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SOME EQUIVALENCE CLASSES OF OPERATORS ON $\mathcal{B}(\mathcal{H})$

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Abstract. Let $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ be the algebra of all linear operators on $\mathcal{B}(\mathcal{H})$ and $\mathcal{P}$ be a property on $\mathcal{B}(\mathcal{H})$. For $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, we say that $\phi_1 \sim_{\mathcal{P}} \phi_2$, whenever $\phi_1(T)$ has property $\mathcal{P}$, if and only if $\phi_2(T)$ has this property. In particular, if $\mathcal{I}$ is the identity map on $\mathcal{B}(\mathcal{H})$, then $\phi \sim_{\mathcal{P}} \mathcal{I}$ means that ϕ preserves property $\mathcal{P}$ in both directions. Each property $\mathcal{P}$ produces an equivalence relation on $\mathcal{L}(\mathcal{B}(\mathcal{H}))$. We study the relation between equivalence classes with respect to different properties such as being Fredholm, semi-Fredholm, compact, finite rank, generalized invertible, or having a specific semi-index.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We denote by $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ the ideals of all finite rank and compact operators in $\mathcal{B}(\mathcal{H})$, respectively. The Calkin algebra of $\mathcal{H}$ is the quotient algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a Fredholm operator if $Im(T)$, the range of T, is closed and both its kernel and co-kernel are finite-dimensional. We recall that $T \in \mathcal{B}(\mathcal{H})$ is called upper (resp. lower) semi-Fredholm if $Im(T)$ is closed and its kernel (resp.

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co-kernel) is finite-dimensional. An operator which is either upper semi-Fredholm or lower semi-Fredholm is called a semi-Fredholm operator. We denote by $\mathcal{UF}(\mathcal{H})$, $\mathcal{LF}(\mathcal{H})$, $\mathcal{SF}(\mathcal{H})$, and $\mathcal{FR}(\mathcal{H})$ the sets of upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm and Fredholm operators, respectively. By Atkinson's Theorem, [4, Theorem 1.4.16], if $\mathcal{H}$ is an infinite-dimensional Hilbert space, then $U \in \mathcal{B}(\mathcal{H})$ is Fredholm if and only if $U+K(\mathcal{H})$ is invertible in the Calkin algebra $\mathcal{C}(\mathcal{H})$. The reader is referred to [4, 6] for more on Fredholm operators. Let $A \in \mathcal{B}(\mathcal{H})$. If there exists $B \in \mathcal{B}(\mathcal{H})$ such that $ABA = A$, then A is called generalized invertible and B is said to be a generalized inverse of A. Note that $A \in \mathcal{B}(\mathcal{H})$ is generalized invertible if and only if $Im(A)$ is closed [5]. The set of generalized invertible elements of $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{G}(\mathcal{H})$.

The nullity (resp. defect) of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined to be $dim(Ker(T))$ (resp. $dim(coker(T))$), denoted by $nul(T)$ (resp. $def(T)$). Now, we define the function $s\text{-}index : \mathcal{B}(\mathcal{H}) \to \{0, \infty\} \cup \mathbb{N}$ as follows:

$$s\text{-}index(T) = \begin{cases} \infty & T \in \mathcal{B}(\mathcal{H}) \backslash \mathcal{SF}(\mathcal{H}), \\ 0 & T \in \mathcal{FR}(\mathcal{H}), \\ nul(T) & T \in \mathcal{UF}(\mathcal{H}) \backslash \mathcal{FR}(\mathcal{H}), \\ def(T) & T \in \mathcal{LF}(\mathcal{H}) \backslash \mathcal{FR}(\mathcal{H}). \end{cases}$$

The number $s\text{-}index(T)$ is called the semi-index of T. Note that for a Fredholm operator T, in general, $s\text{-}index(T)$ does not coincide with the classical index of T which is defined by $nul(T) - def(T)$.

