*Bulletin of the Iranian Mathematical Society Vol.* 39 *No. 1* (*2013*), *pp 65-85.*

# GROUPLIKES

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Communicated by Jamshid Moori

Abstract. We introduce and study an algebraic structure, namely *"Grouplike"*. A grouplike is something between semigroup and group and its axioms are generalizations of the four group axioms. Every grouplike is a semigroup containing the minimum ideal that is also a maximal subgroup (but the converse is not valid). The first idea of grouplikes comes from $b$-parts and $b$-addition of real numbers introduced by the author. Here, the aim is extend the notion of grouplikes (including sub-grouplikes, dual grouplikes, grouplike-homomorphisms with standard kernels, etc.), establish some main results and construct an expanded class. We prove a fundamental structure theorem for a large class of grouplikes, namely *Class United Grouplikes*. Moreover, we obtain some other results for magmas, semigroups and groups in general, exhibit several of their important subsets with related diagrams and give many equivalent conditions for semigroups to be grouplikes. Finally, we point out some directions for further research in grouplikes and semigroup theory.

## 1. **Introduction and preliminaries**

The $b$-parts of real numbers were introduced and studied in [1, 3]. Those have many interesting number theoretic explanations, analytic

MSC(2010): Primary: 20N99; Secondary: 20M99.

Keywords: Grouplike, class united grouplike, identity-like, grouplike homomorphism, real $b$-grouplike.

Received: 30 April 2009, Accepted: 21 June 2010.

and algebraic properties, and their related functions satisfy the real functional equation $f(f(x)+y-f(y))=f(x)$. Also, $b$-addition of two real numbers, that is $b$-decimal part of their ordinary addition, was stated as a new binary operation in $\mathbb{R}$ (see [3, 5]). In fact, $(\mathbb{R},+_b)$ is a semigroup with some additional properties, where $+_b$ is $b$-addition. In [2], the functional equation was generalized to several functional equations on magmas, semigroups and groups such as decomposer and associative functional equations.

1.1. ***B*-parts of real numbers, *b*-addition and *b*-parts functions.** For every real number $a$, denote by $[a]$ the largest integer not exceeding $a$ and put $(a)=a-[a]$ (the decimal part of $a$). Let $b$ be a nonzero constant real number. For any real number $a$, set

$$[a]_b := b[\frac{a}{b}] \ , \ (a)_b := b(\frac{a}{b}).$$

We call the notation $[a]_b$ to be *b-integer part of* $a$ and $(a)_b$ to be *b-decimal part of* $a$. Also, $[a]_b$ and $(a)_b$ are called *b-parts of* $a$.
Clearly, $a=[a]_b+(a)_b$ where $[a]_b \in b\mathbb{Z}=\langle b\rangle$, *and* $(a)_b \in b[0,1)=\{bd|0\le d<1\}$. The $b$-decimal part of $a$ is the remainder of the (generalized) division of $a$ by $b$ and if $b$ is a positive integer, then $[a]_b$ is the same unique integer of the residue class $\{[a]-b+1,\cdots,[a]\}$ (mod $b$) that is divisible by $b$ (see [1, 3, 5]).
We call $(\ )_b$, $[\ ]_b$ to be *b-parts functions*. These functions are idempotent, their compositions are zero, and $(\ )_b$ satisfies the following functional equations:

$$f(f(x)+y-f(y))=f(x)\ ,\ f(x+y-f(y))=f(x)$$
$$f(f(x+y)+z)=f(x+f(y+z)).$$

The additive group of real numbers is equal to direct some of the images of $(\ )_b$ and $[\ ]_b$ that are $\mathbb{R}_b := b[0,1)$ and $b\mathbb{Z}=\langle b\rangle$, respectively (see [2, 5]). Now, for a fixed real number $b\neq 0$ and every $x,y\in\mathbb{R}$, we put

$$x+_b y := (x+y)_b$$

and call it to be $x$ *b-addition* $y$. The magma $(\mathbb{R},+_b)$ is a semigroup with some additional properties, $(\mathbb{R}_b,+_b)$ is the largest subgroup and also the smallest ideal of it. $(\mathbb{R}_b,+_b)$ is a generalization of the group of all least non-negative residues mod $n$ ($\mathbb{Z}_n=\{0,1,2,3,4,...,n-1\}$), and hence we call it the group of all least [respectively largest] non-negative (real) residues mod $b$, if $b>0$ [respectively $b<0$] or *reference b-bounded group*. Also, its every subgroup is called *b-bounded group* and every 1-bounded group is a *decimal group*. Several interesting properties of $b$-bounded groups can be seen in [3, 5].

**Convention.** Here, we consider $X$ as a set with the binary operation "$\cdot$" (magma or binary system $(X, \cdot)$ ) where the product of $x, y \in X$ is denoted by $xy$, $(S, \cdot)$ is a semigroup, and also $(G, \cdot)$ is a group. If the multiplication "$\cdot$" is commutative, then it is denoted by $+$ and is called addition. When we use the notation $H \dot{\leq} K$ [respectively $H \leq K$], it means that $K$ is a semigroup or group and $H$ is its sub-semigroup [respectively subgroup]. A subset $A$ of $X$ is called normal, if $xA = Ax$, for every $x \in X$. Now, by $H \dot{\trianglelefteq} K$ [respectively $H \trianglelefteq K$], we mean $H \dot{\leq} K$ [respectively $H \leq K$] and $H$ is normal. Also, when we use the notation $H \vartriangle S$, it means that $H$ is a two-sided ideal of the semigroup $S$. The two sided unit element of $X$ and $S$ (if it exists) is denoted by 1 and of $G$ by $e = e_G$.

