A GENERALIZATION OF THE PROBABILITY THAT THE COMMUTATOR OF TWO GROUP ELEMENTS IS EQUAL TO A GIVEN ELEMENT

A. M. A. ALGHAMDI AND F. G. RUSSO*

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ABSTRACT. The probability that the commutator of two group elements is equal to a given element has been introduced in literature few years ago. Several authors have investigated this notion with methods of the representation theory and with combinatorial techniques. Here we illustrate that a wider context may be considered and show some structural restrictions on the group.

1. **Different formulations of the commutativity degree**

Given two elements x and y of a group G, several authors studied the probability that a randomly chosen commutator $[x, y]$ of G satisfies a prescribed property. P. Erdős and P. Turán [6] began to investigate the case $[x, y] = 1$, noting some structural restrictions on G from bounds of statistical nature. Their approach involved combinatorial techniques, which were developed successively in [2–5, 7, 9, 10, 12, 13, 15, 17] and extended to the infinite case in [8, 13, 18]. On another hand, P. X. Gallagher [11] investigated the case $[x, y] = 1$, using character theory, and opened another line of research, illustrated in [3, 4, 12, 16, 19]. The literature shows that it is possible to variate the condition on $[x, y]$ involving

arbitrary words, which could not be the commutator word $[x, y]$. From now, all the groups which we consider will be finite.

Given two subgroups H and K of G and two integers $n, m \geq 1$, we define

(1.1) $|H|^n |K|^m \, \mathrm{p}_g^{(n,m)}(H, K) = |\{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times K^m \, |$

$$[x_1, \ldots, x_n, y_1, \ldots, y_m] = g\}|$$

as the *probability that a randomly chosen commutator of weight $n + m$ of $H \times K$ is equal to a given element of G*. Denoting
(1.2)
$$\mathcal{A} = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times K^m \mid [x_1, \ldots, x_n, y_1, \ldots, y_m] = g\},$$

$|\mathcal{A}| = |H|^n \cdot |K|^m \cdot \mathrm{p}_g^{(n,m)}(H, K)$. The case $n = m = 1$ can be found in [4] and is called *generalized commutativity degree of G*. For $n = m = 1$ and $H = K = G$,

(1.3) $\qquad \mathrm{p}_g^{(1,1)}(G, G) = \mathrm{p}_g(G) = \dfrac{|\{(x, y) \in G^2 \mid [x, y] = g\}|}{|G|^2}$

is the *probability that the commutator of two group elements of G is equal to a given element of G* in [16].

It is well known (see for instance [1, Excercise 3, p. 183]) that the function $\psi(g) = |\{(x, y) \in G \times G \mid [x, y] = g\}|$ is a character of G and we have $\psi = \sum\limits_{\chi \in \mathrm{Irr}(G)} \dfrac{|G|}{\chi(1)} \chi$, where $\mathrm{Irr}(G)$ denotes the set of all irreducible complex characters of G. However, the authors in [16, Theorem 2.1] exploited this fact, writing (1.3) as

(1.4) $\qquad\qquad\qquad \mathrm{p}_g(G) = \dfrac{1}{|G|} \sum\limits_{\chi \in \mathrm{Irr}(G)} \dfrac{\chi(g)}{\chi(1)}.$

For terminology and notations in character theory we refer to [14].

Now for $g = 1$,
(1.5)
$$\mathrm{p}_1^{(1,1)}(G, G) = \mathrm{p}_1(G) = \mathrm{d}(G) = \dfrac{|\{(x, y) \in G^2 \mid [x, y] = 1\}|}{|G|^2} = \dfrac{|\mathrm{Irr}(G)|}{|G|}$$

is the *probability of commuting pairs of G* (or briefly the *commutativity degree* of G), largely studied in [2–5, 7, 9–13, 15, 17, 19]. Moreover,
(1.6)
$$\mathrm{p}_1^{(n,1)}(G, G) = \dfrac{|\{(x_1, \ldots, x_n, x_{n+1}) \in G^{n+1} \mid [x_1, \ldots, x_n, x_{n+1}] = 1\}|}{|G|^{n+1}}$$