Let $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ be the set of all linear mappings on $\mathcal{B}(\mathcal{H})$. Recall that $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ is said to be surjective up to finite rank operators if $\mathcal{B}(\mathcal{H}) = Im(\phi) + \mathcal{F}(\mathcal{H})$, and ϕ is said to be surjective up to compact operators if $\mathcal{B}(\mathcal{H}) = Im(\phi) + \mathcal{K}(\mathcal{H})$. Obviously, if ϕ is surjective up to finite rank operators, then it is surjective up to compact operators and each surjective linear map satisfies both of these properties.

Let $\mathcal{P}$ be a property on $\mathcal{B}(\mathcal{H})$. For $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, we say that $\phi_1 \sim_{\mathcal{P}} \phi_2$, whenever $\phi_1(T)$ has property $\mathcal{P}$, if and only if $\phi_2(T)$ has this property. It is easy to see that each property $\mathcal{P}$ produces an equivalence relation on $\mathcal{L}(\mathcal{B}(\mathcal{H}))$. Throughout this paper, we use the following notations for some specific properties:

(i) "f" is the property of "being finite-rank";
(ii) "k" is the property of "being compact";
(iii) "fr" is the property of "being Fredholm";
(iv) "sf" is the property of "being semi-Fredholm";

(v) "g" is the property of "being generalized invertible";
(vi) "si" is the property of "having a specific semi-index".

Let $\mathcal{I}$ denote the identity operator of $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ and $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$. Then, $\phi \sim_{\mathcal{P}} \mathcal{I}$ means that ϕ preserves the property $\mathcal{P}$ in both directions, that is, $\phi(T)$ has property $\mathcal{P}$ if and only if T has this property. Mbekhta and Šemrl in [3] study those ϕ which satisfy $\phi \sim_g \mathcal{I}$ and $\phi \sim_{sf} \mathcal{I}$. In general, if ψ is a linear operator on $\mathcal{B}(\mathcal{H})$, which preserves property $\mathcal{P}$ in both directions, then $\psi\phi \sim_{\mathcal{P}} \phi$, for all $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$. Also, if υ is a linear operator on $\mathcal{B}(\mathcal{H})$, which does not preserve property $\mathcal{P}$ in both directions, then for each surjective linear operator ϕ, $\upsilon\phi \nsim_{\mathcal{P}} \phi$.

In the next section, we study the equivalence classes with respect to the above properties. We show that for surjective up to finite rank mappings ϕ_1 and ϕ_2 in $\mathcal{L}(\mathcal{B}(\mathcal{H}))$, $\phi_1 \sim_g \phi_2$ implies that $\phi_1 \sim_{sf} \phi_2$ and $\phi_1 \sim_f \phi_2$. Also, if ϕ_1, ϕ_2 are linear mappings on $\mathcal{B}(\mathcal{H})$, which are surjective up to compact operators, and $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$, then $\phi_1 \sim_k \phi_2$. It is also proved that $\phi_1 \sim_{si} \phi_2$ implies $\phi_1 \sim_{sf} \phi_2$. We give some examples to illustrate that some of the reverse implications do not hold, in general. We also prove that for surjective linear operators $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, $\phi_1 \sim_{si} \phi_2$ and $\phi_1 \sim_f \phi_2$ imply that $Ker(\phi_1) = Ker(\phi_2)$, and we give an example to show that the converse is not true, in general. Finally, it is proved that if ϕ_1, ϕ_2 are bijections such that $\phi_1 \sim_{sf} \phi_2$, then $\phi_1{\phi_2}^{-1}$ induces a map $\widetilde{\psi} : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$, which is either an automorphism or an anti-automorphism, multiplied by an invertible element $A \in \mathcal{C}(\mathcal{H})$.

2. **The Results**

In the following lemma, $(i) \Leftrightarrow (ii)$ comes from [2, Lemma 2.2] and $(i) \Leftrightarrow (iii)$ comes from [3, Lemma 2.2].

