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ON GENERALIZED LEFT (α, β)-DERIVATIONS IN RINGS

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Abstract. Let R be a 2-torsion free ring and let U be a square closed Lie ideal of R. Suppose that α, β are automorphisms of R. An additive mapping $\delta : R \longrightarrow R$ is said to be a Jordan left (α, β)-derivation of R if $\delta(x^2) = \alpha(x)\delta(x) + \beta(x)\delta(x)$ holds for all $x \in R$. In this paper it is established that if R admits an additive mapping $G : R \longrightarrow R$ satisfying $G(u^2) = \alpha(u)G(u) + \alpha(u)\delta(u)$ for all $u \in U$ and a Jordan left (α, α)-derivation δ; and U has a commutator which is not a left zero divisor, then $G(uv) = \alpha(u)G(v) + \alpha(v)\delta(u)$ for all $u, v \in U$. Finally, in the case of prime ring R it is proved that if $G : R \longrightarrow R$ is an additive mapping satisfying $G(xy) = \alpha(x)G(y) + \beta(y)\delta(x)$ for all $x, y \in R$ and a left (α, β)-derivation δ of R such that G also acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then either R is commutative or $\delta = 0$ on R.

1. **Introduction**

Throughout the present paper, R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$(respectively, $x \circ y$) will denote the commutator $xy - yx$ (respectively, the anti-commutator $xy + yx$). Recall that a ring R is prime if $aRb = \{0\}$

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implies $a = 0$ or $b = 0$. An additive subgroup U of R is said to be a Lie ideal of R if $[U, R] \subseteq U$. A Lie ideal U of R is said to be a square closed Lie ideal if $u^2 \in U$ for all $u \in U$. If $u^2 \in U$ for all $u \in U$, then $uv + vu = (u+v)^2 - u^2 - v^2 \in U$ and $uv - vu \in U$. Hence $2uv \in U$ for all $u, v \in U$. This remark will be freely used throughout the paper. Let α, β be endomorphisms of R. An additive mapping $\delta : R \longrightarrow R$ is said to be a left (α, β)-derivation (respectively, Jordan left (α, β)-derivation) of R if $\delta(xy) = \alpha(x)\delta(y) + \beta(y)\delta(x)$ (respectively, $\delta(x^2) = \alpha(x)\delta(x) + \beta(x)\delta(x)$) holds for all $x, y \in R$. Of course, a left (I, I)-derivation (respectively, Jordan left (I, I)-derivation), where I is the identity map on R, is said to be a left derivation (respectively, Jordan left derivation) of R. The study of left derivation was initiated by Bresar and Vukman in [7] and it was shown that if a prime ring R of characteristic different from 2 and 3 admits a nonzero Jordan left derivation then R must be commutative. Following [8], an additive mapping $F : R \longrightarrow R$ is called a generalized derivation of R if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Inspired by the definition of generalized derivation, Ashraf and Shakir [3] introduced the concepts of generalized left derivation and generalized Jordan left derivation as follows: an additive mapping $G : R \longrightarrow R$ is called a generalized left derivation (respectively, generalized Jordan left derivation) if there exists a Jordan left derivation $\delta : R \longrightarrow R$ such that $G(xy) = xG(y) + y\delta(x)$ (respectively, $G(x^2) = xG(x) + x\delta(x)$) holds for all $x, y \in R$. Motivated by the above definition, we introduce the concept of generalized left (α, β)-derivation and generalized Jordan left (α, β)-derivation as follows: an additive mapping $G : R \longrightarrow R$ is said to be a generalized left (α, β)-derivation (respectively, generalized Jordan left (α, β)-derivation) if there exists a Jordan left (α, β)-derivation $\delta : R \longrightarrow R$ such that $G(xy) = \alpha(x)G(y) + \beta(y)\delta(x)$ (respectively, $G(x^2) = \alpha(x)G(x) + \beta(x)\delta(x)$) holds for all $x, y \in R$. The definition of generalized right (α, β)-derivation (respectively, generalized Jordan right (α, β)-derivation) is self-explanatory. In Section 2, it is shown that every generalized Jordan left (α, α)-derivation on R is a generalized left (α, α)-derivation if the underlying ring R is 2-torsion free and has a commutator which is not a left zero divisor in R. Moreover, in this section we also prove that if U is a square closed Lie ideal of a prime ring R of characteristic different from 2 and $\delta : R \longrightarrow R$ is a Jordan left (α, α)-derivation of R such that R admits an additive mapping $G : R \longrightarrow R$ satisfying $G(uv) = \alpha(u)G(v) + \alpha(v)\delta(u)$ for all $u, v \in U$, then either $\delta(U)=\{0\}$ or $U\subseteq Z(R)$.

