

ITERATIVE METHODS FOR FINDING NEAREST COMMON FIXED POINTS OF A COUNTABLE FAMILY OF QUASI-LIPSCHITZIAN MAPPINGS

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Communicated by Heydar Radjavi

ABSTRACT. We prove a strong convergence result for a sequence generated by Halpern's type iteration for approximating a common fixed point of a countable family of quasi-Lipschitzian mappings in a real Hilbert space. Consequently, we apply our results to the problem of finding a common fixed point of asymptotically nonexpansive mappings, an equilibrium problem, and a variational inequality problem for continuous monotone mappings.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . A mapping $T : C \rightarrow C$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L -Lipschitzian. We denote by $F(T)$ the set of fixed points of T . A mapping T is said to be *quasi-Lipschitzian* if

MSC(2010): Primary: 47H09; Secondary: 47H10.

Keywords: Asymptotically nonexpansive mapping, equilibrium problem, quasi-Lipschitzian mapping, strong convergence theorem, variational inequality problem.

Received: 21 August 2010, Accepted: 24 May 2011.

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$F(T) \neq \emptyset$ and there exists a positive constant L such that

$$\|Tx - y\| \leq L\|x - y\| \quad \text{for all } x \in C \text{ and } y \in F(T).$$

In this case we also say that T is quasi- L -Lipschitzian.

Remark 1.1. *It follows directly from the definitions above that:*

- (i) *If T is L -Lipschitzian with $F(T) \neq \emptyset$, then T is quasi- L -Lipschitzian.*
- (ii) *If T is quasi- L_1 -Lipschitzian and $L_1 < L_2$, then T is quasi- L_2 -Lipschitzian.*
- (iii) *T is (quasi-) 1-Lipschitzian if and only if T is (quasi-) nonexpansive.*

Throughout the paper, we deal with quasi- L -Lipschitzian mappings where $L \geq 1$. There are many methods for approximating fixed points of mappings. In 1953, Mann [12] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$(1.1) \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [24]. Recently, the present authors [16, 17, 18, 19, 20, 21] extended the iteration (1.1) to obtain weak and strong convergence theorems for a countable family of (quasi-) L_n -Lipschitzian mappings $\{T_n\}$ with some appropriate additional conditions by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \quad \text{for all } n \in \mathbb{N},$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. In an infinite-dimensional Hilbert space, strong convergence of Mann iteration is not generally guaranteed [5]. Some attempts to construct an iteration method so that strong convergence is guaranteed have recently been made [3, 8, 10, 13, 14, 15, 27, 28, 29, 30]. Halpern [8] introduced the following iterative scheme for approximating a fixed point of T

$$(1.2) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $x, x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. This iteration process is called Halpern's type iteration. Strong convergence of this type iterative sequence was also studied by Wittmann [28]. In 1996, Bauschke [1] extended the iteration (1.2) to obtain strong convergence theorems

for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ by the following iteration:

$$(1.3) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) T_{n(\bmod N)} x_n \quad \text{for all } n \in \mathbb{N},$$

where $x, x_1 \in C$, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\bmod N$ takes values in $\{1, 2, \dots, N\}$. Recently, O'Hara et al. [22] extended the iteration (1.3) to obtain strong convergence theorems for a countable family of nonexpansive mappings.

In this paper, we establish strong convergence theorem for finding common fixed points of a countable family of quasi- L_n -Lipschitzian mappings in a real Hilbert space. As a consequence, several convergence theorems for quasi-nonexpansive mappings and asymptotically nonexpansive mappings are deduced. Finally, we apply our results to equilibrium problems and variational inequality problems for continuous monotone mappings.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$$

and

$$(2.1) \quad \|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. In particular,

$$(2.2) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$$

for all $x, y \in H$. We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x . It is also known that H satisfies:

- The *Opial's condition* [23], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

- If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that

$$(2.3) \quad \limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - x\|^2 + \|x - y\|^2 \quad \text{for all } y \in H.$$

Let C be a nonempty closed convex subset of H . Then, for any $x \in H$, there exists a unique nearest point in C , denoted by P_Cx , such that

$$\|x - P_Cx\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the *metric projection* of H onto C . We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_Cx \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C.$$

