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TOPOLOGICAL CENTERS OF CERTAIN BANACH MODULE ACTIONS

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Abstract. We study the topological centers of some specific adjoints of a Banach module action. Then, we investigate the Arens regularity and strong irregularity of these actions.

1. Introduction and preliminaries

Based on the celebrated work of R. Arens [1], every bounded bilinear map $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ (on the normed spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$) has two natural but, in general, different extensions f^{***} and f^{r***r} from $\mathcal{X}^{**} \times \mathcal{Y}^{**}$ to $\mathcal{Z}^{**}$. Let us recall these notions with more detail.

For a bounded bilinear map $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, we define the adjoint $f^* : \mathcal{Z}^* \times \mathcal{X} \to \mathcal{Y}^*$ of f by

$$< f^*(z^*, x), y >=< z^*, f(x, y) >, \qquad (x \in \mathcal{X}, y \in \mathcal{Y} \text{ and } z^* \in \mathcal{Z}^*).$$

Continuing this process, we can define the second and the third adjoints f^{**} and f^{***} of f by $f^{**} = (f^*)^* : \mathcal{Y}^{**} \times \mathcal{Z}^* \to \mathcal{X}^*$ and $f^{***} = (f^{**})^* : \mathcal{X}^{**} \times \mathcal{Y}^{**} \to \mathcal{Z}^{**}$, respectively; and so on for the higher rank adjoints of f. One can verify that f^{***} is the unique extension of f which is

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w^*-separately continuous on $\mathcal{X}\times\mathcal{Y}^{**}$. We define the left topological center $Z_\ell(f)$ of f by

$$\begin{aligned} Z_\ell(f) &= \{x^{**}\in\mathcal{X}^{**}; y^{**}\longrightarrow f^{***}(x^{**},y^{**}) \text{ is } w^*-w^*-\text{continuous}\} \\ &= \{x^{**}\in\mathcal{X}^{**}; f^{***}(x^{**},y^{**}) = f^{r***r}(x^{**},y^{**}) \text{ for every } y^{**}\in\mathcal{Y}^{**}\}. \end{aligned}$$

We also denote by f^r the flip map of f, that is, the bounded bilinear map $f^r:\mathcal{Y}\times\mathcal{X}\longrightarrow\mathcal{Z}$ defined by $f^r(y,x)=f(x,y)$ $(x\in\mathcal{X}, y\in\mathcal{Y})$. If we repeat the latter process with f^r instead of f, we come to the bounded bilinear map $f^{r***r}:\mathcal{X}^{**}\times\mathcal{Y}^{**}\to\mathcal{Z}^{**}$, that is, the unique extension of f which is w^*-separately continuous on $\mathcal{X}^{**}\times\mathcal{Y}$. We also define the right topological center $Z_r(f)$ of f by

$$\begin{aligned} Z_r(f) &= \{y^{**}\in\mathcal{Y}^{**}; x^{**}\longrightarrow f^{r***r}(x^{**},y^{**}) \text{ is } w^*-w^*-\text{continuous}\} \\ &= \{y^{**}\in\mathcal{Y}^{**}; f^{***}(x^{**},y^{**}) = f^{r***r}(x^{**},y^{**}) \ \text{ for every } x^{**}\in\mathcal{X}^{**}\}. \end{aligned}$$

From these observations, we have $Z_r(f)=Z_\ell(f^r)$.
An standard argument may also be used to interpret f^{***} and f^{r***r} by the following iterative limit processes,

$$f^{***}(x^{**},y^{**}) = w^*-\lim_\alpha\lim_\beta f(x_\alpha,y_\beta), \text{ and}$$

$$f^{r***r}(x^{**},y^{**}) = w^*-\lim_\beta\lim_\alpha f(x_\alpha,y_\beta),$$

where $\{x_\alpha\}$ and $\{y_\beta\}$ are nets in $\mathcal{X}$ and $\mathcal{Y}$ that converge, in w^*-topologies, to x^{**} and y^{**}, respectively.