$$= \mathrm{d}^{(n)}(G)$$

is the nth *nilpotency degree* of G in $[2,7,9,17,18]$ and

$$(1.7) \quad \mathrm{p}_1^{(n,1)}(H,G) = \frac{|\{(x_1,\ldots,x_n,y) \in H^n \times G \mid [x_1,\ldots,x_n,y] = 1\}|}{|H|^n \; |G|}$$

$$= \mathrm{d}^{(n)}(H,G)$$

is the *relative nth nilpotency degree* of H in G, studied in $[7,9,17,18]$. We may express (1.7) not necessarily with $g = 1$. Assuming that H is normal in G, [4, Equation (4) and Theorem 4.2] imply

$$(1.8) \qquad \mathrm{p}_g^{(1,1)}(H,G) = \frac{|\{(x,y) \in H \times G \mid [x,y] = g\}|}{|H| \; |G|}$$

$$= \frac{1}{|H||G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{|H|\langle \chi_H, \chi_H \rangle}{\chi(1)} \chi(g),$$

where χ_H denotes the restriction of χ to H and $\langle \, , \, \rangle$ the usual inner product. Our purpose is to study $\mathrm{p}_g^{(n,m)}(H,K)$, extending the previous contributions in $[2,4,7,16,17]$. The main results of the present paper are in Section 3, in which the general considerations of Section 2 are applied.

2. **Technical properties and some computations**

We begin with two elementary observations on (1.1).

Remark 2.1. If

$$\mathcal{S} = \{[x_1,\ldots,x_n,y_1,\ldots,y_m] \mid x_1,\ldots,x_n \in H; y_1,\ldots,y_m \in K\},$$

then $\mathrm{p}_g^{(n,m)}(H,K) = 0$ if and only if $g \notin \mathcal{S}$. Moreover, $\mathrm{p}_1^{(n,m)}(H,K) = 1$ if and only if $[\underbrace{H,\ldots,H}_{n-\text{times}}, \underbrace{K,\ldots,K}_{m-\text{times}}] = [{}_n H, \; {}_m K] = 1$.

Remark 2.2. The equation (1.1) assigns by default the map
(2.1)

$$\mathrm{p}_g^{(n,m)} : (x_1,\ldots,x_n,y_1,\ldots,y_m) \in H^n \times K^m \mapsto \mathrm{p}_g^{(n,m)}(H,K) \in [0,1],$$

which is a probability measure on $H^n \times K^m$, satisfying a series of standard properties (see below) such as being multiplicative, symmetric and monotone.

Let E and F be two groups such that $A, C \leq E$ and $B, D \leq F$. Then $A \times B$ and $C \times D$ are subgroups of $E \times F$. The subgroup $A \times \{1_F\}$ can be identified with A and similarly for C. Also $\{1_E\} \times B$ can be identified with B and similarly for D. The fact that (2.1) is multiplicative is described by the next result.

Proposition 2.3. *Let E and F be two groups such that $e \in E$, $f \in F$, $A, C \leq E$ and $B, D \leq F$. Then $A \times B$ and $C \times D$ are subgroups of $E \times F$ such that*

$$\mathrm{p}_{(e,f)}^{(n,m)}(A \times B, C \times D) = \mathrm{p}_{e}^{(n,m)}(A, B) \cdot \mathrm{p}_{f}^{(n,m)}(C, D).$$

Proof. It is enough to note that

$$[([a_1, \ldots, a_n], [c_1, \ldots, c_n]), ([b_1, \ldots, b_m], [d_1, \ldots, d_m])]$$
$$= ([[a_1, \ldots, a_n], [b_1, \ldots, b_m]], [c_1, \ldots, c_n], [d_1, \ldots, d_m]]).$$

$\square$

Proposition 2.3 is true for finitely many factors instead of only two factors and this can be checked with easy computations. Therefore the proof is omitted. The fact that (2.1) is symmetric is described by the next result.