Lemma 2.1 ([29], Lemma 2.1). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\beta_n\}$ is a sequence of real numbers with $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

To deal with a family of mappings, the following conditions are introduced: Let K be a subset of a Banach space, let $\{T_n\}$ be a family of mappings of K into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. $\{T_n\}$ is said to satisfy

- (a) the *ZLC-condition* [31] if for each bounded sequence $\{z_n\}$ in K , there exists a family of nonexpansive mapping of K into itself \mathfrak{T} such that $\|T_n z_n - T(T_n z_n)\| \rightarrow 0$ for all $T \in \mathfrak{T}$ and $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathfrak{T}) \neq \emptyset$, where $F(\mathfrak{T})$ is the set of all common fixed points of all mappings in \mathfrak{T} ;
- (b) the *H-condition* [9] if for each bounded sequence $\{z_n\}$ in K ,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0 \quad \Rightarrow \quad \omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} F(T_n),$$

where $\omega_w\{z_n\}$ denotes the set of all weak subsequential limits of $\{z_n\}$.

Recall that a mapping T is *demi-closed at y* , if $x_n \rightharpoonup x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Lemma 2.2 ([7], Theorem 10.3). *Let K be a nonempty closed convex subset of a reflexive Banach space which satisfies Opial's condition and let T be a nonexpansive mapping of K into itself. Then $I - T$ is demi-closed at zero.*

Lemma 2.3. *Let K be a nonempty closed subset of a reflexive Banach space which satisfies Opial's condition and let $\{T_n\}$ be a family of mappings of K into itself which satisfies the ZLC-condition. Then $\{T_n\}$ satisfies the H-condition.*

Proof. Let $\{z_n\}$ be a bounded sequence in K such that

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0.$$

Since $\{T_n\}$ satisfies the ZLC-condition, there exists a family of nonexpansive mapping of K into itself \mathfrak{T} such that $\|T_n z_n - T(T_n z_n)\| \rightarrow 0$ for all $T \in \mathfrak{T}$ and $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathfrak{T}) \neq \emptyset$. Then

$$\begin{aligned} & \|z_{n+1} - T z_{n+1}\| \\ & \leq \|z_{n+1} - T_n z_n\| + \|T_n z_n - T(T_n z_n)\| + \|T(T_n z_n) - T z_{n+1}\| \\ & \leq 2\|z_{n+1} - T_n z_n\| + \|T_n z_n - T(T_n z_n)\| \rightarrow 0, \end{aligned}$$

for all $T \in \mathfrak{T}$. By Lemma 2.2, $I - T$ is demi-closed at zero. So, we get $\omega_w\{z_n\} \subset F(T)$ for all $T \in \mathfrak{T}$. This implies that $\{T_n\}$ satisfies the H-condition. \square

Lemma 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi- L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in F$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds). If $\{T_n\}$ satisfies the H-condition, then $\bigcap_{n=1}^{\infty} F(T_n)$ is closed and convex.*

Proof. It follows directly from [19, Lemma 2.8]. \square

3. Strong Convergence Theorems

In this section, using the Halpern's type iteration we obtain a strong convergence theorem for a countable family of quasi-Lipschitzian mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi- L_n -Lipschitzian mappings of C into itself with $L_n \geq 1$ and $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in F$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds). Assume that $\{\alpha_n\}$ is a sequence in $(0, 1]$ which satisfies the following conditions:*

- (C1) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (L) $\lim_{n \rightarrow \infty} \frac{L_n - 1}{\alpha_n} = 0$.

Let $\{x_n\}$ be a sequence in C defined as follows: $x, x_1 \in C$ and

$$(3.1) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \quad \text{for all } n \in \mathbb{N}.$$

If $\{T_n\}$ satisfies the H-condition, then the sequence $\{x_n\}$ converges strongly to $P_F x$, where P_F is the projection of H onto F .