A bounded bilinear mapping f is said to be (Arens) regular if $f^{***}=f^{r***r}$. This happens if and only if $Z_\ell(f)=\mathcal{X}^{**}$, or equivalently $Z_r(f)=\mathcal{Y}^{**}$. The map f is said to be left (respectively, right) strongly irregular if $Z_\ell(f)=\mathcal{X}$ (respectively, $Z_r(f)=\mathcal{Y}$).

It is worthwhile mentioning that in the case where π is the multiplication of a Banach algebra $\mathcal{A}$, then π^{***} and π^{r***r} are actually the first and second Arens products, which will be denoted by $\Box$ and $\Diamond$, respectively. We also say that $\mathcal{A}$ is (Arens) regular, left strongly irregular or right strongly irregular if the multiplication π of $\mathcal{A}$ enjoys the corresponding property.

The subject of regularity of bounded bilinear mappings and Banach module actions have been investigated in [3], [6], [7] and [9]. In [7], Eshaghi Gordji and Fillali gave several significant results related to the topological centers of Banach module actions. In [9], the authors have obtained a criterion for the regularity of f, from which they gave several results related to the regularity of Banach module actions with some applications to the second adjoint of a derivation. For a good and rich source of information on this subject, we refer the reader to the Memoire in [5]. We also shall mostly follow [4] as a general reference on Banach algebras.

The remainder of the paper is organized as follows. In Section 1, we study the left topological centers of π_1^{r*} and π_2^*, where $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ are approximately unital left and right Banach $\mathcal{A}-$modules, respectively. We show that π_1^{r*} and π_2^* are permanently left strongly irregular; see Theorem 2.2 below. This result improves some results of [6], [9] and [7]. For instance, it covers [6, Proposition 4.5], [9, Proposition 3.6] and [7, Corollary 2.4] as well. In Section 2, we shall characterize the right topological centers of π_1^{r*} and π_2^* (see Theorem 3.4 below). We apply this fact to determine the topological centers of π^{r*r} and π^* for the multiplication π of a Banach algebra with a bounded approximate identity. Section 3 is devoted to investigation of relationships between the regularity of π_1^{r*}, π_2^*, π_1, π_2 and π, in the case where $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ enjoy some factorization properties and are not necessarily approximately unital.

As already done, throughout the paper we usually identify (an element of) a normed space with its canonical image in its second dual.

2. Left strong irregularity of certain adjoints of a Banach module action

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach space and $\pi_1 : \mathcal{A}\times\mathcal{X} \longrightarrow \mathcal{X}$ be a bounded bilinear mapping. Then, the pair $(\pi_1, \mathcal{X})$ is said to be a left Banach $\mathcal{A}-$module if

$$\pi_1(ab, x) = \pi_1(a, \pi_1(b, x)); \quad (a, b \in \mathcal{A}, x \in \mathcal{X}).$$

A right Banach $\mathcal{A}-$module $(\mathcal{X}, \pi_2)$ can be defined similarly. A triple $(\pi_1, \mathcal{X}, \pi_2)$ is said to be a Banach $\mathcal{A}-$module if $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ are left and right Banach $\mathcal{A}-$modules, respectively, and

$$\pi_1(a, \pi_2(x, b)) = \pi_2(\pi_1(a, x), b); \quad (a, b \in \mathcal{A}, x \in \mathcal{X}).$$

Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be left and right Banach $\mathcal{A}-$modules, respectively. Then, one may verify that $(\pi_1^{***}, \mathcal{X}^{**})$ and $(\mathcal{X}^{**}, \pi_2^{***})$ are left and right Banach $(\mathcal{A}^{**}, \Box)-$modules, respectively. Similarly, $(\pi_1^{r***r}, \mathcal{X}^{**})$

and $(\mathcal{X}^{**}, \pi_2^{r***r})$ are left and right Banach $(\mathcal{A}^{**}, \Diamond)-$modules, respectively.