## 2. **Grouplikes**

Now, we introduce an algebraic structure whose axioms are generalizations of the four group axioms based on ideas from $b$-parts of real numbers, $b$-addition, the semigroup $(\mathbb{R}, +_b)$ and $b$-bounded groups, introduced by the author.

**Definition 2.1.** *Suppose* $\Gamma$ *is a set and* $\cdot$ *is a function defined on* $\Gamma \times \Gamma$. *We call* $(\Gamma, \cdot)$ *a* grouplike, *if it satisfies the following axioms:*
(1) *closure (i.e., the function* $\cdot$ *is from* $\Gamma \times \Gamma$ *to* $\Gamma$*);*
(2) *associativity (i.e.,* $(xy)z = x(yz)$*, for every* $x, y, z \in \Gamma$*);*
(3) *there exists* $\varepsilon \in \Gamma$ *such that*

$$\varepsilon x = \varepsilon^2 x = x\varepsilon^2 = x\varepsilon \ : \ \forall x \in \Gamma;$$

(4) *for every* $\varepsilon$ *satisfying* (3) *and every* $x \in \Gamma$*, there exists* $y \in \Gamma$ *such that*

$$xy = yx = \varepsilon^2.$$

*We call every* $\varepsilon \in \Gamma$ *satisfying the axioms* (3) *and* (4) *an* identity-like*.*
*If* $(\Gamma, \cdot)$ *is grouplike and is not group, then we call it* proper grouplike*.*
*If a semigroup satisfies the axiom* (3) *(equivalently, it contains a central idempotent), then we call it* monoidlike*.*

The following lemma states an important basic property of grouplikes.

**Lemma 2.2.** *Every grouplike contains a unique idempotent identity-like.*

*Proof.* Existence: Choose an arbitrary identity-like $\varepsilon \in \Gamma$. We have $\varepsilon^3 = \varepsilon^2$ and $\varepsilon$ belongs to the center of $\Gamma$. Putting $e = \varepsilon^2$ implies $e$ is a central idempotent and it satisfies the axioms (3), (4) of grouplikes.
Uniqueness: If $e_1$ and $e_2$ are two idempotent identity-likes, then there exist $e'_1$ and $e'_2$ in $\Gamma$ such that $e_1 = e_2 e'_2$, $e_2 = e_1 e'_1$, $e_2 e_1 = e_2^2 e'_2 = e_2 e'_2 = e_1$, and $e_1 e_2 = e_1^2 e'_1 = e_1 e'_1 = e_2$. Therefore, $e_1 = e_2 e_1 = e_1 e_2 = e_2$. □

Now, Lemma 2.2 justifies the following definition.

**Definition 2.3.** *Let $\Gamma$ be a grouplike and let $e$ be the unique idempotent identity-like element of $\Gamma$. Then, we call $e$* standard identity-like *and use the notation $(\Gamma, \cdot, e)$. $\Gamma$ is a* standard grouplike, *if $e$ is the only idempotent of $\Gamma$. $\Gamma$ is a* zero grouplike *if $e$ is a zero of $\Gamma$. Every $y$ that is corresponded to $x$ in axiom (4) is called* inverse-like of $x$ *and is denoted by $x'_e$ or $x'$. By standard mnoidlike, we mean a monidlike with only one idempotent (equivalently, a semigroup with a central idempotent and no other idempotents).*

Regarding the concept of inverse-like in the above definition, note that every identity-like $\varepsilon$ satisfies $\varepsilon^2 = e = e^2$, by Lemma 2.2. So, $y$ is an inverse-like of $x$ (for a given identity-like $\varepsilon$) if and only if $xy = yx = e$. Therefore, we get the following axioms for grouplikes that is very similar to the four groups axioms:
$(i)$ closure;
$(ii)$ associativity;
$(iii)$ there exists a unique element $e \in \Gamma$ such that $ex = xe$ , $e^2 = e$, for all $x \in X$ (i.e., $e$ is its unique central idempotent);
$(iv)$ for every $x \in \Gamma$, there exists $y \in \Gamma$ (not necessarily unique) such that $xy = yx = e$.
(We will minimize these axioms in continuation and give several equivalent conditions for a semigroup to be grouplike.)

**Example 2.4.** *(i) Every group [respectively unipotent monoid] is standard grouplike [respectively standard monoidlike] .*
*(ii) Now, consider the semigroup $(\mathbb{R}, +_b)$. Since $0 +_b 0 = 0$ and $c +_b c = x +_b y$, for all real numbers $x, c$ and $y = 2c - x$, $(\mathbb{R}, +_b, 0)$ is a proper grouplike (we call it* real $b$-grouplike*, and specially* real grouplike *if $b = 1$). Moreover, the set of all identity-likes of $(\mathbb{R}, +_b)$ is $b\mathbb{Z}$ which is also the set of all inverse-likes of $0$.*
*(iii) For an example of finite proper grouplike, consider $\Gamma = \{e, a, \eta, \alpha\}$ and define a binary operation "$\cdot$" by the following multiplication table:*

| $\cdot$ | $e$ | $a$ | $\eta$ | $\alpha$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $e$ | $a$ |
| $a$ | $a$ | $e$ | $a$ | $e$ |
| $\eta$ | $e$ | $a$ | $e$ | $a$ |
| $\alpha$ | $a$ | $e$ | $a$ | $e$ |