Proof. Let $u \in F$. Since $\lim_{n \rightarrow \infty} \frac{L_n - 1}{\alpha_n} = 0$, there exists $N \in \mathbb{N}$ such that $\frac{L_n - 1}{\alpha_n} < \frac{1}{2}$ for all $n \geq N$. Choose a constant $M > 0$ so that

$$\|x_N - u\| \leq M \quad \text{and} \quad \|x - u\| \leq \frac{M}{2}.$$

We proceed by induction to show that $\|x_n - u\| \leq M$ for all $n \geq N$. Assume that $\|x_k - u\| \leq M$ for some $k \geq N$. From the iteration process (3.1), we estimate as follows:

$$\begin{aligned} \|x_{k+1} - u\| &\leq \alpha_k \|x - u\| + (1 - \alpha_k) \|T_k x_k - u\| \\ &\leq \alpha_k \|x - u\| + (1 - \alpha_k) L_k \|x_k - u\| \\ &= \alpha_k \|x - u\| + (1 - \alpha_k)(L_k - 1) \|x_k - u\| + (1 - \alpha_k) \|x_k - u\| \\ &\leq \alpha_k \frac{M}{2} + (1 - \alpha_k) \alpha_k \frac{M}{2} + (1 - \alpha_k) M \\ &\leq \alpha_k \frac{M}{2} + \alpha_k \frac{M}{2} + (1 - \alpha_k) M = M. \end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded and hence so is $\{T_n x_n\}$. So, from $\alpha_n \rightarrow 0$, we get

$$\|x_{n+1} - T_n x_n\| = \alpha_n \|x - T_n x_n\| \rightarrow 0.$$

Since $\{T_n\}$ satisfies the H-condition, $\omega_w \{x_n\} \subset F$. We next show

$$(3.2) \quad \limsup_{n \rightarrow \infty} \langle x - z, x_n - z \rangle \leq 0,$$

where $z := P_F x$. To this end, we choose a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle x - z, x_n - z \rangle = \lim_{i \rightarrow \infty} \langle x - z, x_{n_i} - z \rangle \quad \text{and} \quad x_{n_i} \rightharpoonup w \in F.$$

So, we get

$$\lim_{i \rightarrow \infty} \langle x - z, x_{n_i} - z \rangle = \langle x - z, w - z \rangle \leq 0.$$

Now (3.2) is proved. Finally we prove that $x_n \rightarrow z$. From (2.2), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(x - z) + (1 - \alpha_n)(T_n x_n - z)\|^2 \\ &\leq (1 - \alpha_n)^2 \|T_n x_n - z\|^2 + 2\alpha_n \langle x - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n)^2 L_n^2 \|x_n - z\|^2 + 2\alpha_n \langle x - z, x_{n+1} - z \rangle \\ &= (1 - \alpha_n)^2 \|x_n - z\|^2 + \alpha_n (1 - \alpha_n)^2 \frac{L_n^2 - 1}{\alpha_n} \|x_n - z\|^2 \\ &\quad + 2\alpha_n \langle x - z, x_{n+1} - z \rangle \\ &\leq (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \frac{L_n - 1}{\alpha_n} (L_n + 1) \|x_n - z\|^2 \\ &\quad + 2\alpha_n \langle x - z, x_{n+1} - z \rangle. \end{aligned}$$

Setting

$$\beta_n = \frac{L_n - 1}{\alpha_n} (L_n + 1) \|x_n - z\|^2 + 2 \langle x - z, x_{n+1} - z \rangle,$$

we get

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n) \|x_n - z\|^2 + \alpha_n \beta_n.$$

Since $\lim_{n \rightarrow \infty} \frac{L_n - 1}{\alpha_n} = 0$ by (3.2), $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. By Lemma 2.1, we conclude that $x_n \rightarrow z$. This completes the proof. \square

Remark 3.2. For a given family of quasi- L_n -Lipschitzian mappings, we can always find a sequence $\{\alpha_n\}$ in $(0, 1]$ such that the conditions (C1), (C2) and (L) are satisfied. In fact, if $L_n \rightarrow 1$, we can set $\alpha_n = \max \left\{ \frac{1}{n}, \frac{L_n - 1 + \sqrt{L_n - 1}}{L_n + \sqrt{L_n - 1}} \right\}$.

Setting $L_n \equiv 1$ in Theorem 3.1, we have the following.