In the case where $\mathcal{A}$ enjoys a bounded left approximate identity, we say that $(\pi_1, \mathcal{X})$ is approximately unital if the involved bounded left approximate identity of $\mathcal{A}$ is that of $(\pi_1, \mathcal{X})$. The notion of being approximately unital can be defined similarly for a right Banach $\mathcal{A}-$module $(\mathcal{X}, \pi_2)$. It can be easily verified that a left Banach $\mathcal{A}-$module $(\pi_1, \mathcal{X})$ is approximately unital if the left module action $(\pi_1^{r***r}, \mathcal{X}^{**})$ is unital; that is, $\pi_1^{r***r}(e^{**}, x^{**}) = x^{**}$, for every $x^{**} \in \mathcal{X}^{**}$, in which e^{**} is a w^*-cluster point of the involved left approximate identity of $(\pi_1, \mathcal{X})$. A similar fact is valid for the right Banach $\mathcal{A}-$module $(\mathcal{X}, \pi_2)$. We summarize these observations in the next result.

Proposition 2.1. *Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be left and right Banach $\mathcal{A}-$modules, respectively. Then,*
(i) $(\pi_1, \mathcal{X})$ is approximately unital if $(\pi_1^{r***r}, \mathcal{X}^{**})$ is unital.
(ii) $(\mathcal{X}, \pi_2)$ is approximately unital if $(\mathcal{X}^{**}, \pi_2^{***})$ is unital.

It should be remarked that in contrast to the situation occurring for $(\pi_1^{r***r}, \mathcal{X}^{**})$ and $(\mathcal{X}^{**}, \pi_2^{***})$ in the above result, $(\pi_1^{***}, \mathcal{X}^{**})$ and $(\mathcal{X}^{**}, \pi_2^{r***r})$ are not necessarily unital, in general. For instance, let $\mathcal{A} = K(c_0)$ be the Banach algebra of compact operators on the sequence space c_0. Then, $\mathcal{A}$ enjoys a bounded approximate identity and a direct verification reveals that for the multiplication $\pi_1 = \pi$ of $\mathcal{A}$, $(\pi_1^{***}, \mathcal{A}^{**})$ is not unital. Also, a similar argument shows that for the reversed multiplication $\pi_2 = \pi^r$ on $\mathcal{A}$, $(\mathcal{A}^{**}, \pi_2^{r***r})$ is not unital; for more details, see [8, Example 2.5].

The next result studies the left strong irregularity of π_1^{r*} and π_2^*, from which we improve some older results.

Theorem 2.2. *Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be approximately unital left and right Banach $\mathcal{A}-$modules, respectively. Then, π_1^{r*} and π_2^* are left strongly irregular; that is,*

$$Z_\ell({\pi_1}^{r*}) = \mathcal{X}^* = Z_\ell({\pi_2}^*).$$

Proof. Let $(\mathcal{X}, \pi_2)$ be approximately unital, $x^{***} \in Z_\ell(\pi_2^*)$ and $x^{**} \in \mathcal{X}^{**}$. Then, using Proposition 2.1, there exists $e^{**} \in \mathcal{A}^{**}$ such that

$\pi_2^{***}(x^{**}, e^{**}) = x^{**}$. Now,

$$\begin{aligned}\langle x^{***}, x^{**}\rangle &= \langle x^{***}, \pi_2^{***}(x^{**}, e^{**})\rangle \\ &= \langle \pi_2^{****}(x^{***}, x^{**}), e^{**}\rangle \\ &= \langle \pi_2^{*r***r}(x^{***}, x^{**}), e^{**}\rangle \\ &= \langle x^{**}, \pi_2^{*r**}(x^{***}, e^{**})\rangle\end{aligned}$$

implies that $x^{***} = \pi_2^{*r**}(x^{***}, e^{**}) \in \mathcal{X}^*$. Therefore, $Z_\ell(\pi_2^*) = \mathcal{X}^*$, as required. The other equality needs a similar argument. □

As an immediate consequence of Theorem 2.2, we deduce the next result of [9], (which in turn is a generalization of [6, Proposition 4.5]; see also [2, Theorem 4] and [10, Theorem 3.1].)

Corollary 2.3. ([9, Proposition 3.6]) *Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be approximately unital left and right Banach $\mathcal{A}-$modules, respectively. Then, the following assertions are equivalent:*
(i) π_1^{r*} is regular.
(ii) π_2^* is regular.
(iii) $\mathcal{X}$ is reflexive.