*It is easy to see that* $(\Gamma, \cdot, e)$ *is a grouplike (we call it the* Klein four-grouplike*) and* $\eta$ *is its another identity-like.*
*(iv) Every null semigroup [non-singletons null semigroup] is grouplike [proper grouplike] and all its elements are identity-likes. So, we call it* null grouplike*.*
*(v) Every unipotent semigroup* $S$ *containing the zero* $0$ *is a zero grouplike. Because the zero is an identity-like element and if* $\varepsilon$ *is another element satisfying the axiom* (3)*, then its square is central idempotent and so* $\varepsilon^2 = 0 = x0 = 0x$*, for every* $x \in S$*. Specially,* $([0,1), \cdot)$ *is a standard zero grouplike.*
*(vi) Let* $\Omega$ *be a set with a special element* $0$ *and more than two elements. For every* $x, y \in \Omega$*, put* $xy = x$*, if* $x, y \neq 0$ *and* $xy = 0$*, if* $x = 0$ *or* $y = 0$*. Then,* $\Omega$ *is such zero grouplike which is not standard.*

Consider a magma $(X, \cdot)$. Recall that $Z(X)$ and $It(X) = E(X)$ are the center and the set of all idempotent elements of $X$, respectively (which may be empty). Here, we introduce some important subsets of magmas and we use them specially for studying grouplikes and semigroups:

$$Nz(X) := \{\varepsilon \in Z(X) | \varepsilon x = \varepsilon^2 x = \varepsilon(\varepsilon x) = (x\varepsilon)\varepsilon = x\varepsilon^2, \text{ for all } x \text{ in } X\},$$

$$Sv(X) := \{t \in X | \forall x \in X \ \exists y \in X \text{ such that } t = xy = yx\}.$$

Now, we put

$$Zt(X) := Z(X) \cap It(X)\ ,\ Sq(X) := \{t \in X | t^2 \in Sv(X)\}$$

$$Sz(X) := Sq(X) \cap Z(X)$$

$$Iz(X) := Nz(X) \cap Sq(X)\ ,\ Izt(X) := Iz(X) \cap It(X).$$

We call every element of $Nz(X)$ *neutral-like*, every element of $Sv(X)$ [respectively $Sq(X)$] *solvable* [respectively *square solvable* ], and every element of $Iz(X)$ *identity-like* (this agrees with the definition of identity-like in Definition 2.1). Also, every element of $Zt(X)$ [respectively $Sz(X)$]

is called *central idempotent* [respectively *central square solvable*].
For a fixed $t$ and every $x \in X$, we set

$$Inv_t(x) := \{y \in X | t = yx = xy\},$$

and call its every element *$t$-inverse-like of $x$*. So, in $(\Gamma, \cdot, e)$, the set $Inv(x) := Inv_e(x)$ is the set of all inverse-likes $x'$.
It is easy to show that in every magma $X$,

$$Izt(X) = Iz(X) \cap Sv(X) = Iz(X) \cap Zt(X) \subseteq Zt(X) \cap Sv(X),$$
$$Nz(X) \cap It(X) \subseteq Zt(X).$$

Moreover, if $X$ is alternative, then

$$Nz(X) = \{\varepsilon \in X | x\varepsilon = \varepsilon x = \varepsilon^2 x, \text{ for every } x \text{ in } X\},$$
$$Izt(X) = Zt(X) \cap Sv(X)\ ,\ Nz(X) \cap It(X) = Zt(X).$$

Now, let $X$ be a magma with the zero element. Then,

$$Sv(X) = \{0\} = Izt(X), Sq(X) = \{t \in X | t^2 = 0\}, Iz(X) = Null(X),$$

where $Null(X) := \{t \in X | tx = xt = 0\ ,\ \text{for every x in X}\ \}$. Notice that $X$ is a null semigroup [respectively null-free magma] if and only if $Null(X) = X$ [respectively $Null(X) = \{0\}$], and if $X = S$ is semigroup, then $Null(S)$ is an ideal of $S$.
If $X = S$ is semigroup, then

$$Sv(S) \subseteq Sz(S) \subseteq Sq(S),$$

for if $\delta \in Sv(S)$ and $x \in S$, then there is a $y \in S$ such that $\delta = xy = yx$, and thus

$$\delta x = (xy)x = x(yx) = x\delta\ ,\ \delta^2 = x(\delta y) = (\delta y)x.$$

It is clear that a semigroup $S$ is grouplike [respectively monoidlike] if and only if $\emptyset \neq Nz(S) \subseteq Sq(S)$ [respectively $Zt(S) \neq \emptyset$] if and only if $\emptyset \neq Zt(S) \subseteq Sv(S)$.
For every group $G$, we have

$$It(G) = Nz(G) = Zt(G) = Iz(G) = Izt(G) = \{e\}$$
$$\subseteq Z(G) = Sv(G) = Sz(G) \subseteq Sq(G) \subseteq G.$$