Corollary 3.3. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into itself with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in F$, $\|T_n x - u\| \leq \|x - u\|$ holds). If $\{T_n\}$ satisfies the H-condition, then the sequence $\{x_n\}$ defined by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying (C1) and (C2), converges strongly to $P_F x$.

Remark 3.4. Corollary 3.3 extends and improves Theorem 2.1 of [31] in the following ways:

- (i) Since every nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive, Corollary 3.3 is applicable for a wider class of mappings.

- (ii) *The ZLC-condition is weakened and replaced by the H-condition (see Lemma 2.3).*

4. Applications

In this section, we show that the H-condition studied in the previous section is satisfied by various classes of mappings.

4.1. Convergence theorems for asymptotically nonexpansive mappings. Let C be a subset of a real Hilbert space H . A mapping $T : C \rightarrow C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers such that $k_n \in [1, \infty)$, $k_n \rightarrow 1$, and

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a natural generalization of the class of nonexpansive mappings. They proved that if C is nonempty bounded closed and convex, and T is an asymptotically nonexpansive self-mapping of C , then T has a fixed point.

Lemma 4.1 ([11], Lemma 2.2). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let S and T be two commutative asymptotically nonexpansive mappings of C into itself with asymptotical coefficients $\{s_n\}$ and $\{t_n\}$, respectively. For any $x \in C$ and $n \in \mathbb{N}$, put $R_n x = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x$. Then for each $r > 0$, there holds*

$$\lim_{i \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in C \cap B_r} \|R_n x - S^i(R_n x)\| = 0$$

and

$$\lim_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in C \cap B_r} \|R_n x - T^j(R_n x)\| = 0,$$

where $B_r = \{x \in H : \|x\| \leq r\}$.

From Lemma 4.1, we have the following result.

Lemma 4.2. *Let C, S, T, R_n be the same as Lemma 4.1. Assume that $F(S) \cap F(T) \neq \emptyset$. Then $\{R_n\}$ is a family of L_n -Lipschitzian mappings of C into itself and satisfies the H-condition, where*

$$L_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} s_i t_j.$$

Proof. It is easy to see that $\{R_n\}$ is a family of L_n -Lipschitzian mappings of C into itself. Moreover, by Lemma 4.1, we have $\bigcap_{n=1}^{\infty} F(R_n) = F(S) \cap F(T) \neq \emptyset$. Next, we prove that $\{R_n\}$ satisfies the H-condition. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_{n+1} - R_n z_n\| = 0$ and $z \in \omega_w\{z_n\}$. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $z_{n_k+1} \rightharpoonup z$ and $R_{n_k} z_{n_k} \rightharpoonup z$. Since $\{z_n\}$ is bounded, let $r > 0$ be such that $\{z_n\} \subset C \cap B_r$. Then

$$\begin{aligned} \|z_{n_k+1} - S^i z\| &\leq \|z_{n_k+1} - R_{n_k} z_{n_k}\| + \|R_{n_k} z_{n_k} - S^i(R_{n_k} z_{n_k})\| \\ &\quad + \|S^i(R_{n_k} z_{n_k}) - S^i z_{n_k+1}\| + \|S^i z_{n_k+1} - S^i z\| \\ &\leq (1 + s_i) \|z_{n_k+1} - R_{n_k} z_{n_k}\| + \sup_{x \in C \cap B_r} \|R_{n_k} x - S^i(R_{n_k} x)\| \\ &\quad + s_i \|z_{n_k+1} - z\|. \end{aligned}$$

It follows from $\lim_{i \rightarrow \infty} s_i = 1$ and Lemma 4.1 that

$$(4.1) \quad \limsup_{i \rightarrow \infty} \limsup_{k \rightarrow \infty} \|z_{n_k+1} - S^i z\| \leq \limsup_{k \rightarrow \infty} \|z_{n_k+1} - z\|.$$