3. The right topological centers of π_1^{r*} and π_2^*

Before we proceed to the main result of this section, we need to introduce a set $\mathfrak{M}_\mathcal{X}$ and examine some of its properties. For a normed space $\mathcal{X}$, let $J_\mathcal{X} : \mathcal{X} \to \mathcal{X}^{**}$ denote the canonical embedding of $\mathcal{X}$ into $\mathcal{X}^{**}$, with the second adjoint $(J_\mathcal{X})^{**} : \mathcal{X}^{**} \to \mathcal{X}^{****}$. We define $\mathfrak{M}_\mathcal{X}$ by

$$\mathfrak{M}_\mathcal{X} = \{x^{**} \in \mathcal{X}^{**} : J_{\mathcal{X}^{**}}(x^{**}) = (J_\mathcal{X})^{**}(x^{**})\}.$$

It is routine to verify that $\mathfrak{M}_\mathcal{X}$ is a closed subspace of $\mathcal{X}^{**}$ containing $\mathcal{X}$. It should be mentioned that $\mathfrak{M}_\mathcal{X}$ may lie strictly between $\mathcal{X}$ and $\mathcal{X}^{**}$; as we shall see in Corollary 3.3, this is the case for $\mathcal{X} = c_0$. It would be desirable to characterize those $\mathcal{X}$ for which $\mathcal{X} = \mathfrak{M}_\mathcal{X}$. The next lemma clears the equality $\mathfrak{M}_\mathcal{X} = \mathcal{X}^{**}$.

Lemma 3.1. *For a normed space $\mathcal{X}$, the equality $\mathfrak{M}_\mathcal{X} = \mathcal{X}^{**}$ holds if and only if $\mathcal{X}$ is reflexive.*

Proof. If $\mathcal{X}$ is reflexive, then trivially $\mathfrak{M}_{\mathcal{X}} = \mathcal{X}^{**}$. For the converse, suppose that $x^{**} \in X^{**}$ and $x^{***} \in \mathcal{X}^{***}$. Then,

$$\begin{aligned}\langle x^{***}, x^{**}\rangle &= \langle J_{\mathcal{X}^{**}}(x^{**}), x^{***}\rangle \\ &= \langle (J_{\mathcal{X}})^{**}(x^{**}), x^{***}\rangle \\ &= \langle x^{**}, (J_{\mathcal{X}})^{*}(x^{***})\rangle \\ &= \langle J_{\mathcal{X}^{*}}((J_{\mathcal{X}})^{*}(x^{***})), x^{**}\rangle.\end{aligned}$$

Therefore $x^{***} = J_{\mathcal{X}^*}((J_{\mathcal{X}})^*(x^{***})) \in J_{\mathcal{X}^*}(X^*)$; that is, $\mathcal{X}^*$ is reflexive, and so $\mathcal{X}$ is reflexive. □

Example 3.2. We show that $c_0 \subsetneq \mathfrak{M}_{c_0}$, where c_0 is the Banach space of all sequences converging to zero. Indeed, a direct verification shows that $c \subseteq \mathfrak{M}_{c_0}$, in which c is the Banach space of all convergent sequences. To see this, one may use the direct sum decomposition,

$$\ell^{\infty *} = c^* \oplus c^{\perp},$$

to show that for every $x^{**} \in c \subset \ell^{\infty} = c_0^{**}$ and $x^{***} \in c^{***} = \ell^{\infty *}$,

$$\langle (J_{\mathcal{X}})^{**}(x^{**}), x^{***}\rangle = \langle J_{\mathcal{X}^{**}}(x^{**}), x^{***}\rangle,$$

from which we deduce $x^{**} \in \mathfrak{M}_{c_0}$, as claimed.

Corollary 3.3. $c_0 \subsetneq \mathfrak{M}_{c_0} \subsetneq c_0^{**}$.

The next result, being the main one in this section, characterizes the right topological centers of π_1^{r*} and π_2^*.