Proposition 2.4. *Let H and K be two subgroups of a group G. Then*

$$\mathrm{p}_{g}^{(n,m)}(H, K) = \mathrm{p}_{g^{-1}}^{(n,m)}(K, H).$$

Moreover, if H or K, is normal in G, then

$$\mathrm{p}_{g}^{(n,m)}(H, K) = \mathrm{p}_{g}^{(n,m)}(K, H) = \mathrm{p}_{g^{-1}}^{(n,m)}(H, K).$$

Proof. The commutator rule $[x, y]^{-1} = [y, x]$ implies the first part of the result. Now let H be normal in G, $n \leq m$, and let $\mathcal{A}$ be the set in (1.2) and

$$\mathcal{B} = \{(y_1, \ldots, y_m, x_1, \ldots, x_n) \in K^m \times H^n \mid [y_1, \ldots, y_m, x_1, \ldots, x_n] = g\}.$$

The map

$$\varphi : (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathcal{A} \longmapsto$$
$$(y_1^{-1}, y_2^{-1}, \ldots, y_n^{-1}, y_{n+1}^{-1}, \ldots, y_m^{-1}, y_1 x_1 y_1^{-1}, y_2 x_2 y_2^{-1}, \ldots, y_n x_n y_n^{-1}) \in \mathcal{B}$$

is bijective and so the remaining equalities follow. A similar argument can be applied, when the assumption (H is normal in G) is replaced by (K is normal in G). $\square$

The fact that (2.1) is monotone is more delicate to prove, since this is a situation in which we may find upper bounds for (1.1). Details are given later on. Now we will get another expression for (1.1). We define $\mathrm{Cl}_K([x_1, \ldots, x_n])$ as the *K–conjugacy class of* $[x_1, \ldots, x_n] \in H$.

Proposition 2.5. *Let H and K be two subgroups of a group G. Then,*

$$\mathrm{p}_g^{(n,m)}(H, K) = \frac{1}{|H|^n \, |K|^m} \sum_{\substack{x_1, \ldots, x_n \in H \\ g^{-1}[x_1, \ldots, x_n] \in \mathrm{Cl}_K([x_1, \ldots, x_n])}} |C_K([x_1, \ldots, x_n])|^m.$$

Proof. It is straightforward to check that

$$C_{K^m}([x_1, \ldots, x_n]) = \underbrace{C_K([x_1, \ldots, x_n]) \times \ldots \times C_K([x_1, \ldots, x_n])}_{m-\text{times}}.$$

In particular, $|C_{K^m}([x_1, \ldots, x_n])| = |C_K([x_1, \ldots, x_n])|^m$. Recalling (1.2), we have

$$\mathcal{A} = \bigcup_{[x_1, \ldots, x_n] \in H} \{[x_1, \ldots, x_n]\} \times T_{[x_1, \ldots, x_n]},$$

where

$$T_{[x_1, \ldots, x_n]} = \{(y_1, \ldots, y_m) \in K^m \mid [x_1, \ldots, x_n, y_1, \ldots, y_m] = g\}.$$

Obviously, $T_{[x_1, \ldots, x_n]} \neq \emptyset$ if and only if

$$g^{-1}[x_1, \ldots, x_n] \in \mathrm{Cl}_K([x_1, \ldots, x_n]).$$

Let $T_{[x_1, \ldots, x_n]} \neq \emptyset$. Then $|T_{[x_1, \ldots, x_n]}| = |C_{K^m}([x_1, \ldots, x_n])|$, because the map $\psi : [y_1, \ldots, y_m] \mapsto g\overline{[y_1, \ldots, y_m]}^{-1}[y_1, \ldots, y_m]$ is bijective, where $\overline{[y_1, \ldots, y_m]}$ is a fixed element of $T_{[x_1, \ldots, x_n]}$. We deduce that

$$|\mathcal{A}| = \sum_{[x_1, \ldots, x_n] \in H} |T_{[x_1, \ldots, x_n]}| = \sum_{\substack{x_1, \ldots, x_n \in H \\ g^{-1}[x_1, \ldots, x_n] \in \mathrm{Cl}_K([x_1, \ldots, x_n])}} |C_{K^m}([x_1, \ldots, x_n])|$$

$$= \sum_{\substack{x_1, \ldots, x_n \in H \\ g^{-1}[x_1, \ldots, x_n] \in \mathrm{Cl}_K([x_1, \ldots, x_n])}} |C_K([x_1, \ldots, x_n])|^m$$

and the result follows. $\square$

Special cases of Proposition 2.5 are listed below.