From (2.3) and (2.1), we have

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \|z_{n_k+1} - z\|^2 + \left\| \frac{S^i z - z}{2} \right\|^2 \\ &= \limsup_{k \rightarrow \infty} \left\| z_{n_k+1} - \frac{S^i z + z}{2} \right\|^2 \\ &= \limsup_{k \rightarrow \infty} \left(\frac{1}{2} \|z_{n_k+1} - S^i z\|^2 + \frac{1}{2} \|z_{n_k+1} - z\|^2 - \frac{1}{4} \|S^i z - z\|^2 \right) \\ &\leq \frac{1}{2} \limsup_{k \rightarrow \infty} \|z_{n_k+1} - S^i z\|^2 + \frac{1}{2} \limsup_{k \rightarrow \infty} \|z_{n_k+1} - z\|^2 - \frac{1}{4} \|S^i z - z\|^2 \end{aligned}$$

and hence

$$\|S^i z - z\|^2 \leq \limsup_{k \rightarrow \infty} \|z_{n_k+1} - S^i z\|^2 - \limsup_{k \rightarrow \infty} \|z_{n_k+1} - z\|^2.$$

This together with (4.1) gives

$$\lim_{i \rightarrow \infty} \|S^i z - z\| = 0.$$

Since S is uniformly continuous, $Sz = z$ and then $z \in F(S)$. Similarly, we can get $z \in F(T)$. Hence $z \in F(S) \cap F(T) = \bigcap_{n=1}^{\infty} F(R_n)$. This implies that $\{R_n\}$ satisfies the H-condition. This completes the proof. \square

Applying Theorem 3.1 and Lemma 4.2, we have the following result.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let S and T be two commutative asymptotically non-expansive mappings of C into itself with asymptotical coefficients $\{s_n\}$ and $\{t_n\}$, respectively. Assume that $F := F(S) \cap F(T) \neq \emptyset$. Let $L_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} s_i t_j$ and let $\{x_n\}$ be a sequence in C defined as follows: $x, x_1 \in C$ and*

$$(4.2) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying (C1), (C2) and (L). Then the sequence $\{x_n\}$ converges strongly to $P_F x$.

Setting $L_n \equiv 1$ in Theorem 4.3, we have the following.

Corollary 4.4 ([25], Theorem 1). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let S and T be two commutative non-expansive mappings of C into itself with $F := F(S) \cap F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (4.2), where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying (C1) and (C2), converges strongly to $P_F x$.*

4.2. Some applications for the equilibrium problem. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction of $C \times C$ into \mathbb{R} , where \mathbb{R} is the set of real numbers. The equilibrium problem for $f : C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that

$$(4.3) \quad f(x, y) \geq 0 \quad \text{for all } y \in C.$$

Numerous problems in physics, optimization, and economics reduce to find a solution of (4.3). The set of solutions of (4.3) is denoted by $\text{EP}(f)$. In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme for finding the best approximation to the initial data when $\text{EP}(f)$ is not empty.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions which are generally assumed (see [2]):

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$;
- (A3) f is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} f(tz + (1-t)x, y) \leq f(x, y);$$

- (A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [2, Corollary 1] and [4, Lemma 2.12], we have the following lemma.

Lemma 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then there exists a unique $x^* \in C$ such that*

$$f(x^*, y) + \frac{1}{r} \langle y - x^*, x^* - x \rangle \geq 0 \quad \text{for all } y \in C.$$

Let T_r be a mapping of H into C defined by $T_r(x) = x^*$ for all $x \in H$. Then, the following hold:

(i) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|(T_r x - x) - (T_r y - y)\|^2;$$

(ii) $F(T_r) = \text{EP}(f)$;

(iii) $\text{EP}(f)$ is closed and convex.

We present a convergence theorem for an equilibrium problem with a new control parameter which is complementary to Song and Zheng's result [26, Corollary 5.3].