A derivation $d: R\longrightarrow R$ is said to act as a homomorphism (respectively, anti-homomorphism) on a non-empty subset S of R if $d(xy)=d(x)d(y)$(respectively, $d(xy)=d(y)d(x)$) holds for all $x,y\in S$. The last section of this paper deals with the study of generalized left (α,β)-derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R. The result of this section generalizes the results obtained in [1] and [2].

2. Generalized Jordan left (α,β)-derivation

In an attempt to generalize the result obtained by Bresar and Vukman [7], the first author established that a 2-torsion free prime ring R which admits a nonzero Jordan left (α,α)-derivation must be commutative. Further, as an application of this result, it was shown that if R is a 2-torsion free ring and has a commutator which is not a left zero divisor, then every Jordan left (α,α)-derivation is a left (α,α)-derivation (see [2, Theorem 3.3]). It is obvious to see that every generalized left (α,β)-derivation on a ring R is a generalized Jordan left (α,β)-derivation of R but the converse need not be true in general.

Example 2.1. *Let S be a ring such that square of each element in S is zero, but the product of some nonzero elements in S is nonzero. Next, let $R=\left\{\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix} | x,y\in S\right\}$. Define maps $G,\delta : R\longrightarrow R$ and $\alpha,\ \beta : R\longrightarrow R$ as follows:*

$$G\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & y\\ 0 & 0\end{pmatrix},\ \delta\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix} x & 0\\ 0 & 0\end{pmatrix},$$

and

$$\alpha\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix} x & -y\\ 0 & 0\end{pmatrix},\ \beta\begin{pmatrix} x & y\\ 0 & 0\end{pmatrix}=\begin{pmatrix} -x & -y\\ 0 & 0\end{pmatrix}.$$

Then it is straightforward to check that G is a generalized Jordan left (α,β)- derivation but not a generalized left (α,β)- derivation(for a nonzero left (α,β)-derivation δ).

In the present section our aim is to establish the conditions under which the converse of the above statement is true.

Theorem 2.2. *Let R be a 2-torsion free ring and let U be a square closed Lie ideal of R. Suppose that α is an automorphism of R and $\delta : R \longrightarrow R$ is a Jordan left (α, α)-derivation of R. Suppose further that U has a commutator which is not a left zero divisor. If $G : R \longrightarrow R$ is an additive mapping satisfying $G(u^2) = \alpha(u)G(u) + \alpha(u)\delta(u)$ for all $u \in U$, then $G(uv) = \alpha(u)G(v) + \alpha(v)\delta(u)$ for all $u, v \in U$.*

We begin our discussion with the following known lemmas. Lemma 2.3 is essentially proved in [6] while the proof of Lemma 2.4 runs exactly on the same lines as that of Lemma 2.3 of [4]. We skip the details of the proof just to avoid repetition.

Lemma 2.3. *Let R be a prime ring such that $char R \neq 2$, and let U be a Lie ideal of R such that $U \not\subseteq Z(R)$. If $a, b \in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2.4. *Let R be a 2-torsion free ring and let U be a square closed Lie ideal of R. Suppose that α is an endomorphism of R and $\delta : R \longrightarrow R$ is an additive mapping satisfying $\delta(u^2) = 2\alpha(u)\delta(u)$ for all $u \in U$. Then for all $u, v \in U$*

(i) $\alpha([u, v])\delta([u, v]) = 0$,
(ii) $\alpha(u^2v - 2uvu + vu^2)\delta(v) = 0$.