Corollary 2.6. *In* Proposition 2.5 *, if* $m = 1$ *and* $G = K$, *then*

$$\mathrm{p}_g^{(n,1)}(H,G) = \frac{1}{|H|^n \ |G|} \sum_{\substack{x_1,\ldots,x_n \in H \\ g^{-1}[x_1,\ldots,x_n] \in \mathrm{Cl}_G([x_1,\ldots,x_n])}} |C_G([x_1,\ldots,x_n])|.$$

Corollary 2.7 (See [4], Theorem 2.3). *In* Proposition 2.5 *, if* $m = n = 1$, *then*

$$\mathrm{p}_g^{(1,1)}(H,K) = \frac{1}{|H| \ |K|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{Cl}_K(x)}} |C_K(x)|.$$

In particular, if $G = K$, *then*

$$\mathrm{p}_g^{(1,1)}(H,G) = \frac{1}{|H| \ |G|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{Cl}_G(x)}} |C_G(x)|.$$

Corollary 2.8 (See [7], Proof of Lemma 4.2). *In* Proposition 2.5 *, if* $m = 1$ *and* $G = K$, *then*

$$\mathrm{p}_1^{(n,1)}(H,G) = \mathrm{d}^{(n)}(H,G) = \frac{1}{|H|^n \ |G|} \sum_{x_1,\ldots,x_n \in H} |C_G([x_1,\ldots,x_n])|.$$

Corollary 2.9. *In* Proposition 2.5 *, if* $C_K([x_1,\ldots,x_n]) = 1$, *then*

$$\mathrm{p}_1^{(n,m)}(H,K) = \frac{1}{|H|^n} + \frac{1}{|K|^m} - \frac{1}{|H|^n \ |K|^m}.$$

The result [4, Proposition 3.4] follows from Corollary 2.9, when $m = n = 1$.

Remark 2.10. Equation (1.7) makes the study of $\mathrm{p}_1^{(n,1)}(H,G)$ and that of $\mathrm{d}^{(n)}(H,G)$ equivalent. This is illustrated in Corollary 2.8 and noted here for the first time. Therefore, there are many information from $[2,7,9,17]$ and $[3,4,16]$ which can be connected. It is relevant to point out that these concepts were treated independently and with different methods in the last few years.

Let χ be a character of G and θ a character of $H \leq G$. The *Frobenius Reciprocity Law* [14, Lemma 5.2] gives a link between the restriction χ_H of χ to H and the induced character θ^G of θ. Therefore $\langle \chi, \theta^G \rangle_G = \langle \chi_H, \theta \rangle_H$. Write this number as $e_{(\chi,\theta)} = \langle \chi, \theta^G \rangle_G = \langle \chi_H, \theta \rangle_H$. If $e_{(\chi,\theta)} = 0$, then θ does not appear in χ_H and so χ does not appear in θ^G. Recall from [14] that, if $e_{(\chi,\theta)} \neq 0$, then χ *covers* θ (or

θ belongs to the constituents of χ_H). In particular, if $\theta = \chi_H$, then $e_{(\chi, \chi_H)} = \langle \chi, (\chi_H)^G \rangle_G = \langle \chi_H, \chi_H \rangle_H$. From a classic relation (see [14, Lemma 2.29]), $e_{(\chi, \chi_H)} = \langle \chi, (\chi_H)^G \rangle_G = \langle \chi_H, \chi_H \rangle_H \leq |G : H| \langle \chi, \chi \rangle_G = |G : H| e_{(\chi, \chi)}$ and the equality holds if and only if $\chi(x) = 0$ for all $x \in G - H$. In particular, if $\chi \in \mathrm{Irr}(G)$, then $\langle \chi_H, \chi_H \rangle_H = |G : H|$ if and only if $\chi(x) = 0$, for all $x \in G - H$. Therefore the following result is straightforward.

Corollary 2.11. *If H is a subgroup of a group G, then*

$$\mathrm{p}_g^{(1,1)}(H, G) \leq |G : H| \, \mathrm{p}_1(G),$$

and the equality holds if and only if all characters vanish on $G - H$.