Lemma 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)-(A4) and $\text{EP}(f) \neq \emptyset$. If $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\lim_{n \rightarrow \infty} r_n = \infty$, then $\{T_{r_n}\}$ is a family of firmly nonexpansive mappings of H into C with $\bigcap_{n=1}^{\infty} F(T_{r_n}) = \text{EP}(f)$ and satisfies the H-condition.*

Proof. We note that $\bigcap_{n=1}^{\infty} F(T_{r_n}) = \text{EP}(f) \neq \emptyset$. Let $\{z_n\}$ be a bounded sequence in H such that $\lim_{n \rightarrow \infty} \|z_{n+1} - T_{r_n} z_n\| = 0$ and $z \in \omega_w \{z_n\}$. For each $n \in \mathbb{N}$, let $y_n = T_{r_n} z_n$. Then $z_{n_i+1} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$ for some subsequence $\{n_i\}$ of $\{n\}$. We note that $\{z_n - y_n\}$ is bounded. Since $\lim_{n \rightarrow \infty} r_n = \infty$, we have

$$(4.4) \quad \frac{z_n - y_n}{r_n} \rightarrow 0.$$

Notice that

$$f(y_n, y) + \frac{1}{r_n} \langle y - y_n, y_n - z_n \rangle \geq 0 \quad \text{for all } y \in C.$$

So, from (A2), we have

$$\left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle \geq f(y, y_n) \quad \text{for all } y \in C.$$

In particular,

$$\left\langle y - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{r_{n_i}} \right\rangle \geq f(y, y_{n_i}) \quad \text{for all } y \in C.$$

This together with (4.4), (A4) and $y_{n_i} \rightarrow z$ gives

$$0 \geq f(y, z) \quad \text{for all } y \in C.$$

Then, for $t \in (0, 1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty + (1-t)z, ty + (1-t)z) \\ &\leq tf(ty + (1-t)z, y) + (1-t)f(ty + (1-t)z, z) \\ &\leq tf(ty + (1-t)z, y) \end{aligned}$$

hence

$$f(ty + (1-t)z, y) \geq 0.$$

Letting $t \rightarrow 0^+$ and using (A3), we get

$$f(z, y) \geq 0 \quad \text{for all } y \in C$$

and hence $z \in \text{EP}(f) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. This implies that $\{T_{r_n}\}$ satisfies the H-condition. This completes the proof. \square

Using Corollary 3.3, we have the following theorem.

Theorem 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)-(A4) and $\text{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined as follows: $x, x_1 \in H$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_{r_n}x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying (C1) and (C2), and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{\text{EP}(f)}x$.

4.3. Some applications for the variational inequality problem.

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . Let $A : C \rightarrow H$ be a mapping. The classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \quad \text{for all } y \in C.$$

The set of solutions of classical variational inequality problem is denoted by $\text{VIP}(C, A)$.

The following lemma were also given in Nilsrakoo and Saejung [16].

Lemma 4.8 ([16], Lemmas 19, 20). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let A be a continuous monotone mapping of C into H , that is,*

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \text{for all } x, y \in C.$$

Define $f : C \times C \rightarrow \mathbb{R}$ as follows

$$f(x, y) = \langle Ax, y - x \rangle \quad \text{for all } x, y \in C.$$

Then, the following hold:

- (i) f satisfies (A1)-(A4) and $\text{VIP}(C, A) = \text{EP}(f)$;
- (ii) for $x \in H$, $u \in C$ and $r > 0$,

$$f(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0 \quad \text{for all } y \in C \quad \Leftrightarrow \quad u = P_C(x - rAu).$$

Using Theorem 4.7 and Lemma 4.8, we have the following theorem.

Theorem 4.9. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let A be a continuous monotone mapping of C into H such that $\text{VIP}(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x, x_1 \in C$ and*

$$\begin{cases} u_n = P_C(x_n - r_n Au_n) \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) u_n \end{cases} \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$ satisfying (C1) and (C2), and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\lim_{n \rightarrow \infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{\text{VIP}(C, A)}x$.

5. Conclusion

We propose the Halpern's type iteration to obtain a strong convergence theorem for a common fixed point of a countable family of certain quasi-Lipschitzian mappings in a real Hilbert space. We assume that the family of mappings satisfies the H-condition introduced by Hirstoaga in [9]. This is not restrictive because there are many examples satisfying the H-condition. Applications for equilibrium problems and variational inequality problems are also discussed.

Acknowledgments

The authors thank the referee and the editor for their comments and helpful suggestions. The research work of the second author was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission, through the Cluster of Research to Enhance the Quality of Basic Education.

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