Lemma 2.5. *Let R be a 2-torsion free ring and let U be a square closed Lie ideal of R. Suppose that α is an endomorphism of R and $\delta : R \longrightarrow R$ is a Jordan left (α, α)-derivation of R. If $G : R \longrightarrow R$ is an additive mapping satisfying $G(u^2) = \alpha(u)G(u) + \alpha(u)\delta(u)$ for all $u \in U$, then for all $u, v, w \in U$*

$$\begin{aligned}
(i)\ G(uv + vu) &= \alpha(u)G(v) + \alpha(v)G(u) + \alpha(u)\delta(v) + \alpha(v)\delta(u),\\
(ii)\ G(uvu) &= \alpha(uv)G(u) + 2\alpha(uv)\delta(u) + \alpha(u^2)\delta(v)\\
&\quad -\alpha(vu)\delta(u),\\
(iii)\ G(uvw + wvu) &= \alpha(uv)G(w) + \alpha(wv)G(u) + 2\alpha(uv)\delta(w)\\
&\quad +2\alpha(wv)\delta(u) + \alpha(uw)\delta(v) + \alpha(wu)\delta(v)\\
&\quad -\alpha(vu)\delta(w) - \alpha(vw)\delta(u).
\end{aligned}$$

Proof. (i) We have

$$G(u^2)=\alpha(u)G(u)+\alpha(u)\delta(u) \text{ for all } u\in U. \tag{2.1}$$

Linearizing (2.1), we get the required result.

(ii) Since $uv+vu=(u+v)^2-u^2-v^2\in U$, replacing v by $uv+vu$ in (i), we get

$$\begin{aligned} G(u(uv+vu)+(uv+vu)u) &= \alpha(u)G(uv+vu)+\alpha(uv+vu)G(u)\\ &\quad +\alpha(u)\delta(uv+vu)+\alpha(uv+vu)\delta(u). \end{aligned} \tag{2.2}$$

Since δ is a Jordan left (α,α)-derivation, $\delta(u^2)=2\alpha(u)\delta(u)$ and hence linearizing this relation, we find that $\delta(uv+vu)=2\alpha(u)\delta(v)+2\alpha(v)\delta(u)$ for all $u,v\in U$. Now using relation (i) in (2.2) , we find that

$$\begin{aligned} G(u(uv+vu)+(uv+vu)u) &= \alpha(u^2)G(v)+2\alpha(uv)G(u)\\ &\quad +\alpha(vu)G(u)+4\alpha(uv)\delta(u)\\ &\quad +3\alpha(u^2)\delta(v)+\alpha(vu)\delta(u). \end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned} G(u(uv+vu)+(uv+vu)u) &= 2G(uvu)+G(u^2v+vu^2)\\ &= 2G(uvu)+\alpha(u^2)G(v)+\alpha(vu)G(u)\\ &\quad +\alpha(vu)\delta(u)+\alpha(u^2)\delta(v)\\ &\quad +2\alpha(vu)\delta(u). \end{aligned} \tag{2.4}$$

Comparing (2.3) and (2.4), we get the required result.

(iii) Linearizing (ii), we find that

$$\begin{aligned} G((u+w)v(u+w)) &= \alpha(uv)G(u)+\alpha(uv)G(w)+\alpha(wv)G(u)\\ &\quad +\alpha(wv)G(w)+2\alpha(uv)\delta(u)+2\alpha(uv)\delta(w)\\ &\quad +2\alpha(wv)\delta(u)+2\alpha(wv)\delta(w)+\alpha(u^2)\delta(v)\\ &\quad +\alpha(uw)\delta(v)+\alpha(wu)\delta(v)+\alpha(w^2)\delta(v)\\ &\quad -\alpha(vu)\delta(u)-\alpha(vu)\delta(w)\\ &\quad -\alpha(vw)\delta(u)-\alpha(vw)\delta(w). \end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned} G((u+w)v(u+w)) &= G(uvu)+G(wvw)+G(uvw+wvu) \\ &= \alpha(uv)G(u)+2\alpha(uv)\delta(u)+\alpha(u^2)\delta(v) \\ &\quad -\alpha(vu)\delta(u)+\alpha(wv)G(w)+2\alpha(wv)\delta(w) \\ &\quad +\alpha(w^2)\delta(v)-\alpha(vw)\delta(w)+G(uvw+wvu). \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we get the required result. □

We are now well equipped to prove our theorem:

Proof of Theorem 2.2. Replacing w by $uv - vu$ in part (iii) of Lemma 2.5 , we get

$$\begin{aligned} G(uv(uv-vu)+(uv-vu)vu) &= \alpha(uv)G(uv)-\alpha(uv)G(vu) \\ &\quad +\alpha([u,v])\alpha(v)G(u) \\ &\quad +\alpha([u,v])\delta([u,v])+\alpha(uv)\delta([u,v]) \\ &\quad +2\alpha([u,v])\alpha(v)\delta(u) \\ &\quad +\alpha(u)\alpha([u,v])\delta(v) \\ &\quad +\alpha([u,v])\alpha(u)\delta(v) \\ &\quad -\alpha(v)\alpha([u,v])\delta(u). \end{aligned}$$