Lemma 2.1. *Let $K \in \mathcal{B}(\mathcal{H})$. Then, the following are equivalent.*
(i) *K is compact.*
(ii) *for every $B \in \mathcal{FR}(\mathcal{H})$, we have $B + K \in \mathcal{FR}(\mathcal{H})$.*
(iii) *for every $B \in \mathcal{SF}(\mathcal{H})$, we have $B + K \in \mathcal{SF}(\mathcal{H})$.*

Take $C = \{T \in \mathcal{B}(\mathcal{H}) \mid$ for every operator $A \in \mathcal{B}(\mathcal{H})$ with $Im(A)$ not closed, there exists $\lambda \in \mathbb{C}$ such that $A + \lambda T \neq 0$ and $Im(A + \lambda T)$ is closed$\}$. It is proved in [1, Lemma 3.1] that $C = \mathcal{SF}(\mathcal{H})$.

We recall that if $T \in \mathcal{G}(\mathcal{H})$, then, for each finite rank operator F, we have $T + F \in \mathcal{G}(\mathcal{H})$.

Theorem 2.2. *Let $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be linear mappings. Then,*
(i) $\phi_1 \sim_{si} \phi_2 \quad \Rightarrow \quad \phi_1 \sim_{sf} \phi_2$;
if ϕ_1, ϕ_2 are surjective up to finite rank operators, then
(ii) $\phi_1 \sim_g \phi_2 \quad \Rightarrow \quad \phi_1 \sim_{sf} \phi_2$;
(iii) $\phi_1 \sim_g \phi_2 \quad \Rightarrow \quad \phi_1 \sim_f \phi_2$;
if ϕ_1, ϕ_2 are surjective up to compact operators, then
(iv) $\phi_1 \sim_{fr} \phi_2 \quad \Rightarrow \quad \phi_1 \sim_k \phi_2$;
(v) $\phi_1 \sim_{sf} \phi_2 \quad \Rightarrow \quad \phi_1 \sim_k \phi_2$.

Proof. (i) It is trivial by the definition of semi-index.
(ii) Suppose that $\phi_1(T)$ is a semi-Fredholm operator. We show that $\phi_2(T) \in C$. Suppose that $B \in \mathcal{B}(\mathcal{H})$ is not generalized invertible, or equivalently $Im(B)$ is not closed. Since ϕ_2 is surjective up to finite rank operators, there exist $A \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ such that $\phi_2(A) = B+F$. Since F is finite rank, $Im(\phi_2(A))$ is not closed and it follows that the range of $\phi_1(A)$ is not closed. Since $\phi_1(T)$ is semi-Fredholm, we have $\phi_1(T) \in C$. Thus, there exists $\alpha \in \mathbb{C}$ such that $Im(\phi_1(\alpha T + A))$ is closed. It follows that the range of $\phi_2(\alpha T + A) = \alpha\phi_2(T) + B + F$ is also closed, which implies that $Im(\alpha\phi_2(T) + B)$ is closed. Note that $\alpha\phi_2(T) + B \neq 0$. Otherwise, $Im(\alpha\phi_2(T)) = Im(B)$ is not closed, which contradicts $\phi_1 \sim_g \phi_2$. Therefore, $\phi_2(T) \in C$.
(iii) Let $\phi_1 \sim_g \phi_2$. Suppose that $\phi_1(T) \in \mathcal{F}(\mathcal{H})$, but $\phi_2(T)$ is not finite-rank. Therefore, the range of $\phi_1(T)$ is closed, but it is not semi-Fredholm. By the fact that $\phi_1 \sim_g \phi_2$, $Im(\phi_2(T))$ is closed. Also, by (ii), $\phi_2(T)$ is not semi-Fredholm. Take $S = \phi_2(T)$. Then, both $Ker(S)$ and $Im(S)^{\perp}$ are infinite-dimensional and we can define a bounded linear bijection $S' : Ker(S) \to Im(S)^{\perp}$. Extend S' on $\mathcal{H}$ by $S'(x) = 0$, for all $x \in Ker(S)^{\perp}$, and denote this extension by S' as well. Since S is not finite rank, S' is not a semi-Fredholm operator on $\mathcal{H}$. Now, take $\widetilde{T} \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ such that $\phi_2(\widetilde{T}) = S' + F$. We have $S + S'$ is a bijective bounded linear operator on $\mathcal{H}$, and hence it is Fredholm. Therefore, $\phi_2(T + \widetilde{T}) = S + S' + F \in \mathcal{FR}(\mathcal{H})$. On the other hand, $\phi_1(T + \widetilde{T})$ is not semi-Fredholm. Otherwise, $\phi_1(\widetilde{T})$ must be semi-Fredholm and it follows by (ii) that $S' + F = \phi_2(\widetilde{T})$ is semi-Fredholm, which is not correct. Thus, $\phi_1 \nsim_{sf} \phi_2$, a contradiction with (ii), and so $\phi_1 \sim_f \phi_2$.
(iv) Suppose that $\phi_1(T)$ is compact. Let S be an arbitrary Fredholm operator. Since ϕ_2 is surjective up to compact operators, there exist $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ such that $\phi_2(A) = S+K$. Obviously, $\phi_2(A) \in$