Theorem 3.4. *Let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be approximately unital left and right Banach $\mathcal{A}-$modules, respectively. Then,*

$$Z_r(\pi_1^{r*}) = \mathfrak{M}_{\mathcal{X}} = Z_r(\pi_2^*).$$

Proof. We shall only prove $\mathfrak{M}_{\mathcal{X}} = Z_r(\pi_2^*)$; the other equality needs a similar argument. Let $x^{**} \in \mathfrak{M}_{\mathcal{X}}$. Then, for every $x^{***} \in \mathcal{X}^{***}$, $a^{**} \in \mathcal{A}^{**}$ and bounded nets $\{x_\alpha\} \subseteq \mathcal{X}$, $\{x_\beta^*\} \subseteq \mathcal{X}^*$, w^*-converging to

x^{**} and x^{***}, respectively, we have,

$$\begin{aligned}
\langle \pi_2^{****}(x^{***}, x^{**}), a^{**}\rangle &= \langle \pi_2^{*****}(a^{**}, x^{***}), x^{**}\rangle \\
&= \langle J_{\mathcal{X}^{**}}(x^{**}), \pi_2^{*****}(a^{**}, x^{***})\rangle \\
&= \langle (J_{\mathcal{X}})^{**}(x^{**}), \pi_2^{*****}(a^{**}, x^{***})\rangle \\
&= \langle x^{**}, (J_{\mathcal{X}})^{*}(\pi_2^{*****}(a^{**}, x^{***}))\rangle \\
&= \lim_{\alpha} \langle (J_{\mathcal{X}})^{*}(\pi_2^{*****}(a^{**}, x^{***})), x_\alpha\rangle \\
&= \lim_{\alpha} \langle \pi_2^{*****}(a^{**}, x^{***}), J_{\mathcal{X}}(x_\alpha)\rangle \\
&= \lim_{\alpha} \langle \pi_2^{****}(x^{***}, x_\alpha), a^{**}\rangle \\
&= \lim_{\alpha} \langle x^{***}, \pi_2^{***}(x_\alpha, a^{**})\rangle \\
&= \lim_{\alpha}\lim_{\beta} \langle \pi_2^{***}(x_\alpha, a^{**}), x_\beta^{*}\rangle \\
&= \lim_{\alpha}\lim_{\beta} \langle \pi_2^{**}(a^{**}, x_\beta^{*}), x_\alpha\rangle \\
&= \lim_{\alpha}\lim_{\beta} \langle a^{**}, \pi_2^{*}(x_\beta^{*}, x_\alpha)\rangle \\
&= \langle \pi_2^{*r****r}(x^{***}, x^{**}), a^{**}\rangle.
\end{aligned}$$

We thus have $x^{**} \in Z_r(\pi_2^*)$; that is, $\mathfrak{M}_{\mathcal{X}} \subseteq Z_r(\pi_2^*)$. To prove the reverse inclusion, let $x^{**} \in Z_r(\pi_2^*)$. As $(\mathcal{X}, \pi_2)$ is approximately unital, by Proposition 2.1, there exists a bounded right approximate identity $\{e_\lambda\} \subseteq \mathcal{A}$ for $(\mathcal{X}, \pi_2)$ with $e^{**} \in \mathcal{A}^{**}$ as a w^*-cluster point of $\{e_\lambda\}$ such that $\pi_2^{***}(x^{**}, e^{**}) = x^{**}$. Let $x^{***} \in X^{***}$, $\{x_\alpha\} \subseteq \mathcal{X}$ and $\{x_\beta^*\} \subseteq \mathcal{X}^*$ be bounded nets that are w^*-convergent to x^{**} and x^{***}, respectively. Then,

$$\begin{aligned}
\langle J_{\mathcal{X}^{**}}(x^{**}), x^{***}\rangle &= \langle x^{***}, x^{**}\rangle \\
&= \langle x^{***}, \pi_2^{***}(x^{**}, e^{**})\rangle \\
&= \langle \pi_2^{****}(x^{***}, x^{**}), e^{**}\rangle \\
&= \langle \pi_2^{*r****r}(x^{***}, x^{**}), e^{**}\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\gamma} \langle \pi_2^{*}(x_\beta^{*}, x_\alpha), e_\gamma\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\gamma} \langle x_\beta^{*}, \pi_2(x_\alpha, e_\gamma)\rangle \\
&= \lim_{\alpha}\lim_{\beta} \langle x_\beta^{*}, x_\alpha\rangle \\
&= \lim_{\alpha}\lim_{\beta} \langle J_{\mathcal{X}}(x_\alpha), x_\beta^{*}\rangle
\end{aligned}$$