Moreover, if $G$ is commutative, then $Sv(G) = Sz(G) = Sq(G) = G$.
For an example of groups, if $G = S_3 =< a, b | a^3 = b^2 = 1\ ,\ bab = a^{-1} >$, then

$$Sv(S_3) = Z(S_3) = Sz(S_3) = \{1\}\ ,\ Sq(S_3) = \{1, b, ab, ba\} \dot{\not\leq} S_3.$$

Also, for an example of commutative semigroups (grouplikes), consider

$$Sv(\mathbb{R}, +_b) = \mathbb{R}_b = \mathbb{R} +_b \mathbb{R}\ ,\ Sq(\mathbb{R}, +_b) = \mathbb{R} = Sz(\mathbb{R}, +_b),$$

$$Iz(\mathbb{R}, +_b) = Nz(\mathbb{R}, +_b) = b\mathbb{Z},$$

$$It(\mathbb{R}, +_b) = Zt(\mathbb{R}, +_b) = Izt(\mathbb{R}, +_b) = \{0\} = \mathbb{R}_b \cap b\mathbb{Z}.$$

We will continue this topic in the last section (specially their interesting useful diagrams, for semigroups and grouplikes, will be shown in figures 1-4). Now, we return to grouplikes.

**Lemma 2.5.** *(Basic properties of grouplikes) For every grouplike* $(\Gamma, \cdot, e)$*, we have:*
*(i)* $Iz(\Gamma) \cup It(\Gamma) \subseteq Inv(e)$ *and*

$$Izt(\Gamma) = Zt(\Gamma) = Iz^2(\Gamma) = Iz^{(2)}(\Gamma) = \{e\},$$

*and so every grouplike which* $It(\Gamma) \subseteq Z(\Gamma)$ *(e.g., every commutative grouplike) is standard and* $Iz(\Gamma)$ *is a null sub-semigroup of* $\Gamma$ *and* $e$ *is its zero.*
*(ii) The equation* $tx = ty$ *[respectively* $xt = yt$*] implies* $ex = ey$ *[respectively* $xe = ye$*], for every* $t, x, y$ *in* $\Gamma$ *(we call it left [respectively right] e-cancelation property).*
*(iii) If* $x \in \Gamma$ *and* $\varepsilon \in Iz(\Gamma)$*, then* $\varepsilon x = ex = xe = x\varepsilon$*, and so*

$$Inv(\varepsilon) = Inv(e)\ ,\ \varepsilon\Gamma = e\Gamma \ \ : \ \ \forall \varepsilon \in Iz(\Gamma).$$

*Also, if an element* $x_0$ *satisfies* $\varepsilon_1 x_0 = \varepsilon_2$*, for some* $\varepsilon_1, \varepsilon_2 \in Iz(\Gamma)$*, then* $\varepsilon_2 = e$ *and* $x_0 \in Inv(e)$*. Hence, for every* $\varepsilon_1, \varepsilon_2 \in Iz(\Gamma)$*, the equation* $\varepsilon_1 x = \varepsilon_2$ *has a solution in* $\Gamma$ *if and only if* $\varepsilon_2 = e$*.*
*(iv) For every* $x, y$*,*

$$\exists t_0 \in \Gamma\ :\ t_0 x = t_0 y \ \Leftrightarrow\ \exists \varepsilon_1, \varepsilon_2 \in Iz(\Gamma)\ :\ \varepsilon_1 x = \varepsilon_2 y \ \Leftrightarrow\ ex = ey.$$

*(v) In each of the following descriptions,* $e$ *is the unique element of* $\Gamma$:
*- There exists a unique solvable identity-like in* $\Gamma$*.*
*- There exists a unique central idempotent element in* $\Gamma$*.*
*- There exists a unique idempotent identity-like in* $\Gamma$*.*
*- There exists the least idempotent in* $\Gamma$*.*

*Proof.* If $\varepsilon_1, \varepsilon_2 \in Iz(\Gamma)$, then $\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2 \in Z(\Gamma)$ and $(\varepsilon_1\varepsilon_2)^2 = \varepsilon_1^2\varepsilon_2^2 = \varepsilon_1\varepsilon_2^2 = \varepsilon_1\varepsilon_2$. For every $x \in \Gamma$, there exists $y \in \Gamma$ such that $\varepsilon_1^2 = yx = xy$, so $\varepsilon_1\varepsilon_2 = (\varepsilon_1\varepsilon_2)^2 = \varepsilon_2\varepsilon_1^2 = (\varepsilon_2 y)x = x(\varepsilon_2 y)$, and thus $\varepsilon_1\varepsilon_2 \in$

$Sq(\Gamma)$. Therefore, $\varepsilon_1\varepsilon_2$ is an idempotent identity-like of $\Gamma$ and Lemma 2.2 implies $\varepsilon_1\varepsilon_2 = e$. Specially, we have $\varepsilon^2 = e = \varepsilon e = e\varepsilon$, for every $\varepsilon \in Iz(\Gamma)$. Also, it is easy to see that $It(\Gamma) \subseteq Inv(e)$ (thus $e \leq \delta$, for all idempotents $\delta$) and

$$Zt(\Gamma) = Nz(\Gamma) \cap It(\Gamma) = Iz(\Gamma) \cap It(\Gamma) = Izt(\Gamma) = \{e\}.$$

So, we have established the proof of (i). Other parts are concluded from (i), Lemma 2.2 and axioms (3) and (4) of the definition. □