$\mathcal{FR}(\mathcal{H})$, and since $\phi_1 \sim_{fr} \phi_2$, we have $\phi_1(A)$ is Fredholm. On the other hand, $\phi_1(T)$ is compact and so by Lemma 2.1, $\phi_1(T+A) \in \mathcal{FR}(\mathcal{H})$. Thus, $\phi_2(T+A)$ is Fredholm, and it follows that $\phi_2(T)+S = \phi_2(T+A)-K$ is also a Fredholm operator and Lemma 2.1 implies that $\phi_2(T)$ is compact.
(v) The proof is similar to the one given in (iv). $\square$

In what follows we give some examples to show that in Theorem 2.2 some of the reverse implications do not hold, in general.

Example 2.3. *Suppose that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective linear map and $S \in \mathcal{B}(\mathcal{H})$ is a lower semi-Fredholm operator such that s-index$(S) = 1$. Since ϕ is surjective, there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\phi(T) = S$. Now, if $A \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator with $def(A) = 2$, then $\phi \sim_{sf} L_A\phi$, but $\phi \nsim_{si} L_A\phi$, since s-index$(A\phi(T)) \geq 2 > 1$. Here $L_A : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the left multiplier operator defined by $L_A(S) = AS$.*

Example 2.4. *We show that, in general, $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$ does not imply $\phi_1 \sim_g \phi_2$. Let S be a surjective bounded linear map on $\mathcal{H}$ such that $dim(Ker(S)) = 1$. Note that $S \in \mathcal{FR}(\mathcal{H})$. Let $P : \mathcal{H} \to Ker(S)$ be the projection of $\mathcal{H}$ onto $Ker(S)$. Take $\phi_0 = L_S$. Since S has a bounded right inverse, ϕ_0 is surjective. Extend $\{P\}$ to a vector space basis $\{T_\alpha\}$ for $\mathcal{B}(\mathcal{H})$. Suppose that $K \in \mathcal{K}(\mathcal{H})$ has a non-closed range. Define a linear map $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ by*

$$\lambda(T_\alpha) = \begin{cases} K & T_\alpha = P \\ 0 & T_\alpha \neq P. \end{cases}$$

