\|(C_N^l)^t x_N\right\|_{w,p} = p.$$

Proof. For $x \geq 0$, $\|(C_N^l)^t x\|_{w,p}^p \leq \|\left(C_N^1\right) x\|_{w,p}^p$. Applying Lemma 2.1, we have

$$L_{w,p}((C_N^l)^t) \leq L_{w,p^*}((C_N^1)^t) = p.$$

The left side in (2.2) follows from the observation,

$$\|(C_N^l)^t x\|_{w,p}^p \leq \|(C_N^1)^t) x\|_{w,p}^p = \|C(1)^t x'\|_{w,p}^p \qquad (x \geq 0).$$

Hence, to prove (i) it is suffices to show the right side of (2-2). Assume that $l \leq N$, $x \geq 0$ and $x \downarrow$. Applying definition of x'_k, we get

$$
\begin{aligned}
\|C(1)^t x'\|_{w,p}^p &= \sum_{k=0}^{N-1} w_k \left(\sum_{n=N-1}^{\infty} \tfrac{x'_n}{n+1}\right)^p + \sum_{k=N}^{\infty} w_k \left(\sum_{n=k}^{\infty} \tfrac{x'_n}{n+1}\right)^p \\
&\leq \sum_{k=0}^{N} w_k \left(\sum_{n=0}^{\infty} \tfrac{x_n}{n+N}\right)^p + \sum_{k=N+1}^{\infty} w_k \left(\sum_{n=k-N+1}^{\infty} \tfrac{x_n}{n+N}\right)^p \qquad (2.3)\\
&= \Sigma_1 + \Sigma_2.
\end{aligned}
$$

We know that $a^p + b^p \geq (a+b)^p$, for all $a, b \geq 0$. Hence,

$$
\begin{aligned}
\Sigma_1 \leq \sum_{k=0}^{N-l} w_k \left(\sum_{n=0}^{\infty} c^l_{n,k} x_n\right)^p + \sum_{k=N-l+1}^{N} w_k \Bigg\{ &\left(\sum_{n=0}^{k-N+l-1} \tfrac{x_n}{n+N}\right)^p \\
&+ \left(\sum_{n=k-N+l}^{\infty} c^l_{n,k} x_n\right)^p \Bigg\}. \qquad (2.4)
\end{aligned}
$$

The monotonicity of x_n implies that $\sum\limits_{n=0}^{k-N+l-1} \frac{x_n}{n+N} \leq \left({}^{l}\!/\!_{N}\right) x_0$, for all $N - l < k \leq N$. Inserting this into (2.4), yields:

$$
\Sigma_1 \leq \sum_{k=0}^{N-l} w_k \left(\sum_{n=0}^{\infty} c^l_{n,k} x_n\right)^p + \frac{l^{p+1} x_0^p}{N^p} + \sum_{k=N-l+1}^{N} w_k \left(\sum_{n=0}^{\infty} c^l_{n,k} x_n\right)^p. \qquad (2.5)
$$

In the same way as in (2.4), one can show

$$
\begin{aligned}
\Sigma_2 &\leq \sum_{k=N+1}^{\infty} w_k \left\{ \left(\sum_{n=k-N+1}^{k-N+l-1} \tfrac{x_n}{n+N}\right)^p + \left(\sum_{n=k-N+l}^{\infty} c^l_{n,k} x_n\right)^p \right\} \qquad (2.6)\\
&\leq \tfrac{l^p}{N^p} \sum_{k=N+1}^{\infty} w_k x^p_{k-N+1} + \sum_{k=N+1}^{\infty} w_k \left(\sum_{n=0}^{\infty} c^l_{n,k} x_n\right)^p.
\end{aligned}
$$

Putting (2.3), (2.5) and (2.6) together, yields:

$$
\left\|C(1)^t x'\right\|_{w,p}^p \leq \left\|\left(C^l_N\right)^t x\right\|_{w,p}^p + \frac{l^p(l+1)}{N^p} \|x\|_{w,p}^p.
$$

This completes the proof of (i).