The above lemma contains two important facts, $\varepsilon\Gamma = e\Gamma$ and the equivalent relation $ex = ey$, in grouplikes. These induce two groups that are very useful for studying grouplikes. Let us introduce them here. For every $x, y \in \Gamma$, we define

$$x \sim_e y \ \Leftrightarrow \ ex = ey.$$

Note that $\sim_e$ is a semigroup congruence and $\Gamma/\sim_e$ (the set of all equivalent classes $\overline{x}$ that obtained from $\sim_e$) is its quotient semigroup with the binary operation $\circ$, defined by $\overline{x} \circ \overline{y} = \overline{xy}$ . For, if $\overline{x_1} = \overline{y_1}$ and $\overline{x_2} = \overline{y_2}$, then $ex_1 = ey_1$ and $ex_2 = ey_2$, so $(ex_1)(ex_2) = (ey_1)(ey_2)$, thus $ex_1x_2 = ey_1y_2$, and so $\overline{x_1x_2} = \overline{y_1y_2}$.

**Theorem 2.6.** *The quotient semigroup $(\Gamma/\sim_e, \circ)$ is a group and $e\Gamma$ is a <u>maximal</u> subgroup (as the sense of subgroup of a semigroup) and also <u>minimum</u> ideal of $\Gamma$ . Moreover, $\Gamma/\sim_e \cong e\Gamma$ and*

$$(2.1) \qquad (ex)^{-1} = ex' = e(ex)' \ , \ (ex)^{-1} \sim_e x' \sim_e (ex)' \ : \ \forall x \in \Gamma,$$

*where $x'$ [respectively $(ex)'$] is every inverse-like of $x$ [respectively $ex$] in $\Gamma$ and $(ex)^{-1}$ is the inverse of $ex$ in the group $e\Gamma$.*

*Proof.* The identities $(ex)(ey) = exy$, $e(ex) = ex = (ex)e$ and $e = (ex)(ex') = (ex')(ex)$ imply $e\Gamma$ is subgroup of $\Gamma$ with the identity $e$ and (2.1) holds. Now, if $e\Gamma \subseteq G \leq \Gamma$, then $e \in G$, so $e = e_G$ and $G = e_G G = eG \subseteq e\Gamma$. Therefore, $e\Gamma$ is a maximal subgroup. Also, $e\Gamma \vartriangle \Gamma$, clearly, and if $I \vartriangle \Gamma$, then $e \in I$, by the axiom (4) of grouplikes and the identity $\varepsilon^2 = e$, so $e\Gamma \subseteq I$. It is clear that $\overline{e}$ is the identity element of $\Gamma/\sim_e$ and $\overline{(ex)^{-1}} = \overline{x'} = \overline{(ex)'}$ is the same inverse of $\overline{x}$ for a given $\overline{x} \in \Gamma/\sim_e$. Therefore, $(\Gamma/\sim_e, \circ)$ is a group. Now, define the map $\Psi : e\Gamma \to \Gamma/\sim_e$, by $\Psi(ex) = \overline{x}$, for every $x \in \Gamma$. It is a surjective well-defined map and $\Psi(ex) = \Psi(ey)$ if and only if $ex = ey$, for every $x, y \in \Gamma$. Also,

$$\Psi((ex)(ey)) = \Psi(e(xy)) = \overline{xy} = \overline{(ex)(ey)} = \Psi(x) \circ \Psi(y).$$

Therefore, the proof is complete. □

**Note.** First notice that in general $e\Gamma$ is not the largest subgroup of $\Gamma$ (consider the grouplike $\Omega$ in Example 2.4), but it is so if $\Gamma$ is a standard grouplike. In fact,

$$e\Gamma \leq_{\max} \Gamma \;\Leftrightarrow\; \Gamma \text{ is standard } \;\Leftrightarrow It(\Gamma) = Zt(\Gamma).$$

$$\Leftrightarrow\; \Gamma \text{ contains the largest subgroup}$$

Regarding the above theorem, it can be shown that for a semigroup $S$ for which $Zt(S)$ is a singleton, we have

$$S \text{ is grouplike } \;\Leftrightarrow\; S \text{ contains an ideal subgroup.}$$

$$\Leftrightarrow\; S \text{ contains the least ideal which is also a maximal subgroup.}$$

Therefore, in general, we have

$$S \text{ is standard grouplike}$$

$$\Leftrightarrow\; S \text{ has an ideal subgroup containing all its idempotents}$$

$$\Leftrightarrow\; S \text{ contains the least ideal which is also the largest subgroup.}$$

Considering the property $Iz(\Gamma) \subseteq Inv(e)$, we consider two hypothesis for grouplikes:

($\mathbf{H_1}$) (The identity-like hypothesis) $exy = xy$, for every $x, y \in \Gamma$ (equivalently, $e$ is a bi-identity of $\Gamma$).
($\mathbf{H_2}$) (The inverse-like hypothesis) $Inv(e) = Iz(\Gamma)$ (equivalently, $Inv(e) \subseteq Iz(\Gamma)$).

By Lemma 2.5, in every grouplike, the equation $exy = xy$ holds, for all $x, y$ such that $x$ or $y$ belongs to $e\Gamma \cup Iz(\Gamma)$.
Now, we prove that $(H_1)$ implies $(H_2)$.