Now, define $\phi_1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $\phi_0(T) + \lambda(T)$. Then, by Lemma 2.1, $\phi_1 \sim_{sf} \phi_0$. Also, ϕ_1 is surjective. To see this, take $T \in \mathcal{B}(\mathcal{H})$. There exists $U \in \mathcal{B}(\mathcal{H})$ such that $\phi_0(U) = T$. Since $\{T_\alpha\}$ is a vector space basis for $\mathcal{B}(\mathcal{H})$, there exist $\beta_1, ..., \beta_n \in \mathbb{C}$ and $T_{\alpha_1}, ..., T_{\alpha_n} \in \{T_\alpha\}$ such that $U = \sum_{i=1}^n \beta_i T_{\alpha_i}$. If for each $1 \leq j \leq n$, $T_{\alpha_j} \neq P$, then $\phi_1(U) = \phi_0(U) = T$. Otherwise, if for some $1 \leq j \leq n$, $T_{\alpha_j} = P$, then take $U' = U - \beta_j T_{\alpha_j}$ and we have $\phi_0(U) = \phi_0(U')$, and therefore, $\phi_1(U') = \phi_0(U') + \lambda(U') = T$.

Finally, $\phi_1(P) = K$, which is not generalized invertible since $Im(K)$ is not closed, while $\phi_0(P) = 0$ is generalized invertible and this shows $\phi_1 \nsim_g \phi_0$.

We do not know any example of two linear mappings $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, which are surjective up to compact operators, $\phi_1 \sim_k \phi_2$ but $\phi_1 \nsim_{fr} \phi_2$ or $\phi_1 \nsim_{sf} \phi_2$. As seen in the above examples, the multiplier operator L_T

for a suitable T plays an important role. But, here we show that if ϕ_1 and ϕ_2 satisfy the above mentioned conditions, they can not be related by $\phi_2 = L_A R_B \phi_1 + \lambda$, where λ is a linear mapping from $\mathcal{B}(\mathcal{H})$ to $\mathcal{K}(\mathcal{H})$. Here, R_B denotes the right multiplier operator $T \mapsto TB$ on $\mathcal{B}(\mathcal{H})$.

Proposition 2.5. *Let $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ be surjective up to compact operators and $\phi_2 = L_A R_B \phi_1 + \lambda$, where, $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is a linear mapping. If $\phi_1 \sim_k \phi_2$, then A and B are Fredholm operators, and hence $\phi_1 \sim_{fr} \phi_2$ and $\phi_1 \sim_{sf} \phi_2$.*

Proof. Let $\phi_2 = L_A R_B \phi_1 + \lambda$. For $i = 1, 2$, consider $\tau_i : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ defined by $\tau_i(T) = \pi \circ \phi_i(T)$, where, $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is the canonical quotient map. It is easy to check that $\tau_2(T) = a\tau_1(T)b$, for all $T \in \mathcal{B}(\mathcal{H})$, where, $a = \pi(A), b = \pi(B)$.

The condition that ϕ_1, ϕ_2 are surjective up to compact operators implies that τ_1, τ_2 are surjective. The condition $\phi_1 \sim_k \phi_2$ says that $\tau_1(T) = 0$ if and only if $\tau_2(T) = 0$ if and only if $a\tau_1(T)b = 0$. Since τ_1 is onto, this in turn says that with $x \in \mathcal{C}(\mathcal{H})$, $axb = 0$ if and only if $x = 0$.

Now, τ_2 is onto, and so $azb = \pi(I)$, for some $z \in \mathcal{C}(\mathcal{H})$. Thus, there exists $Z \in \mathcal{B}(\mathcal{H})$ such that AZB is a Fredholm operator, which shows that A and B are semi-Fredholm. If A were not Fredholm, then (since $a = \pi(A)$ is right invertible), we must have $nul(A) = \infty$. Let $P \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection of $\mathcal{H}$ onto $Ker(A)$. Then, $p = \pi(P) \neq 0$, but $apb = \pi(APB) = \pi(0B) = 0$, which is a contradiction. Thus, A is Fredholm. Finally, since AZB is a Fredholm operator, we have that $B^*Z^*A^*$ is also a Fredholm operator. The same argument implies that B^*, and hence B is a Fredholm operator.

□

In the sequel, we explore the consequences when both ϕ_1 and ϕ_2 are in certain equivalence classes.