$$\begin{aligned} &= \lim_{\alpha}\langle x^{***}, J_X(x_\alpha)\rangle \\ &= \lim_{\alpha}\langle (J_{\mathcal{X}})^*(x^{***}), x_\alpha\rangle \\ &= \langle x^{**}, (J_{\mathcal{X}})^*(x^{***})\rangle \\ &= \langle (J_{\mathcal{X}})^{**}(x^{**}), x^{***}\rangle. \end{aligned}$$

Therefore, $x^{**} \in \mathfrak{M}_{\mathcal{X}}$, completing the proof of the equality $\mathfrak{M}_{\mathcal{X}} = Z_r(\pi_2^*)$. □

As a rapid consequence of theorems 2.2 and 3.4, in the next result we determine the topological centres of module actions of $\mathcal{A}$ on $\mathcal{A}^*$.

Proposition 3.5 (See [7, Theorem 2.1, Corollaries 2.1 and 2.4])**.** *For the multiplication π of a Banach algebra $\mathcal{A}$ with a bounded approximate identity, we have,*

$$Z_\ell(\pi^{r*r}) = \mathfrak{M}_{\mathcal{A}} = Z_r(\pi^*) \quad \text{and} \quad Z_r(\pi^{r*r}) = \mathcal{A}^* = Z_\ell(\pi^*).$$

In particular, π^{r*r} is regular if and only if π^* is regular if and only if $\mathcal{A}$ is reflexive.

In the following, we have a more illuminating example characterizing the right topological centers of π_1^{r*} and π_2^*.

Example 3.6. Let $\mathcal{A}$ be a Banach space such that $\mathcal{A} \subsetneq \mathfrak{M}_{\mathcal{A}} \subsetneq \mathcal{A}^{**}$ (such as c_0). Fix $e \in \mathcal{A}$ and $e^* \in \mathcal{A}^*$ such that $\| e^* \| \leq 1$ and $\langle e^*, e\rangle = 1$. Then, the multiplication $\pi_1(a, b) = \langle e^*, a\rangle b$ turns $\mathcal{A}$ into a Banach algebra with e as a left identity; similarly, $\pi_2(a, b) = \pi_1^r(a, b) = \langle e^*, b\rangle a$ turns $\mathcal{A}$ into a Banach algebra with e as a right identity (see [6, Example 4.7]). As theorems 2.2 and 3.4 demonstrate, we have,

$$Z_\ell({\pi_1}^{r*}) = \mathcal{A}^* = Z_\ell({\pi_2}^*) \quad \text{and} \quad Z_r({\pi_1}^{r*}) = \mathfrak{M}_{\mathcal{A}} = Z_r({\pi_2}^*).$$

Moreover, for $\pi_1^* = \pi_2^{r*} : \mathcal{A}^* \times \mathcal{A} \to \mathcal{A}^*$ one may verify that $\pi_1^*(a^*, a) = \langle e^*, a\rangle a^*$; and this equality reveals that π_1^* is regular. In other words, $Z_\ell({\pi_1}^*) = \mathcal{A}^{***}$ and $Z_r({\pi_1}^*) = \mathcal{A}^{**}$. Therefore, $\pi_1^* = \pi_2^{r*}$ is neither left nor right strongly irregular. Note that neither $(\mathcal{A}, \pi_1)$ nor $(\pi_2, \mathcal{A})$ is approximately unital.

4. **(Arens) regularity of factorizable Banach module actions**

A bounded bilinear mapping $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is said to factor if it is onto, that is $f(\mathcal{X} \times \mathcal{Y}) = \mathcal{Z}$. As a consequence of the so-called Cohen's Factorization Theorem (see for example [4]), every approximately unital (left or right) Banach $\mathcal{A}-$module $(\pi_1, \mathcal{X})$ or $(\mathcal{X}, \pi_2)$ factors. Moreover, as Proposition 2.1 demonstrates, in this case both $(\pi_1^{r***r}, \mathcal{X}^{**})$ and $(\mathcal{X}^{**}, \pi_2^{***})$ are unital and thus factor.