**Theorem 2.7.** *In every grouplike, the identity-like hypothesis implies the inverse-likes hypothesis.*

*Proof.* Define another binary operation in $\Gamma$ by $x \centerdot y = exy$. This is associative, $x \centerdot (y \centerdot z) = (x \centerdot y) \centerdot z = exyz$, and $\varepsilon \centerdot x = e \centerdot x = ex$, for every $x \in \Gamma$. Thus, $Iz(\Gamma, \cdot) \subseteq Iz(\Gamma, \centerdot)$, and so $e \in Zt(\Gamma, \centerdot)$. Also, if $\delta \in Zt(\Gamma, \centerdot)$, then $e\delta \in Zt(\Gamma, \cdot)$ and $\delta = e\delta^2$, and so $e\delta \in Sv(\Gamma, \cdot)$. Thus,

$$\delta = e\delta^2 = (e\delta)^2 = (e\delta y)x = (e\delta)xy = x(e\delta)y = x(e\delta y),$$

for some $y \in \Gamma$. Therefore, $(\Gamma, \centerdot)$ is grouplike. Now, if $\delta \in Iz(\Gamma, \centerdot)$, then $e\delta \in Iz(\Gamma, \cdot)$, so $e = e(e\delta) = e\delta$ and $\delta \in Inv(e, \cdot)$. Conversely, if $\delta \in Inv(e, \cdot)$, then $e\delta = e$, $\delta \in Z(\Gamma, \centerdot)$ (because $\delta \centerdot x = e\delta x = ex = xe = x(e\delta) = ex\delta = x \centerdot \delta$), and $\delta \centerdot \delta \centerdot x = e\delta^2 x = ex = e\delta x = \delta \centerdot x$. Also, putting $y = \delta^2 x'$, we have

$$y \centerdot x = e\delta^2 x' x = \delta^2 e = \delta \centerdot \delta$$

$$x \centerdot y = x \centerdot (\delta^2 x') = x \centerdot (\delta^2 \centerdot x') = x \centerdot (x' \centerdot \delta^2) = e\delta^2 = \delta \centerdot \delta.$$

Therefore, $Inv(e, \cdot) = Iz(\Gamma, \centerdot)$. But, $(H_1)$ implies $\centerdot = \cdot$, and so $Inv(e, \cdot) = Iz(\Gamma, \centerdot) = Iz(\Gamma, \cdot)$. □

**Remark 2.8.** *There exists a short proof for the above theorem, but we give the proof for introducing the dual grouplike as follows.*
*For every grouplike* $(\Gamma, \cdot, e)$, *we have the grouplike* $(\Gamma, \centerdot, e)$, *where* $x \cdot^e y := x \centerdot y = exy$.

**Dual grouplike:** *We call the grouplike* $(\Gamma, \cdot^e, e)$ dual grouplike of $(\Gamma, \cdot, e)$. *It is an important fact that in the dual grouplikes, the two hypothesis and the following identities hold:*

$$Iz(\Gamma, \cdot) \subseteq Iz(\Gamma, \cdot^e) = Inv(e, \cdot) = Inv(e, \cdot^e) = (\overline{e}, \cdot) = (\overline{e}, \cdot^e),$$

*where* $(\overline{e}, \cdot)$ *[respectively* $(\overline{e}, \cdot^e)$*] is the class congruence of* $e$ *in* $(\Gamma, \cdot)$ *[respectively* $(\Gamma, \cdot^e)$*].*
*We will show that* $(H_1)$ *is correct only for* class united grouplikes *which is a very vast class of grouplikes and contains a large class of grouplikes namely,* $f$-grouplikes. *However, the followings are some examples of grouplikes which do not satisfy the hypotheses* $(H_1)$ *and* $(H_2)$:
*All non-null semigroups* $S$ *such that* $Zt(S) = \{0\}$ *are such grouplikes (note that* $Inv(e) = Inv(0) = S$ , $Iz(S) = Null(S) \neq S$ *and* $Izt(S) = Sv(S) = \{0\}$*). Specially, the semigroup* $S = \{0\} \cup (1, +\infty)$ *which is a sub-semigroup of the multiplicative real semigroup. So, it is a standard grouplike that does not satisfy* $(H_2)$ *and* $(H_1)$. *Also,* $\Omega$ *in Example* 2.4 *is a non-standard grouplike which does not satisfy* $(H_2)$.
*Note that if* $\Gamma$ *satisfies the hypothesis*$(H_2)$, *then* $\Gamma$ *is standard (because* $It(\Gamma) \subseteq Inv(e) = Iz(\Gamma) \subseteq Z(\Gamma)$ *so* $It(\Gamma) = Zt(\Gamma) = \{e\}$*).*
*More explanations are mentioned in the last section of this paper. Of course, the following question remains unanswered here*:
**Question.** *Does the inverse-like hypothesis* $(H_2)$ *imply the identity-like hypothesis* $(H_1)$*? (the answer seems to be negative.)*

Now, we state several equivalent conditions of $(H_1)$ for grouplikes.