Theorem 2.6. *If $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are surjective linear maps such that $\phi_1 \sim_{si} \phi_2$ and $\phi_1 \sim_f \phi_2$, then $Ker(\phi_1) = Ker(\phi_2)$.*

Proof. Let $T \in Ker(\phi_1)$. Since $\phi_1 \sim_f \phi_2$, $\phi_2(T)$ is finite-rank, then it is not a semi-Fredholm operator. It follows that $null(\phi_2(T)) = \infty = def(\phi_2(T))$. Now, we can write $Ker(\phi_2(T)) = \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are infinite-dimensional closed subspaces of $Ker(\phi_2(T))$. Define a bounded linear bijection $T' : \mathcal{N} \to Im(\phi_2(T))^\perp$. Extend T' on $\mathcal{H}$ by $T'(x) = 0$, for all $x \in (Ker(\phi_2(T))^\perp \oplus \mathcal{M}$ and denote this extension by T' as well. Clearly, T' is a semi-Fredholm operator and

$s\text{-}index(T') = rank(\phi_2(T))$. We have $\phi_2(T) + T'$ is surjective and $\mathcal{M} \subseteq Ker(\phi_2(T) + T')$. Thus, it is a semi-Fredholm operator with $s\text{-}index(\phi_2(T) + T') = 0$. Since ϕ_2 is surjective, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $\phi_2(S) = T'$. Therefore, $rank(\phi_2(T)) = s\text{-}index(\phi_2(S)) = s\text{-}index(\phi_1(S)) = s\text{-}index(\phi_1(S+T)) = s\text{-}index(\phi_2(S+T)) = 0$. It follows that $Ker(\phi_1) \subseteq Ker(\phi_2)$. The reverse inclusion follows similarly and we have the result. □

Corollary 2.7. *If ϕ is a surjective linear map on $\mathcal{B}(\mathcal{H})$ that preserves finite rank operators and semi-index property in both directions, then ϕ is injective.*

As a consequence, by Theorem 2.2 (*iii*), if ϕ is a surjective linear map on $\mathcal{B}(\mathcal{H})$ that preserves generalized invertible operators and semi-index property in both directions, then it is injective.

Remark 2.8. (*i*) *In general, if surjective linear maps $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ have the same kernels, then it may happen that $\phi_1 \nsim_f \phi_2$, and hence the converse of Theorem 2.6 does not hold. To see this, take $e \in \mathcal{H}$ with $\|e\| = 1$. It is clear that I and $e \otimes e$ are linearly independent. We can extend $\{I, e \otimes e\}$ to a basis for the vector space $\mathcal{B}(\mathcal{H})$. Now, define $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\phi(I) = e \otimes e$, $\phi(e \otimes e) = I$ and $\phi(T) = T$, for every T in the basis with $T \neq I$, $T \neq e \otimes e$. Extend ϕ to a bijection on $\mathcal{B}(\mathcal{H})$, by linearity. Hence, $Ker(\phi) = 0 = Ker(\mathcal{I})$, but we do not have $\phi \sim_f \mathcal{I}$, since $\phi(I) = e \otimes e$. It also follows that $\phi \nsim_g \mathcal{I}$, since otherwise by Theorem 2.2 (iii), we must have $\phi \sim_f \mathcal{I}$.*
(*ii*) *The condition $\phi_1 \sim_{si} \phi_2$ in Theorem 2.6 can not be omitted. Suppose that $S \in \mathcal{B}(\mathcal{H})$ is surjective and $dim(Ker(S)) = 1$. Thus, $\mathcal{I} \sim_f L_S$, $\mathcal{I} \nsim_{si} L_S$, and clearly $Ker(\mathcal{I}) \neq Ker(L_S)$. We do not know any example of surjective linear operators ϕ_1, ϕ_2 on $\mathcal{B}(\mathcal{H})$ such that $\phi_1 \sim_{si} \phi_2$, but $\phi \nsim_f \phi_2$. So, at this point we do not know whether the case $\phi_1 \sim_{si} \phi_2$, $\phi_1 \nsim_f \phi_2$ happens or not.*

Now, we consider the case $Ker(\phi_1) = \{0\} = Ker(\phi_2)$.