However, many natural occurring Banach modules which factor are approximately unital; but this is not the case, in general. For instance, one may refer to [11] (see also [4]), for a wide variety of Banach algebras and Banach modules which enjoy some types of factorization properties but are not approximately unital. Here, we present some miscellaneous results on the regularity of π_1^{r*} and π_2^{*} for the case where the module actions $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ are not necessarily approximately unital. It should be remarked that in the case where the involved module actions are approximately unital, then these results can be derived as straight corollaries of our results given in the former sections.

Proposition 4.1. *Let $\mathcal{A}$ be Arens regular and let $(\pi_1, \mathcal{X})$ and $(\mathcal{X}, \pi_2)$ be left and right Banach $\mathcal{A}-$modules, respectively.*

(i) If $(\pi_1^{r***r}, \mathcal{X}^{**})$ factors, then the regularity of π_1^{r*} implies that of π_1.

(ii) If $(\mathcal{X}^{**}, \pi_2^{***})$ factors, then the regularity of π_2^{*} implies that of π_2.

Proof. We only give a proof for (ii). For each $x^{**} \in \mathcal{X}^{**}$ there exist $y^{**} \in \mathcal{X}^{**}$ and $b^{**} \in \mathcal{A}^{**}$ such that $x^{**} = \pi_2^{***}(y^{**}, b^{**})$. Let $a^{**} \in A^{**}$ and let $\{a_\alpha\} \subseteq \mathcal{A}$, $\{b_\beta\} \subseteq \mathcal{A}$ and $\{y_\gamma\} \subseteq \mathcal{X}$ be bounded nets w^*-converging to a^{**}, b^{**} and y^{**}, respectively. Then, for each $x^* \in \mathcal{X}^*$, $\pi_2^{r*}(x^*, a_\alpha)$ converges, in w^*-topology, to $\pi_2^{r****}(x^*, a^{**})$ and

$$\begin{aligned}
\langle \pi_2^{r***r}(x^{**}, a^{**}), x^* \rangle &= \langle \pi_2^{r****}(x^*, a^{**}), \pi_2^{***}(y^{**}, b^{**}) \rangle \\
&= \langle \pi_2^{****}(\pi_2^{r****}(x^*, a^{**}), y^{**}), b^{**} \rangle \\
&= \langle \pi_2^{*r***r}(\pi_2^{r****}(x^*, a^{**}), y^{**}), b^{**} \rangle \\
&= \lim_\gamma \lim_\alpha \langle \pi_2^*(\pi_2^{r*}(x^*, a_\alpha), y_\gamma), b^{**} \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle \pi_2^*(\pi_2^{r*}(x^*, a_\alpha), y_\gamma), b_\beta \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle \pi_2^{r*}(x^*, a_\alpha), \pi_2(y_\gamma, b_\beta) \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle x^*, \pi_2(\pi_2(y_\gamma, b_\beta), a_\alpha) \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle x^*, \pi_2(y_\gamma, b_\beta a_\alpha) \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle \pi_2^*(x^*, y_\gamma), b_\beta a_\alpha \rangle \\
&= \lim_\gamma \langle \pi_2^*(x^*, y_\gamma), b^{**} \Diamond\, a^{**}) \rangle \\
&= \langle \pi_2^{***}(y^{**}, b^{**} \Diamond\, a^{**}), x^* \rangle \\
&= \langle \pi_2^{***}(y^{**}, b^{**} \Box\, a^{**}), x^* \rangle \\
&= \langle \pi_2^{***}(\pi_2^{***}(y^{**}, b^{**}), a^{**}), x^* \rangle \\
&= \langle \pi_2^{***}(x^{**}, a^{**}), x^* \rangle.
\end{aligned}$$

Therefore, $\pi_2^{r***r}(x^{**}, a^{**}) = \pi_2^{***}(x^{**}, a^{**})$, for all $a^{**} \in \mathcal{A}^{**}, x^{**} \in \mathcal{X}^{**}$, which meaning that π_2 is regular.