**Lemma 2.9.** *For every grouplike* $(\Gamma, \cdot, e)$ *the following statements are equivalent:*
*(i)* $\Gamma$ *satisfies the identity-like hypothesis;*
*(ii)* $\varepsilon xy = xy$*, for every* $x, y \in \Gamma$ *and* $\varepsilon \in Iz(\Gamma)$*;*
*(iii)* $\varepsilon_0 xy = xy$*, for every* $x, y \in \Gamma$ *and some* $\varepsilon_0 \in Iz(\Gamma)$*;*
*(iv)* $e\Gamma = \Gamma^2$*;*
*(v)* $Iz(\Gamma)\Gamma = \Gamma^2$*;*
*(vi)* $\Gamma$ *is equal to its dual grouplike;*
*(vii)* $\Gamma$ *is a class united grouplike (see Theorem* 2.11*);*
*(viii)* $\Gamma^2$ *is a subgroup of* $\Gamma$*;*
*(ix)* $x\Gamma = \Gamma^2$ *[*$\Gamma x = \Gamma^2$*], for every* $x \in \Gamma$*.*
*Moreover, if* $(H_2)$ *holds, then the following conditions are equivalent to (i)-(ix):*
*(x)* $\gamma xy = xy$ *[*$xy = xy\gamma$*], for every* $x, y \in \Gamma$ *and some* $\gamma \in \Gamma$*;*
*(xi)* $\Gamma^2$ *has a left [right] identity;*
*(xii)* $\Gamma^2$ *is a sub-monoid of* $\Gamma$ *(i.e.* $\Gamma^2 \dot{\leq} \Gamma$ *and* $\Gamma^2$ *is a monoid).*

*Proof.* If $xy \in Iz(\Gamma)\Gamma^2$ or $xy \in e\Gamma$, then $xy = ez$, for some $z \in \Gamma$ so $exy = ez = xy$.
If $\Gamma^2$ is group, then $e\Gamma \leq \Gamma^2 \leq \Gamma$ and so $\Gamma^2 = e\Gamma$ (because $e\Gamma$ is a maximal subgroup).
If $(H_1)$ holds and $x \in \Gamma$, then $\Gamma^2 = e\Gamma = (xx')\Gamma = x(x'\Gamma) \subseteq x\Gamma$. So $x\Gamma = \Gamma^2$.
Also, if (x) and the hypothesis $(H_2)$ hold, then putting $x = y = e$ we have $\gamma e = e$ thus $\gamma \in Inv(e) = Iz(\Gamma)$. Therefore $xy = \gamma xy = exy$, for every $x, y$.
The above explanations together with Lemma 2.5 and Theorem 2.11 complete the proof. □

**Note.** If $(\Gamma, \cdot, e)$ satisfies the identity-like hypothesis, then it satisfies the weak cancelation properties, which are

$$txy = tzw \ \Rightarrow \ xy = zw \ , \ xyt = zwt \ \Rightarrow \ xy = zw \ : \quad \forall x, y, z, w, t \in \Gamma.$$

Also, we have

$$x \sim_e y \ \Leftrightarrow \ zx = zy \ \Leftrightarrow \ xz = yz,$$

for every $x, y, z \in \Gamma$.

2.1. **Class united grouplikes.** We call $\mathcal{G}$ a *class group,* if $\mathcal{G}$ is a group for which all its elements are nonempty disjoint sets. Every quotient group is a class group and it can be shown that for every cardinal number $a \neq 0$, there exists a class group $\mathcal{G}$ with the cardinal number $a$ (if we

accept the generalized continuum hypothesis).
We call every function $\Psi : \cup\mathcal{G} \to \mathcal{G}$ a class function, if $x \in \Psi(x)$, for every $x \in \cup\mathcal{G}$. Because of our assumption for $\mathcal{G}$, the surjective class function $\Psi$ always exists and is unique. We use the notation $\Psi(x) = A_x$, when $A \in \mathcal{G}$ and $x \in A = \Psi(x)$. Now, if $\varphi$ is a choice function from $\mathcal{G}$ to $\cup\mathcal{G}$ ($\varphi(A) \in A$), then it is injective, $\Psi\varphi = 1_{\mathcal{G}}$, $\varphi = \varphi\Psi\varphi$ and $\Psi = \Psi\varphi\Psi$ (so, $A_x = A_{\varphi(A_x)}$, for every $x \in \cup\mathcal{G}$).
Now, let $E$ be the identity element of the class group $(\mathcal{G}, \circ)$ and define the binary operation $\centerdot = \centerdot^{\varphi}$ in $\cup\mathcal{G}$ by

$$x \centerdot^{\varphi} y = x \centerdot y = \varphi(\Psi(x) \circ \Psi(y)) = \varphi(A_x \circ A_y) \quad : \quad \forall x, y \in \cup\mathcal{G},$$

where $\Psi$ is the unique class function and $\varphi$ is an arbitrary choice function.
First, we have

$$x \centerdot (y \centerdot z) = \varphi(A_x \circ A_{\varphi(A_y \circ A_z)}) = \varphi(A_x \circ (A_y \circ A_z))$$
$$= \varphi((A_x \circ A_y) \circ A_z) = \varphi(A_{\varphi(A_x \circ A_y)} \circ A_z) = (x \centerdot y) \centerdot z.$$