Proposition 2.9. *Let $\mathcal{P}$ be a property on $\mathcal{B}(\mathcal{H})$. If $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are bijective linear maps such that $\phi_1 \sim_{\mathcal{P}} \phi_2$, then ${\phi_1\phi_2}^{-1}$ preserves property $\mathcal{P}$ in both directions.*

Proof. Let $\mathcal{M}_{\mathcal{P}} = \{T \in \mathcal{B}(\mathcal{H}) : T \quad \text{has property} \quad \mathcal{P}\}$. Then, ${\phi_1}^{-1}(\mathcal{M}_{\mathcal{P}}) = {\phi_2}^{-1}(\mathcal{M}_{\mathcal{P}})$, and hence $\mathcal{M}_{\mathcal{P}} = \phi_1{\phi_2}^{-1}(\mathcal{M}_{\mathcal{P}})$. It follows that $\phi_1{\phi_2}^{-1}$ preserves $\mathcal{P}$ in both directions. □

The following theorem was proved by Mbekhta and Šemrl [3, Theorem 1.2].

Theorem 2.10. *Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map preserving semi-Fredholm operators in both directions. Suppose that ϕ is surjective up to compact operators. Then,*

$$\phi(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H}),$$

and the induced map $\widetilde{\phi} : \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $a \in \mathcal{C}(\mathcal{H})$.

Corollary 2.11. *Suppose that ϕ_1, ϕ_2 are bijective linear maps on $\mathcal{B}(\mathcal{H})$ such that $\phi_1 \sim_{sf} \phi_2$. Take $\psi = \phi_1 {\phi_2}^{-1}$. Then, $\psi(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$ and the induced map $\widetilde{\psi}$ on $\mathcal{C}(\mathcal{H})$ is an automorphism or an anti-automorphism multiplied by an invertible element $a \in \mathcal{C}(\mathcal{H})$.*

Note that, by Theorem 2.2 (ii), we have the same result for $\phi_1 \sim_g \phi_2$. Now, a question comes to mind: *Is it possible to identify the equivalence class of ϕ with respect to a property $\mathcal{P}$*?

Remark 2.12. (i) *Let $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the linear map which takes T to its transpose with respect to a given basis for $\mathcal{H}$. If $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, then it is easy to see that $\tau\phi \sim_g \phi$, and hence $\tau\phi \sim_{sf} \phi$, $\tau\phi \sim_k \phi$, and $\tau\phi \sim_f \phi$. Also, $\tau\phi \sim_{fr} \phi$ and $\tau\phi \sim_{si} \phi$.*
(ii) *Let A and B be Fredholm operators and $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ be a linear map. If $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ are related as $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau\phi_1 + \lambda$, then it is easy to see that $\phi_1 \sim_{sf} \phi_2$, $\phi_1 \sim_{fr} \phi_2$.*
(iii) *Let A and B be Fredholm operators and $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ be a linear map. If $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ are such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau\phi_1 + \lambda$, then $\phi_1 \sim_g \phi_2$.*
(iv) *Let $A, B \in \mathcal{B}(\mathcal{H})$ be invertible operators. If $\phi_2 = L_A R_B \phi_1$ or $\phi_2 = L_A R_B \tau\phi_1$, then it is easy to see that $\phi_1 \sim_{si} \phi_2$.*

Question 2.13. Let $\phi_1 \sim_g \phi_2$. Are there A, $B \in \mathcal{FR}(\mathcal{H})$ and a linear map $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau\phi_1 + \lambda$?

Question 2.14. Let $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$. Are there A, $B \in \mathcal{FR}(\mathcal{H})$ and a linear map $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau\phi_1 + \lambda$?

Question 2.15. Let $\phi_1 \sim_{si} \phi_2$. Are there invertible operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1$ or $\phi_2 = L_A R_B \tau \phi_1$?

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