Using Lemma 2.4 (i) in the above relation we have

$$\begin{aligned} G(uv(uv-vu)+(uv-vu)vu) &= \alpha(uv)G(uv)-\alpha(uv)G(vu) \\ &\quad +\alpha([u,v])\alpha(v)G(u) \\ &\quad +\alpha(uv)\delta([u,v]) \\ &\quad +2\alpha([u,v])\alpha(v)\delta(u) \\ &\quad +\alpha(u)\alpha([u,v])\delta(v) \\ &\quad +\alpha([u,v])\alpha(u)\delta(v) \\ &\quad -\alpha(v)\alpha([u,v])\delta(u). \end{aligned}$$

Adding and substracting $\alpha(v)\alpha([u,v])\delta(u)$ in the right hand side of the above relation, we get

$$\begin{aligned} G(uv(uv-vu)+(uv-vu)vu) &= \alpha(uv)G(uv)-\alpha(uv)G(vu) \\ &\quad +\alpha([u,v])\alpha(v)G(u)+\alpha(uv)\delta([u,v]) \\ &\quad +2\alpha([u,v])\alpha(v)\delta(u) \\ &\quad +\alpha(u)\alpha([u,v])\delta(v)) \\ &\quad +\alpha([u,v])\alpha(u)\delta(v) \\ &\quad -2\alpha(v)\alpha([u,v])\delta(u) \\ &\quad +\alpha(v)\alpha([u,v])\delta(u). \end{aligned} \tag{2.7}$$

Since

$$\begin{aligned} 4G(uv(uv-vu)+(uv-vu)vu) &= G((2uv)^2-(2vu)^2) \\ &= 4\{\alpha(uv)G(uv)+\alpha(uv)\delta(uv) \\ &\quad -\alpha(vu)G(vu)-\alpha(vu)\delta(vu)\} \end{aligned}$$

and R is 2-torsion free, the above relation yields that

$$\begin{aligned} G(uv(uv-vu)+(uv-vu)vu) &= \alpha(uv)G(uv)+\alpha(uv)\delta(uv) \\ &\quad -\alpha(vu)G(vu)-\alpha(vu)\delta(vu). \end{aligned} \tag{2.8}$$

Comparing (2.7) and (2.8) , we find that

$$\begin{aligned} 0 &= \alpha([v,u])G(vu)+\alpha([u,v])\alpha(v)G(u)+\alpha([u,v])\alpha(u)\delta(v) \\ &\quad +2\alpha([u,v])\alpha(v)\delta(u)-2\alpha(v)\alpha([u,v])\delta(u)+\alpha(u)\alpha([u,v])\delta(v) \\ &\quad +\alpha(v)\alpha([u,v])\delta(u)+\alpha(vu)\delta(vu)-\alpha(uv)\delta(vu). \end{aligned} \tag{2.9}$$

In view of [2, Theorem 3.3], every Jordan left (α,α)-derivation is a left (α,α)-derivation. Hence by using Lemma 2.4(ii) in (2.9), we have

$$\begin{aligned} 0 &= \alpha(u)\alpha([u,v])\delta(v)+\alpha(v)\alpha([u,v])\delta(u)+\alpha(vu)\delta(vu) \\ &\quad -\alpha(uv)\delta(vu) \\ &= \alpha(u^2v-2uvu+vu^2)\delta(v)-\alpha(v^2u-2vuv+uv^2)\delta(u). \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} 0 &= 2\alpha([u,v])\alpha(v)\delta(u)-2\alpha(v)\alpha([u,v])\delta(u) \\ &= 2\alpha(v^2u-2vuv+uv^2)\delta(u). \end{aligned} \tag{2.11}$$

Now in view of (2.9) , (2.10) and (2.11) , we find that

$$\alpha([v,u])G(vu)+\alpha([u,v])\alpha(v)G(u)+\alpha([u,v])\alpha(u)\delta(v)=0 \text{ for all } u,v\in U.$$