At this point, [4, Theorem 4.2] becomes

$$(2.2) \quad \zeta(g) = |H| \sum_{\chi \in \mathrm{Irr}(G)} \frac{e_{(\chi_H, \chi_H)}}{\chi(1)} \cdot \chi(g) = |\{(x, y) \in H \times G \mid [x, y] = g\}|$$

$$= \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{Cl}_G(x)}} |C_G(x)|,$$

where $\zeta(g)$ is the number of solutions $(x, y) \in H \times G$ of the equation $[x, y] = g$. Note that (2.2) and [1, Excercise 3, p. 183] give a short argument to prove that $\zeta(g)$ is a character of G with respect to the argument in [4, Corollary 4.3]. The equation (1.8) becomes

$$\mathrm{p}_g^{(1,1)}(H, G) = \frac{\zeta(g)}{|H| \, |G|}.$$

For the general case, when $n > 1$, $m > 1$ and $G = K$, we have

$$\mathrm{p}_g^{(n,m)}(H, G) = \frac{\zeta^{(n,m)}(g)}{|H|^n \, |G|^m}$$

$$= \frac{1}{|H|^n \, |G|^m} \left(\sum_{\substack{x_1, \ldots, x_n \in H \\ g^{-1}[x_1, \ldots, x_n] \in \mathrm{Cl}_G([x_1, \ldots, x_n])}} |C_G([x_1, \ldots, x_n])|^m \right),$$

where

$$\zeta^{(n,m)}(g) = \sum_{\substack{x_1, \ldots, x_n \in H \\ g^{-1}[x_1, \ldots, x_n] \in \mathrm{Cl}_G([x_1, \ldots, x_n])}} |C_G([x_1, \ldots, x_n])|^m$$

is the number of solutions $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in H^n \times G^m$ of the equation $[x_1, \ldots, x_n, y_1, \ldots, y_m] = g$.

Remark 2.12. There are many evidences from the computations that $\zeta^{(n,m)}(g)$ is a character of G. The research report in [20] contains some similar computations in this direction.

Now we may prove upper bounds for (1.1), finding that (2.1) is monotone.

Proposition 2.13. *Let H and K be two subgroups of a group G. If $H \leq K$, then*
$$\mathrm{p}_g^{(n,m)}(H,G) \geq \mathrm{p}_g^{(n,m)}(K,G).$$
The equality holds if and only if $\mathrm{Cl}_H(x) = \mathrm{Cl}_K(x)$ for all $x \in G$.

Proof. We note that $\frac{1}{|K|} \leq \frac{1}{|H|}$ and then $\frac{1}{|K|^n} \leq \frac{1}{|H|^n}$. By Proposition 2.5, we have

$$|G|^m \cdot \mathrm{p}_g^{(n,m)}(K,G) = \frac{1}{|K|^n} \sum_{\substack{x_1,\ldots,x_n \in K \\ g^{-1}[x_1,\ldots,x_n] \in \mathrm{Cl}_G([x_1,\ldots,x_n])}} |C_G([x_1,\ldots,x_n])|$$

$$\leq \frac{1}{|H|^n} \sum_{\substack{x_1,\ldots,x_n \in K \\ g^{-1}[x_1,\ldots,x_n] \in \mathrm{Cl}_G([x_1,\ldots,x_n])}} |C_G([x_1,\ldots,x_n])|.$$

In particular, the last relation is true for $x_1,\ldots,x_n \in H \leq K$, and by continuing this process,

$$= \frac{1}{|H|^n} \sum_{\substack{x_1,\ldots,x_n \in H \\ g^{-1}[x_1,\ldots,x_n] \in \mathrm{Cl}_G([x_1,\ldots,x_n])}} |C_G([x_1,\ldots,x_n])| = |G|^m \cdot p_g^{(n,m)}(H,G).$$

The rest of the proof is clear. $\square$

The next result shows an upper bound, which generalizes [7, Theorem 4.6].

Proposition 2.14. *Let N be a normal subgroup of a group G and H a subgroup of G such that $H \geq N$, then*

$$\mathrm{p}_g^{(n,m)}(H,G) \leq \mathrm{p}_g^{(n,m)}\left(\frac{H}{N}, \frac{G}{N}\right).$$

Moreover, if $N \cap [_nH, \ _mG] = 1$, then the equality holds.