Proposition 4.2. *Let $\mathcal{A}$ be a Banach algebra and $(\pi_1, \mathcal{X}, \pi_2)$ be a Banach $\mathcal{A}-$module.*

(i) If $(\pi_1^{r***r}, \mathcal{X}^{**})$ factors, π_1 and π_1^{r*} are regular, then π_2 is regular.

(ii) If $(\mathcal{X}^{**}, \pi_2^{***})$ factors, π_2 and π_2^* are regular, then π_1 is regular.

Proof. As the proof is similar to that of Proposition 4.1, we only give a brief explanation for part (i). For each $x^{**} \in \mathcal{X}^{**}$, there exist $y^{**} \in \mathcal{X}^{**}$ and $b^{**} \in \mathcal{A}^{**}$ such that $x^{**} = \pi_1^{r***r}(b^{**}, y^{**})$. Let $a^{**} \in A^{**}$ and let $\{a_\alpha\} \subseteq \mathcal{A}$, $\{b_\beta\} \subseteq \mathcal{A}$ and $\{y_\gamma\} \subseteq \mathcal{X}$ be bounded nets, w^*-converging to a^{**}, b^{**} and y^{**}, respectively. Using the regularity of π_1^{r*} and π_1, for each $x^* \in \mathcal{X}^*$,

$$\begin{aligned}
\langle \pi_2^{r***r}(x^{**}, a^{**}), x^* \rangle &= \langle \pi_2^{r****}(x^*, a^{**}), \pi_1^{r***}(y^{**}, b^{**}) \rangle \\
&= \langle \pi_1^{r****}(\pi_2^{r****}(x^*, a^{**}), y^{**}), b^{**} \rangle \\
&= \langle \pi_1^{r*r****r}(\pi_2^{r****}(x^*, a^{**}), y^{**}), b^{**} \rangle \\
&= \lim_\gamma \lim_\alpha \langle \pi_1^{r*}(\pi_2^{r*}(x^*, a_\alpha), y_\gamma), b^{**} \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle \pi_1^{r*}(\pi_2^{r*}(x^*, a_\alpha), y_\gamma), b_\beta \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle \pi_2^{r*}(x^*, a_\alpha), \pi_1(b_\beta, y_\gamma) \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle x^*, \pi_2(\pi_1(b_\beta, y_\gamma), a_\alpha) \rangle \\
&= \lim_\gamma \lim_\alpha \lim_\beta \langle x^*, \pi_1(b_\beta, \pi_2(y_\gamma, a_\alpha)) \rangle \\
&= \langle \pi_1^{r***r}(b^{**}, \pi_2^{***}(y^{**}, a^{**})), x^* \rangle \\
&= \langle \pi_1^{***}(b^{**}, \pi_2^{***}(y^{**}, a^{**})), x^* \rangle \\
&= \langle \pi_2^{***}(x^{**}, a^{**}), x^* \rangle.
\end{aligned}$$

Therefore, π_2 is regular, as required. □

As an immediate consequence of propositions 4.1 and 4.2, we have the next corollaries.

Corollary 4.3. *Let $(\pi_1, \mathcal{X}, \pi_2)$ be a Banach $\mathcal{A}-$module such that $(\pi_1^{r***r}, \mathcal{X}^{**})$ and $(\mathcal{X}^{**}, \pi_2^{***})$ factor. We have:*
(i) If either π and π_1^{r*} or π_2 and π_2^* are regular, then so is π_1.
(ii) If either π and π_2^* or π_1 and π_1^{r*} are regular, then so is π_2.

Corollary 4.4. *Let $\mathcal{A}$ be (Arens) regular and let $(\pi_1, \mathcal{X}, \pi_2)$ be a Banach $\mathcal{A}-$module such that $(\pi_1^{r***r}, \mathcal{X}^{**})$ and $(\mathcal{X}^{**}, \pi_2^{***})$ factor. If either π_2^* or π_1^{r*} is regular, then both π_1 and π_2 are regular.*

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