This implies that $\alpha([u,v])(G(uv)-\alpha(u)G(v)-\alpha(v)\delta(u))=0$ for all $u,v\in U$. Now define a map $H:U\times U\longrightarrow R$ such that $H(u,v)=G(uv)-\alpha(u)G(v)-\alpha(v)\delta(u)$. Since G and δ both are additive, we find that H is additive in both arguments. Hence the latter relation can be written as $\alpha([u,v])H(u,v)=0$ for all $u,v\in U$. Since α is an automorphism, we find that

$$[u,v]\alpha^{-1}(H(u,v))=0 \text{ for all } u,v\in U. \tag{2.12}$$

Now let a,b be fixed elements of U such that $[a,b]c=0$ implies that $c=0$. Then (2.11) yields that $\alpha^{-1}(H(a,b))=0$, and hence

$$H(a,b)=0. \tag{2.13}$$

Replacing u by $u+a$ in (2.12) and using (2.12) , we get

$$(2.14)\qquad [u,v]\alpha^{-1}(H(a,v))+[a,v]\alpha^{-1}(H(u,v))=0.$$

Again replace v by b in (2.14) , to get $[a,b]\alpha^{-1}(H(u,b))=0$. Since $[a,b]$ is not a left zero divisor, we have

$$(2.15)\qquad \alpha^{-1}(H(u,b))=0 \text{ for all } u\in U.$$

Replacing v by $v+b$ in (2.14) and using (2.13) , (2.14) and (2.15) , we get

$$(2.16)\qquad [a,b]\alpha^{-1}(H(u,v))+[u,b]\alpha^{-1}(H(a,v))=0.$$

Substituting a for u in (2.16) and using the fact that R is 2-torsion free, we get $[a,b]\alpha^{-1}(H(a,v))=0$ and hence

$$(2.17)\qquad \alpha^{-1}(H(a,v))=0.$$

Comparing (2.16) and (2.17), we have $[a,b]\alpha^{-1}(H(u,v))=0$ for all $u,v\in U$ and hence $H(u,v)=0$ for all $u,v\in U$. This completes the proof of our theorem. □

Corollary 2.6. *Let R be a 2-torsion free ring. Suppose that α is an automorphism of R and R has a commutator which is not a left zero divisor. If $\delta: R\longrightarrow R$ is a Jordan left (α,α)-derivation of R, then δ is a left (α,α)-derivation of R.*

Theorem 2.7. *Let R be a prime ring such that $char(R)\neq 2$ and U be a square closed Lie ideal of R. Let α be an automorphism of R and $\delta: R\longrightarrow R$ be a Jordan left (α,α)-derivation of R. If $G: R\longrightarrow R$ is an additive mapping satisfying $G(uv)=\alpha(u)G(v)+\alpha(v)\delta(u)$ for all $u,v\in U$, then either $\delta(U)=\{0\}$ or $U\subseteq Z(R)$.*

Proof. Let us suppose that $U\not\subseteq Z(R)$. We have

$$G(uv)=\alpha(u)G(v)+\alpha(v)\delta(u) \text{ for all } u\in U.$$

Replacing u by u^2 in the above relation, we have

$$(2.18)\qquad G(u^2v)=\alpha(u^2)G(v)+2\alpha(vu)\delta(u) \text{ for all } u\in U.$$

On the other hand,

$$\begin{aligned} 2G(u^2v) &= G(u(2uv)) \\ &= 2\{\alpha(u^2)G(v)+2\alpha(uv)\delta(u)\}. \end{aligned}$$

Since $char(R)\neq 2$, we get

$$(2.19)\qquad G(u^2v)=\alpha(u^2)G(v)+2\alpha(uv)\delta(u).$$

Comparing (2.18) and (2.19), we get $2\alpha([u, v])\delta(u) = 0$ for all $u, v \in U$. Since $char(R) \neq 2$ and α is an automorphism, we get