Proof. The normality of N allows us to consider the quotient G/N. Then

$$|H|^n \ |G|^m \ \mathrm{p}_g^{(n,m)}(H,G) = |\mathcal{A}|$$

$$= |\{(x_1,\ldots,x_n,y_1,\ldots,y_m) \in H^n \times G^m \mid [x_1,\ldots,x_n,y_1,\ldots,y_m] \cdot g^{-1} = 1\}|$$

$$= |\{(x_1,\ldots,x_n,y_1,\ldots,y_m) \in H^n \times G^m \mid [x_1,\ldots,x_n,y_1,\ldots,y_m,g^{-1}] = 1\}|$$

$$= \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \sum_{y_1 \in G} \cdots \sum_{y_m \in G} |C_G([x_1,\ldots,x_n,y_1,\ldots,y_m])|$$

$$= \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \sum_{y_1 \in G} \cdots \sum_{y_m \in G}$$

$$\frac{|C_G([x_1,\ldots,x_n,y_1,\ldots,y_m])N| \cdot |C_N([x_1,\ldots,x_n,y_1,\ldots,y_m])|}{|N|}$$

$$\leq \sum_{x_1 \in H} \cdots \sum_{x_n \in H} \sum_{y_1 \in G} \cdots \sum_{y_m \in G} |C_{G/N}([x_1N,\ldots,x_nN,y_1N,\ldots,y_mN])|$$

$$\cdot |C_N([x_1,\ldots,x_n,y_1,\ldots,y_m])|$$

$$= \sum_{S_1 \in H/N} \sum_{x_1 \in S_1} \cdots \sum_{S_n \in H/N} \sum_{x_n \in S_n} \sum_{T_1 \in G/N} \sum_{y_1 \in T_1} \cdots \sum_{T_m \in G/N} \sum_{y_m \in T_m}$$

$$|C_{G/N}([S_1,\ldots,S_n,T_1,\ldots,T_m])| \cdot |C_N([x_1,\ldots,y_m])|$$

$$= \Big(\sum_{S_1 \in H/N} \cdots \sum_{S_n \in H/N} \sum_{T_1 \in G/N} \cdots \sum_{T_m \in G/N} |C_{G/N}([S_1,\ldots,S_n,T_1,\ldots,T_m])| \Big)$$

$$\cdot \Big(\sum_{x_1 \in S_1} \cdots \sum_{x_n \in S_n} \sum_{y_1 \in T_1} \cdots \sum_{y_m \in T_m} |C_N([x_1,\ldots,y_m])| \Big)$$

$$\leq |N|^{n+m} \sum_{S_1 \in H/N} \cdots \sum_{S_n \in H/N} \sum_{T_1 \in G/N} \cdots \sum_{T_m \in G/N}$$

$$|C_{G/N}([S_1,\ldots,S_n,T_1,\ldots,T_m])|$$

$$= \Big|\frac{H}{N}\Big|^n \Big|\frac{G}{N}\Big|^m p_g^{(n,m)}\Big(\frac{H}{N},\frac{G}{N}\Big) |N|^{n+m}$$

$$= |H|^n |G|^m p_g^{(n,m)}\Big(\frac{H}{N},\frac{G}{N}\Big).$$

The condition of equality in the above relations is satisfied exactly when $N \cap [{}_nH,\ {}_mG] = 1$. The result follows. $\qquad\square$

Corollary 2.15. *A special case of* Proposition 2.14 *is* $p_g(G) \leq p_g(G/N)$.

Corollary 2.16 (See [7], Theorem 4.6)**.** *In* Proposition 2.14 *, if* $m = 1$ *and* $g = 1$, *then*

$$d^{(n)}(H,G) \leq d^{(n)}(H/N,G/N).$$

3. Some upper and lower bounds

A relation between (1.1)–(1.8) is described below.

Theorem 3.1. *If H and K are subgroups of a group G, then*

$$\mathrm{p}_g^{(n,m)}(G,G) \leq \mathrm{p}_g^{(n,m)}(H,K) \leq \mathrm{p}_1^{(n,m)}(H,K) \leq \mathrm{p}_1^{(n,m)}(H,G) \leq \mathrm{p}_1^{(n,m)}(H,H).$$

