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MULTIPLICATIVE BIJECTIVE MAPS ON STANDARD OPERATOR ALGEBRAS

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Abstract. We provide an elementary proof of the fact that every bijective multiplicative map $\pi : \mathcal{A} \to \mathcal{B}$ of standard operator algebras on real normed spaces X and Y, is respectively of the form $\pi(A) = TAT^{-1}$ *and* $A \in \mathcal{A}$, where $T : X \to Y$ is a bounded invertible linear operator.

Šemrl proved the following theorem for the infinite dimensional real and complex Banach spaces by using projective geometry [6] and automatic continuity [5]. Here, we give an elementary proof of the theorem for any real normed space of dimension at least two. We note that in our proof, we do not use the completeness of X and Y.

Let X and Y be normed spaces. Denote by $B(X)$, the algebra of all bounded linear operators on X. A subalgebra of $B(X)$ which contains $F(X)$ (the ideal of all finite rank operators in $B(X)$) is called a standard operator algebra on X.

Theorem. *Let X and Y be real normed spaces, at least two-dimensional, and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Assume that $\pi : \mathcal{A} \to \mathcal{B}$ is a bijective map satisfying*

$$\pi(AB) = \pi(A)\pi(B), \quad \textit{for every } A, B \in \mathcal{A}.$$

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Then, $\pi(A) = TAT^{-1}, A \in \mathcal{A}$, where $T : X \to Y$ is a bounded invertible linear operator. In particular, π is continuous.

Proof. Let $P \in \mathcal{A}$ be a rank one idempotent. Then, $\pi(P)$ is rank one idempotent, as well. It follows that a nonzero idempotent $P \in B(X)$ has rank one if and only if for every nonzero idempotent $Q \in B(X)$, the equality $PQ = Q$ implies $P = Q$.

We fix a unit vector $z \in X$ and a functional $g \in X'$ with $g(z) = 1$. Then, $\pi(z \otimes g) = u \otimes h$, where $u \in Y$ and $h \in Y'$ with $h(u) = 1$. We define $T : X \to Y$ by

$$T(x) = \pi(x \otimes g)u, \quad x \in X.$$

For any $A \in \mathcal{A}$, we have

$$TAx = \pi(A(x \otimes g))u = \pi(A)\pi(x \otimes g)u = \pi(A)Tx.$$

Therefore,

$$TA = \pi(A)T. \quad (*)$$

We observe that $T \neq 0$, by $T(z) = u$, and so the above equality implies that T is a bijective map from X onto Y.

Since $\pi(P)$ is a rank one idempotent, there exists a suitable number $U_P(\lambda)$ for any number λ such that

$$\pi(P)\pi(\lambda P)\pi(P) = U_P(\lambda)\pi(P).$$

Therefore, $\pi(\lambda P) = U_P(\lambda)\pi(P)$. It is easy to see that U_P does not depend on $\pi(P)$; i.e., $U_P = U_Q$, for all rank one idempotents P and Q. Hence, we use U instead of U_P.

In fact, if R is a rank one idempotent such that $RP \neq 0, RQ \neq 0$, then $U_R(\lambda)\pi(R)\pi(P) = \pi((\lambda R)P) = \pi(R(\lambda P)) = U_P(\lambda)\pi(R)\pi(P)$ and so $U_R(\lambda) = U_P(\lambda)$.

Clearly, U is a one-to-one multiplicative map on $\mathbb{R}$, $U(1) = 1$ and $U(-1) = -1$, since π is a multiplicative bijective map.

Also, we have $T(\lambda P) = U(\lambda)TP$, for any rank one idempotent P in $\mathcal{A}$. Therefore, $T(-P) = -TP$ and so $T(-x) = -T(x)$, for any x in X. We show that T is an additive map, and it follows that U is additive.

Suppose first that $x_1, x_2 \in X$ are linearly independent. We put $y_1 = T(x_1)$, $y_2 = T(x_2)$ and distinguish two cases.

(1) $T^{-1}(y_1 + y_2) = x_3$ is linearly independent of x_1, x_2.

(2) The contrary to (1) occurs.

In the first case, we can find an operator $A \in \mathcal{A}$ such that

$$A(x_1) = x_1, A(x_2) = x_2, A(x_3) = x_1 + x_2.$$

Then, we have

$$
\begin{aligned}
T(x_1+x_2) &= T(A(x_3)) = \pi(A)T(x_3) = \pi(A)(T(x_1)+T(x_2)) \\
&= T(x_1)+T(x_2).
\end{aligned}
$$

In the second case, let $x_3 = \lambda_1 x_1 + \lambda_2 x_2$, for some $\lambda_1, \lambda_2 \in \mathbb{R}$ and $P_1 = x_1 \otimes f_1, P_2 = x_2 \otimes f_2$, where $f_1(x_1) = f_2(x_2) = 1$. We find an operator $A \in \mathcal{A}$ such that $A(x_1) = x_1, A(x_2) = 0$. Then, we have

$$
\begin{aligned}
\pi(A)(T(x_1)+T(x_2)) &= \pi(A)T(x_3) = TA(\lambda_1 x_1 + \lambda_2 x_2) \\
&= T(\lambda_1 x_1) = U(\lambda_1)T(x_1).
\end{aligned}
$$

On the other hand, $\pi(A)(T(x_1)+T(x_2)) = TA(x_1)+TA(x_2) = T(x_1)$. Then, $U(\lambda_1) = 1$ and $\lambda_1 = 1$, since U is one-to-one.

In the same way, we obtain $\lambda_2 = 1$; i.e., $T(x_1+x_2) = T(x_1)+T(x_2)$.

Now, for each $0 \neq x \in X$ and $-1 \neq r \in \mathbb{R}$, assume that $y \in X$ be linear by independent of x. Then, $\{x+y, rx-y\}$ is a linear independent set, and therefore,

$$
\begin{aligned}
T(x+rx) &= T(x+y+rx-y) = T(x+y)+T(rx-y) \\
&= T(x)+T(y)+T(rx)-T(y) = T(x)+T(rx).
\end{aligned}
$$

Also, if $r=-1$, then $T(x+rx) = T(x)+T(rx)$, since $T(-x) = -T(x)$. It follows that T and so U are additive. Further more U is multiplicative and $U(1) = 1$. Therefore, $U(\lambda) \equiv \lambda$, by the Darboux theorem. Then, T and so π are linear operators. Now, similar to [1], it can be proved that T is bounded. $\square$

Remark. *It is essential that the normed spaces X and Y be at least two dimensional. For instance, let $X = Y = \mathbb{R}$ and consequently, $\mathcal{A} = \mathcal{B} = \mathbb{R}$. The map $\pi : \mathcal{A} \to \mathcal{B}$, related by $\pi(x) = x^3$, for $x \in \mathbb{R}$, is a bijective multiplicative map, but π is not a linear (or even additive) map.*

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