$$[u, v]\alpha^{-1}(\delta(u)) = 0.$$

Replacing v by $2vw$ in the above expression for any $w \in U$, we find that $[u, v]w\alpha^{-1}(\delta(u)) = 0$ for all $u, v, w \in U$. This implies that $[u, v]U\alpha^{-1}(\delta(u)) = \{0\}$. By Lemma 2.3, for each fixed $u \in U$ either $[u, v] = 0$ or $\alpha^{-1}(\delta(u)) = 0$ for all $v \in U$. Now we put $A = \{u \in U \mid [u, v] = 0 \text{ for any } v \in U\}$ and $B = \{u \in U \mid \alpha^{-1}(\delta(u)) = 0\}$. Clearly A and B are additive subgroups of U whose union is U and hence by Brauer's trick either $U = A$ or $U = B$. If $U = A$, then $[u, v] = 0$ for all $u, v \in U$ and hence U is commutative. If U is commutative then using similar arguments as used in the last paragraph of the proof of Lemma 1.3 of Herstein [9]; it can be easily seen that U is central, i.e., $U \subseteq Z(R)$, a contradiction. On the other hand, we have $\alpha^{-1}(\delta(u)) = 0$ for all $u \in U$. Since α is an automorphism, the last relation forces that $\delta(u) = 0$ for all $u \in U$, i.e., $\delta(U) = \{0\}$. □

Remark 2.8. *The results of this section are still open for generalized Jordan left (α, β)-derivations in rings.*

3. Generalized left (α, β)-derivation

Let S be a nonempty subset of a ring R and $d : R \longrightarrow R$ a derivation of R. If $d(xy) = d(x)d(y)$(respectively, $d(xy) = d(y)d(x)$) holds for all $x, y \in S$, then d is said to act as a homomorphism (respectively, anti-homomorphism) on S. In the year 1989, Bell and Kappe [5] proved that if K is a nonzero right ideal of a prime ring R and $d : R \longrightarrow R$ is a derivation of R such that d acts as a homomorphism or as an anti-homomorphism on K, then $d = 0$ on R. In [2], Ashraf proved that if $\delta : R \longrightarrow R$ is a left (α, β)-derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then $\delta = 0$ on R. This result was further extended for generalized left derivations in [1]. Now in this section we study generalized left (α, β)-derivations of a prime ring R with associated left (α, β)-derivation δ, which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal of R. The main result of this section generalizes the results obtained in [1] and [2].

Theorem 3.1. *Let R be a prime ring and let I be a nonzero ideal of R. Suppose that α, β are automorphisms of R and $G : R \longrightarrow R$ is a generalized left (α, β)-derivation of R with associated left (α, β)-derivation δ.*

(*i*) *If G acts as a homomorphism on I, then either R is commutative or $\delta = 0$ on R.*

(*ii*) *If G acts as an anti-homomorphism on I, then either R is commutative or $\delta = 0$ on R.*

Proof. (*i*) We have

$$\begin{aligned} G(uv) &= G(u)G(v) \\ &= \alpha(u)G(v) + \beta(v)\delta(u) \text{ for all } u, v \in I. \end{aligned} \tag{3.1}$$

Using (3.1) we have

$$\begin{aligned} G(uvw) &= G(u(vw)) \\ &= \alpha(u)G(vw) + \beta(vw)\delta(u) \text{ for all } u, v, w \in I. \end{aligned} \tag{3.2}$$

On the other hand, we find that

$$\begin{aligned} G(uvw) &= G((uv)w)) \\ &= G(uv)G(w) = \alpha(u)G(v)G(w) + \beta(v)\delta(u)G(w). \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) and using (3.1), we get

$$\beta(vw)\delta(u) = \beta(v)\delta(u)G(w) \text{ for all } u, v, w \in I. \tag{3.4}$$

This implies that $\beta(v)\{\beta(w)\delta(u) - \delta(u)G(w)\} = 0$ for all $u, v, w \in I$. This can be written as $v\beta^{-1}\{\beta(w)\delta(u) - \delta(u)G(w)\} = 0$ for all $u, v, w \in I$. Now replacing v by vr for any $r \in R$, we find that

$$vR\beta^{-1}(\beta(w)\delta(u) - \delta(u)G(w)) = \{0\} \text{ for all } u, v, w \in I.$$

Since I is nonzero and R is prime, the last expression gives that

$$\beta(w)\delta(u) = \delta(u)G(w) \text{ for all } u, w \in I. \tag{3.5}$$

Replacing u by uv for any $v \in I$ in (3.5), we have

$$\begin{aligned} \beta(w)\alpha(u)\delta(v) + \beta(w)\beta(v)\delta(u) &= \alpha(u)\delta(v)G(w) \\ &\quad + \beta(v)\delta(u)G(w). \end{aligned} \tag{3.6}$$

Using (3.5) in (3.6), we find that

$$[\beta(w), \beta(v)]\delta(u) + [\beta(w), \alpha(u)]\delta(v) = 0. \tag{3.7}$$