Proof. From Proposition 2.13, $\mathrm{p}_g^{(n,m)}(G,G) \leq \mathrm{p}_g^{(n,m)}(G,H)$. From Proposition 2.5, one has

$$\mathrm{p}_g^{(n,m)}(H,K) = \frac{1}{|H|^n\,|K|^m} \sum_{\substack{x_1,\ldots,x_n \in H \\ g^{-1}[x_1,\ldots,x_n] \in \mathrm{Cl}_K([x_1,\ldots,x_n])}} |C_K([x_1,\ldots,x_n])|^m,$$

and for $g = 1$ we get

$$\mathrm{p}_g^{(n,m)}(H,K) \leq \frac{1}{|H|^n\,|K|^m} \sum_{x_1,\ldots,x_n \in H} |C_K([x_1,\ldots,x_n])|^m = \mathrm{p}_1^{(n,m)}(H,K),$$

where in the last passage Proposition 2.5 is again used. From

$$C_K([x_1,\ldots,x_n]) \subseteq C_G([x_1,\ldots,x_n]),$$

we deduce

$$\mathrm{p}_g^{(n,m)}(H,K) \leq \sum_{x_1,\ldots,x_n \in H} |C_G([x_1,\ldots,x_n])|^m = \mathrm{p}_1^{(n,m)}(H,G).$$

Applying Proposition 2.4, $\mathrm{p}_1^{(n,m)}(H,G) = \mathrm{p}_1^{(n,m)}(G,H)$ and so we have $\mathrm{p}_1^{(n,m)}(G,H) \leq \mathrm{p}_1^{(n,m)}(H,H)$ by Proposition 2.13.   □

Corollary 3.2. *Let H and K be subgroups of a group G. If G has trivial center, then*

$$\mathrm{p}_g^{(n,1)}(H,K) \leq \frac{2^n - 1}{2^n}.$$

Proof. It follows from Theorem 3.1 and [7, Theorem 5.3].   □

Another significant restriction is the following.

Theorem 3.3. *Let H and K be two subgroups of a group G and let p be the smallest prime divisor of $|G|$. Then*

(i) $\mathrm{p}_g^{(n,m)}(H,K) \leq \frac{2p^n + p - 2}{p^{m+n}}$;

(ii) $\mathrm{p}_g^{(n,m)}(H,K) \geq \frac{(1-p)|Y_{H^n}| + p|H^n|}{|H^n|\,|K^m|} - \frac{(|K|+p)|C_H(K)|^n}{|H^n|\,|K^m|}$;

where $Y_{H^n} = \{[x_1,\ldots,x_n] \in H^n \mid C_K([x_1,\ldots,x_n]) = 1\}$.

Proof. If $[_nH, \ _mK] = 1$, then $C_{H^n}(K^m) = H^n$ and Y_{H^n} is equal to H^n or it is the empty set provided that K^m is trivial or nontrivial, respectively. Assume that $[_nH, \ _mK] \neq 1$. Then

$$Y_{H^n} \cap C_{H^n}(K^m) = Y_{H^n} \cap (C_H(K^m) \times \ldots \times C_H(K^m))$$

$$= Y_{H^n} \cap (C_H(K) \times C_H(K) \times \ldots \times C_H(K)) = Y_{H^n} \cap (C_H(K))^{nm} \neq \emptyset,$$

and

$$\sum_{x_1,\ldots,x_n \in H} |C_{K^m}([x_1,\ldots,x_n])| = \sum_{x_1,\ldots,x_n \in H} |C_K([x_1,\ldots,x_n])|^m$$

$$\sum_{x_1,\ldots,x_n \in Y_{H^n}} |C_K([x_1,\ldots,x_n])|^m + \sum_{x_1,\ldots,x_n \in C_{H^n}(K)} |C_K([x_1,\ldots,x_n])|^m$$

$$+ \sum_{x_1,\ldots,x_n \in H^n - (Y_{H^n} \cup C_{H^n}(K))} |C_K([x_1,\ldots,x_n])|^m$$

$$= |Y_{H^n}| + |K| \ |C_H(K)|^n + \sum_{x_1,\ldots,x_n \in H^n - (Y_{H^n} \cup C_{H^n}(K))} |C_K([x_1,\ldots,x_n])|^m.$$