(ii). Let $x_0 = x_1 = ... = x_{\left[\frac{1}{p}\right]+1} = e_0$, where, $e_0 = (1, 0, 0, ...)$. For each $N > \frac{1}{p} + 1$, it follows from the case $r = N - 1$ of Lemma 2.1

that there exist x_N with the properties: $x_N \geq 0$, $x_N \downarrow$, $\|x_N\|_{w,p} = 1$, $\|x_N\|_{w,1} \leq \frac{1}{N}$ and

$$p - \frac{1}{N} \leq \left\| \left(C_N^l\right)^t x_N \right\|_{w,p} \leq p + \frac{1}{N}.$$

Obviously,

$$\lim_{N\to\infty} \|x_N\|_{w,1} = 0, \lim_{N\to\infty} \left\| \left(C_N^l\right)^t x_N \right\|_{w,p} = p.$$

Applying (2.2), we get

$$\left\| \left(C_N^l\right)^t x_N \right\|_{w,p}^p \leq \|C(1)^t x'_N\|_{w,p}^p = \left\| \left(C_N^1\right)^t x_N \right\|_{w,p}^p$$

$$\leq \left\| \left(C_N^l\right)^t x_N \right\|_{w,p}^p + \tfrac{l^p(l+1)}{N^p}. \quad (N \geq l)$$

Making $N \to \infty$, it follows that

$$\lim_{N\to\infty} \left\| \left(C_N^l\right)^t x_N \right\|_{w,p} = \lim_{N\to\infty} \left\| \left(C_N^1\right)^t x_N \right\|_{w,p}^p = p.$$

This completes the proof. $\square$

Note that, in general, $L_{w,p}((C_N^l)^t) \neq p$. In fact, we have $L_{w,p}((C_N^N)^t) \leq \frac{1}{N} < p$, if $N > \frac{1}{p}$. One can see this by considering the definition of C_N^N.

Theorem 2.3. *Let $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$ and $A = (a_{n,k})_{n,k\geq 0}$ be a lower triangular matrix with $A \geq 0$. Then,*

$$pM^{p-1}(\inf_{n\geq 0} \sum_{k=0}^{n} a_{n,k}) \leq L_{w,p}(A^t). \tag{2.7}$$

Also, the same inequality holds, if $L_{w,p}(A^t)$ is replaced by $L_{w,p^*}(A)$. Here, M is defined by (1.1).

Proof. Applying Proposition 4.3.6 of [3], we have $L_{w,p}\left(A^t\right) = L_{w,p^*}\left(A\right)$, and so it suffices to prove (2.7). Let $x \geq 0$ with $\|x\|_{w,p} = 1$. Since

$p-1<0$, from Lemma 2.7.18 of [3] with (1.1) and Fubini's theorem, it follows that:

$$\begin{aligned}
\|A^t x\|_{w,p}^p &= \sum_{k=0}^{\infty} w_k \left(\sum_{n=k}^{\infty} a_{n,k}x_n\right)^p \\
&\geq p\left\{\sum_{k=0}^{\infty} w_k \sum_{j=k}^{\infty} a_{j,k}x_j \left(\sum_{n=j}^{\infty} a_{n,k}x_n\right)^{p-1}\right\} \\
&\geq pM^{p-1} \sum_{k=0}^{\infty} w_k \sum_{j=k}^{\infty} a_{j,k}x_j \left(\sum_{n=j}^{\infty} a_{n,j}x_n\right)^{p-1} \qquad (2.8)\\
&\geq pM^{p-1} \sum_{j=0}^{\infty} w_j x_j \left(\sum_{n=j}^{\infty} a_{n,j}x_n\right)^{p-1} \left(\sum_{k=0}^{j} a_{j,k}\right) \\
&\geq pM^{p-1} \left(\inf_{j\geq 0} \sum_{k=0}^{j} a_{j,k}\right) \left\{\sum_{j=0}^{\infty} w_j x_j \left(\sum_{n=j}^{\infty} a_{n,j}x_n\right)^{p-1}\right\}.
\end{aligned}$$

Applying Hölder's inequality, we deduce that

$$\begin{aligned}
\sum_{j=0}^{\infty} w_j x_j \left(\sum_{n=j}^{\infty} a_{n,j}x_n\right)^{p-1} &= \sum_{j=0}^{\infty} w_j^{\frac{1}{p}} x_j \left(w_j^{\frac{1}{p^*(p-1)}} \sum_{n=j}^{\infty} a_{n,j}x_n\right)^{p-1} \\
&\geq \left(\sum_{j=0}^{\infty} w_j x_j^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{\infty} \left(w_k^{\frac{1}{p}} \sum_{j=k}^{\infty} a_{j,k}x_j\right)^p\right)^{\frac{1}{p^*}} \\
&= \|x\|_{w,p} \left\|A^t x\right\|_{w,p}^{p-1}.
\end{aligned}$$

Inserting this estimate into the corresponding term in (2.8), gives

$$\left\|A^t x\right\|_{w,p} \geq pM^{p-1} \left(\inf_{j\geq 0} \sum_{k=0}^{j} a_{j,k}\right) \|x\|_{w,p}.$$