Hence in particular, we find that

$$[\beta(v), \alpha(u)]\delta(v) = 0 \text{ for all } u, v \in I. \tag{3.8}$$

Replacing u by ru in (3.8) for any $r \in R$ and using (3.8) in the relation so obtained, we get

$$[\beta(v), \alpha(r)]\alpha(u)\delta(v) = 0 \text{ for all } u, v \in I.$$

The above relation implies that $\alpha^{-1}([\beta(v), \alpha(r)])u\alpha^{-1}(\delta(v)) = 0$ for all $u, v \in I$ and $r \in R$. This can be rewritten as $\alpha^{-1}([\beta(v), \alpha(r)])IR\alpha^{-1}(\delta(v)) = \{0\}$ for all $v \in I$ and $r \in R$. Since R is prime, we find that for each fixed $v \in I$ either $\alpha^{-1}([\beta(v), \alpha(r)])I = \{0\}$ or $\alpha^{-1}(\delta(v)) = 0$ for all $r \in R$. Now if we put $A = \{v \in I \mid \alpha^{-1}([\beta(v), \alpha(r)])I = \{0\} \text{ for all } r \in R\}$ and $B = \{v \in I \mid \alpha^{-1}(\delta(v)) = 0\}$. Then clearly A and B are additive subgroups of I whose union is I. Hence either $A = I$ or $B = I$. If $A = I$, we find that $\alpha^{-1}([\beta(v), r'])I = \{0\}$ for every $v \in I$ and $r' \in R$. This shows that $\alpha^{-1}([\beta(v), r'])RI = \{0\}$. This implies that $[\beta(v), r'] = 0$, as α is an automorphism of R and $I \neq \{0\}$. Since β is an automorphism, this implies that I is central and hence R is commutative. If $B = I$, then $\alpha^{-1}(\delta(v)) = 0$ for all $v \in I$. Since α is an automorphism, we find that $\delta(v) = 0$ for all $v \in I$. Thus for any $r \in R$, $\delta(rv) = 0$, i.e., $\beta(v)\delta(r) = 0$ or $I\beta^{-1}(\delta(r)) = \{0\}$. Since I is nonzero, the last relation yields that $\delta(r) = 0$, i.e., $\delta = 0$ on R.

(ii) We have

$$\begin{aligned} G(uv) &= G(v)G(u) \\ &= \alpha(u)G(v) + \beta(v)\delta(u) \text{ for all } u, v \in I. \end{aligned} \tag{3.9}$$

Replacing v by uv in (3.9), we have

$$\begin{aligned} G(u^2v) &= G(uv)G(u) \\ &= \alpha(u)G(uv) + \beta(uv)\delta(u) \text{ for all } u, v \in I. \end{aligned} \tag{3.10}$$

Using (3.9) in (3.10), we find that

$$\begin{aligned} &\alpha(u)G(v)G(u) + \beta(v)\delta(u)G(u) \\ &= \alpha(u)G(uv) + \beta(uv)\delta(u). \end{aligned} \tag{3.11}$$

Again using (3.9) in (3.11), we get

$$\beta(uv)\delta(u) = \beta(v)\delta(u)G(u) \text{ for all } u, v \in I. \tag{3.12}$$

Replacing v by rv for any $r \in R$ in (3.12), we obtain that

$$\beta(u)\beta(r)\beta(v)\delta(u) = \beta(r)\beta(v)\delta(u)G(u) \text{ for all } u, v \in I. \tag{3.13}$$

Multiplying (3.12) by $\beta(r)$ from the left, we have

$$\beta(r)\beta(u)\beta(v)\delta(u) = \beta(r)\beta(v)\delta(u)G(u). \tag{3.14}$$

Comparing (3.13) and (3.14), we obtain

$$[\beta(u), \beta(r)]\beta(v)\delta(u) = 0 \text{ for all } u, v \in I,\ r \in R.$$

The last expression can be rewritten as $[u, r]I\beta^{-1}(\delta(u)) = \{0\}$, *i.e.*, $[u, r]IR\beta^{-1}(\delta(u)) = \{0\}$ for all $u \in I$ and $r \in R$. This implies that for each fixed $u \in I$ either $[u, r]I = \{0\}$ or $\delta(u) = 0$. Now using similar techniques as above, we get the required result. □

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