Since

$$p^m \leq |C_K([x_1,\ldots,x_n])|^m \leq \frac{|K^m|}{p^m},$$

we have $|Y_{H^n}| \leq |H^n|$ and $p^n \leq |C_H(K)|^n \leq \frac{|H^n|}{p^n}$. It follows that

$$|H^n| \cdot |K^m| \cdot \mathrm{p}_g^{(n,m)} \ \leq \ |Y_{H^n}| + |K| \ |C_H(K)|^n + (|H^n|$$

$$-(|Y_{H^n}| + |C_H(K)|^n) \cdot \frac{|K^m|}{p^m}$$

and then
(3.1)
$$\mathrm{p}_g^{(n,m)}(H,K) \leq \frac{|Y_{H^n}|}{|H^n| \ |K^m|} + \frac{|K| \ |C_H(K)|^n}{|H^n| \ |K^m|} + \frac{1}{p^m} - \frac{|Y_{H^n}|}{p^m \ |H^n|} - \frac{|C_H(K)|^n}{p^m \ |H^n|}$$

$$\frac{1}{p^m} + \frac{1}{p^{m+n-1}} + \frac{1}{p^m} - \frac{1}{p^{m+n}} - \frac{1}{p^{m+n}} = \frac{2p^n + p - 2}{p^{m+n}}.$$

Hence (i) follows. On the other hand, we may continue in the opposite direction

$$|H^n| \cdot |K^m| \cdot \mathrm{p}_g^{(n,m)} \ \geq \ |Y_{H^n}| + |K| \ |C_H(K)|^n$$

$$+p \ (|H^n| - (|Y_{H^n}| + |C_H(K)|^n)$$

and then

$$\mathrm{p}_g^{(n,m)}(H, K) \geq \frac{(1-p)|Y_{H^n}|}{|H^n|\,|K^m|} + \frac{p}{|K^m|} - \frac{(|K|+p)|C_H(K)|^n}{|H^n|\,|K^m|}.$$

Thus (ii) follows. $\square$

The bound in Theorem 3.3 (i) is a little bit different from the bound in [4, Corollary 3.9], where it is proved that $\mathrm{p}_g^{(1,1)}(H, K) \leq \frac{2p-1}{p^2}$ and in particular $\mathrm{p}_g^{(1,1)}(H, K) \leq \frac{3}{4}$. We conclude the following structural restriction.

Corollary 3.4. *In* Theorem 3.3, *if* $\mathrm{p}_g^{(n,m)}(H, K) = \frac{2p^n+p-2}{p^{m+n}}$ *and* $p \neq 2$, *then*

$$\frac{p \cdot p^{\frac{1}{n}}}{(p-2)^{\frac{1}{n}}} \geq |H : C_H(K)|.$$

Proof. Looking at (3.1) and the proof of Theorem 3.3 (i), we deduce

$$\frac{2p^n + p - 2}{p^{m+n}} \leq \frac{|Y_{H^n}|}{|H^n|\,|K^m|} + \frac{|K||C_H(K)|^n}{|H^n|\,|K^m|} + \frac{1}{p^m}$$

$$\leq \frac{1}{p^m} + \frac{1}{p^{m-1}}\left|\frac{C_H(K)}{H}\right|^n + \frac{1}{p^m} = \frac{1}{p^{m-1}}\left(\frac{2}{p} + |\frac{C_H(K)}{H}|^n\right),$$

and then

$$\frac{2p^n + p - 2}{p^{n+1}} \leq \frac{2}{p} + \left|\frac{C_H(K)}{H}\right|^n.$$

We conclude that

$$\frac{2p^n + p - 2}{p^{n+1}} - \frac{2}{p} = \frac{p-2}{p^{n+1}} \leq \left|\frac{C_H(K)}{H}\right|^n,$$

and then

$$\frac{p^{n+1}}{p-2} \geq \left|\frac{H}{C_H(K)}\right|^n.$$

Extracting the n-th root, we have

$$\frac{p \cdot p^{\frac{1}{n}}}{(p-2)^{\frac{1}{n}}} \geq |H : C_H(K)|.$$

Now the result follows. $\square$

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Ahmad M. A. Alghamdi
Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Alqura
University, P.O. Box 14035, 21955, Makkah, Saudi Arabia
Email: amghamdi@uqu.edu.sa

Francesco G. Russo
Dipartimento di Matematica e Informatica, Universitá degli Studi di Palermo, via
Archirafi 34, 90123, Palermo, Italy
and
Department of Mathematics, Universiti Teknologi Malaysia, Johor Bahru, Malaysia
Email: francescog.russo@yahoo.com