This leads us to the lower estimate in (2.7). □

Theorem 2.3 has some applications. For example, consider the weighted mean matrix, say $(A_{W\ l}^{WM})$, associated with the sequence $W' = (w'_n)_{n=0}^{\infty}$, where, $l = 0, 1, 2, \cdots$, $w'_0 = w'_1 = \cdots = w'_l = 1$ and $w'_n = \frac{1}{2}$, for $n > l$. Applying inequality (2.7) for $M = 2$, we have

$$L_{w,p}((A_{W\ l}^{WM})^t) \geq p2^{p-1}.$$

Next, consider the Nörlund matrix (A_W^{NM}), where, $w' = (w'_n)_{n=0}^{\infty}$ is a non-negative sequence with $w'_0 > 0$ and $W'_n = \sum_{k=0}^{n} w'_k$. If $w'_n \downarrow$, then

$M = 1$. Applying (2.7), we deduce that

$$L_{w,p}((A_W^{NW})^t) \geq p.$$

In general, for the summability matrix A (see [1]), with increasing rows $M = 1$, we observe that (2.7) has the following form:

$$p \leq L_{w,p}(A^t) = L_{w,p^*}(A). \tag{2.9}$$

Inequality (2.9) is an analogue of ([4], Theorem 4.2), obtained by a different way.

Theorem 2.4. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_0 > 0$ *and* $w'_n \geq 0$, *for all* $n \geq 1$ *and also* $\lim_{n\to\infty} W'_n = \infty$. *Then, the following assertions are true:*

(i) $L_{w,p}\left(\left(A_W^{NM}\right)^t\right) = L_{w,p^*}\left(A_W^{NM}\right) \leq p\left(\lim_{l\to\infty} K(l)\right)$,

where, $\qquad K(l) := \sup_{n\geq 0,\ N\geq l,\ l\leq k\leq n+N} \frac{(n+N+1)w'_k}{W'_{n+N}}.$

(ii) $L_{w,p}\left(\left(A_W^{WM}\right)^t\right) = L_{w,p^*}\left(A_W^{WM}\right) \leq p\left(\lim_{l\to\infty} k(l)\right)$,

where, $\qquad k(l) := \sup_{n\geq 0,\ l\leq k\leq n} \frac{(n+1)w'_k}{W'_n}.$

Obviously, $k(l) \leq K(l)$, for all $l \geq 1$. Since $k(l) \downarrow$ and $K(l) \downarrow$, then the limits in (i) and (ii) can be replaced by $\inf_{l\in\mathbb{N}}$. We have

$$K(l) \leq \left(\sup_{n\geq l} w'_n\right) / \left(\inf_{n\geq l} \frac{W'_n}{n+1}\right).$$

Proof. Let x_N and x'_N be defined as in Lemma 2.2. Since $a^p + b^p \geq (a+b)^p$, for all $a, b \geq 0$, we deduce that

$$\left\|\left(A_W^{NM}\right)^t x'_N\right\|_{w,p}^p \leq \left\|\left(A_1^l\right)^t x'_N\right\|_{w,p}^p + \left\|\left(A_2^l\right)^t x'_N\right\|_{w,p}^p \quad (N \geq 0), \tag{2.10}$$

where, $A_2^l = A_W^{NM} - A_1^l$ and $A_1^l = (a_{n,k})_{n,k\geq 0}$ is the matrix obtained from A_W^{NW} by replacing the (n,k)th entry of A_W^{NW} with 0, for all n, k, with $n - l < k \leq n$. Consider $N \geq l + 1$. Obviously, $a_{n+N-1,k} \leq K(l)/n + N$, for $0 \leq k < n + N - l$, and $a_{n+N-1,k} = 0$, for $k \geq n + N - l$. This implies that

$$\left\|\left(A_1^l\right)^t x'_N\right\|_{w,p}^p \leq K(l)^p \left\|\left(C_N^l\right)^t x_N\right\|_{w,p}^p. \tag{2.11}$$

On the other hand, it follows from the definition of A_2^l that

$$\left\|\left(A_2^l\right)^t x'_N\right\|_{w,p}^p \leq l \left(\frac{\max\{w'_0, w'_1, ..., w'_{l-1}\}}{W'_{N-1}}\right)^p \|x_N\|_{w,p}^p. \tag{2.12}$$

Putting (2.10) and (2.11) together with (2.12), yields:

$$\begin{aligned}\left\|\left(A_W^{NM}\right)^t x'_N\right\|_{w,p}^p \leq & \ (K(l))^p \left\|\left(C_N^l\right)^t x_N\right\|_{w,p}^p \\ & + \ l \left(\tfrac{\max\{w'_0, w'_1, ..., w'_{l-1}\}}{W'_{N-1}}\right)^p \|x_N\|_{w,p}^p.\end{aligned}$$

We have $\|x_N\|_{w,p} = 1$ and $W'_N \to \infty$, as $N \to \infty$, and applying Lemma 2.2(ii), we get $L_{w,p}((A_W^{NM})^t) \leq pK(l)$. Hence,

$$L_{w,p}((A_W^{NM})^t) \leq p(\inf_{l\in\mathbb{N}} K(l)) = p \lim_{l\to\infty} K(l).$$

This proves (i).

Now, consider (ii). Let $\{x_N^m\}_{m=0}^{\infty}$ be the corresponding sequence given in Lemma 2.2. Similar to A_W^{NM}, write $A_W^{WM} = A_1^l + A_2^l$, where, A_1^l is the matrix obtained from A_W^{WM} by replacing the (n,k)th entry of A_W^{WM} with 0, for all $n \geq 0$ and $0 \leq k < l$. As seen above, one can easily derive:

$$\begin{aligned}\left\|\left(A_W^{WM}\right)^t (x_N^m)'\right\|_{w,p}^p \leq & \ \left\|(A_1^l)^t (x_N^m)'\right\|_{w,p}^p + \left\|(A_2^l)^t (x_N^m)'\right\|_{w,p}^p \\ \leq & \ (k(l))^p \left\|\left(C_N^1\right) x_N^m\right\|_{w,p}^p \\ & + \ l(\tfrac{max\{w'_0, w'_1, ..., w'_{l-1}\}}{W'_{N-1}})^p \|x_N^m\|_{w,p}^p,\end{aligned}$$

which gives $L_{w,p}(A_W^{WM})^t) \leq pk(l)$, for all $l \in \mathbb{N}$. Therefore,

$$L_{w,p}((A_W^{WM})^t) \leq p(\inf_{l\in\mathbb{N}} k(l)) = p \lim_{l\to\infty} k(l).$$

This completes the proof of the (ii). □

Applying (2.9) for the summability matrix A, with increasing rows, we have

$$p \leq L_{w,p}(A^t) = L_{w,p^*}(A).$$

Also, applying Theorem 2.4(i), we deduce the following corollaries.

Corollary 2.5. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_n \downarrow \alpha$ *and* $\alpha > 0$. *Then,*

$$L_{w,p}((A_W^{NM})^t) = L_{w,p^*}((A_W^{NM})) = p.$$

Remark 2.6. *The case* $\alpha = 0$ *in Corollary 2.5 is false. In general, a counterexample is the Nörlund matrix* (A_W^{NM}), *where,* $w'_0 = 1$, $w'_n \downarrow 0$, $\inf\limits_{k\geq 0} \frac{w'_0}{w'_0+\ldots+w'_k} > p$. *For this matrix,* $\alpha = 0$, *but*

$$\begin{aligned} L_{w,p}((A_W^{NM})^t) &\geq \inf_{\|x\|_{w,p}=1,\, x\geq 0} \left(\sum_{n=0}^{\infty} w_n (a_{n,n} x_n)^p\right)^{1/p} \\ &\geq \inf_{k\geq 0} \frac{w'_0}{w'_0+\ldots+w'_k} \\ &> p. \end{aligned}$$

In ([5], Theorem 4.1), the upper bound of $L_{w,p}(A^t)$ is established for those summability matrices A, whose rows are decreasing, where, such matrices, $L_{w,p}(A^t) \leq p$. For this of type matrix, applying (2.7), we have

$$pM^{p-1} \leq L_{w,p}(A^t) \leq p.$$

Also, we have the following results for particular cases of such matrices.

Corollary 2.7. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_n \downarrow \alpha$ *and* $\alpha \geq 0$. *Then,*

$$p(\frac{w'_0}{\alpha})^{p-1} \leq L_{w,p}\left(\left(A_W^{WM}\right)^t\right) = L_{w,p^*}\left(A_W^{WM}\right) \leq p.$$

Corollary 2.8. *Let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $w'_n \uparrow \alpha$ *and* $w'_0 > 0$. *Then,*

$$p(\frac{\alpha}{w'_0})^{p-1} \leq L_{w,p}\left(\left(A_W^{NM}\right)^t\right) = L_{w,p^*}\left(A_W^{NM}\right) \leq p.$$

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