Therefore, $x \centerdot (y \centerdot z) = (x \centerdot y) \centerdot z = \varphi(A_x \circ A_y \circ A_z)$. Now, if $\varepsilon \in E$, then $\varepsilon \centerdot \varepsilon = \varphi(E)$. Putting $e = \varphi(E)$, we have $e \centerdot e = e$ and $e \centerdot x = x \centerdot e = \varphi(A_x)$, for every $x \in \mathcal{G}$. So, $e \in Zt(\cup\mathcal{G}, \centerdot)$. Now, if $\delta \in Zt(\cup\mathcal{G}, \centerdot)$, then $\delta = \delta \centerdot \delta = \varphi(A_\delta \circ A_\delta) \in A_\delta \circ A_\delta$. If $x \in \cup\mathcal{G}$, then putting $y = \varphi(A_\delta \circ A_\delta \circ B)$, where $B$ is the inverse element of $A = A_x$ in $\mathcal{G}$, then we have

$$x \centerdot y = \varphi(A_x \circ A_{\varphi(A_\delta \circ A_\delta \circ B)}) = \varphi(A_x \circ A_{\varphi(B \circ A_\delta \circ A_\delta)})$$
$$= \varphi((A_x \circ B) \circ (A_\delta \circ A_\delta)) = \varphi(A_\delta \circ A_\delta) = \delta.$$

Similarly, $y \centerdot x = \delta$. Therefore, $\emptyset \neq Zt(\cup\mathcal{G}, \centerdot) \subseteq Sv(\cup\mathcal{G}, \centerdot)$, and so $(\cup\mathcal{G}, \centerdot)$ is a grouplike. Hence, we can state the following definition.

**Definition 2.10.** *Let $(\mathcal{G}, \circ)$ be a class group with the identity element $E$ and $\Psi_{\mathcal{G}} : \cup\mathcal{G} \to \mathcal{G}$ be the unique class function and $\varphi$ be a given choice function from $\mathcal{G}$. We call the algebraic structure $(\cup\mathcal{G}, \centerdot^{\varphi}, \varphi(E))$* $\varphi$-class united grouplike. *Also, we say a grouplike $(\Gamma, \cdot)$ is* class united, *if there exists a class group $(\mathcal{G}, \circ)$ and a choice function $\varphi$ such that $\cup\mathcal{G} = \Gamma$ and $\centerdot^{\varphi} = \cdot$ .*

**Theorem 2.11.** *(A) For every class united grouplike the identity-like and the inverse-like hypotheses hold. Hence, a grouplike is class united if and only if it satisfies the hypothesis $(H_1)$.*
*(B) (General form of grouplikes satisfying the identity-like hypothesis) A magma $(\Gamma, \cdot)$ is a grouplike with the identity-like hypothesis if and only*

*if it there exist a class group $\mathcal{G}$ and a choice function $\varphi : \mathcal{G} \to \cup\mathcal{G}$ such that $\Gamma = \cup\mathcal{G}$ and $\cdot = \centerdot^{\varphi}$.*

*Proof.* (A): Consider $(\mathcal{G}, \circ)$, $E$, $\varphi$, $\centerdot = \centerdot^{\varphi}$ as above, and put $e = \varphi(E)$. Then

(2.2) $e \centerdot x \centerdot y = \varphi(A_e \circ A_x \circ A_y) = \varphi(E \circ A_x \circ A_y) = \varphi(A_x \circ A_y) = x \centerdot y,$

for every $x, y \in \cup\mathcal{G}$. Hence, if $(\Gamma, \cdot)$ is class united, then its standard identity-like is equal to $\varphi(E)$, and so (2.2) guarantees that $\Gamma$ satisfies $(H_1)$. Conversely, if in $(\Gamma, \cdot, e)$ the hypothesis $(H_1)$ holds, then putting $(\mathcal{G}, \circ) = (\overline{\Gamma}, \circ)$, $\varphi(\overline{x}) = ex$, we have $\cup\mathcal{G} = \Gamma$, $\varphi$ is a choice function from $\mathcal{G}$ and $\centerdot^{\varphi} = \cdot$, because

$$x \centerdot^{\varphi} y = \varphi(\overline{x} \circ \overline{y}) = \varphi(\overline{xy}) = exy = x \cdot y,$$

for every $x, y \in \cup\mathcal{G} = \Gamma$. So, $\Gamma$ is class united.
Part (B) is concluded from (A) and Definition 2.10. □

**Note.** Theorem 2.11 implies that every dual grouplike is class united. Hence, every grouplike gives us a class united grouplike.

**Theorem 2.12.** *Let $a$ be a given infinite [respectively finite] cardinal number and $(G, \cdot, e)$ be an arbitrary group such that $a \geq |G|$ [respectively $a > |G|$]. Then, there exists a proper grouplike $(\Gamma, \centerdot, e)$ with cardinal $a$ such that $G \subsetneq \Gamma$, $\centerdot|_{G \times G} = \cdot$, $G = e \centerdot \Gamma$ and $G$ is homomorphic image of $\Gamma$ under a semigroup endomorphism in $\Gamma$. Moreover, if $b$ is an arbitrary cardinal number such that $b \leq a$ [respectively $b < a - |G| + 1$], then there exists such proper grouplike for which $|Iz(\Gamma)| = b$. Also, we can find such a $\Gamma$ for which $(\Gamma, \centerdot, \epsilon)$ is proper grouplike with $\epsilon \neq e \in Iz(\Gamma)$.*

*Proof.* For a given group $G$, consider an arbitrary family of disjoint sets $\{A^g\}_{g \in G}$ such that $A^g \cap G = \emptyset$, for every $g \in G$. Now, put $G_g = A^g \cup \{g\}$ $(\forall g \in G)$ and $\mathcal{G} = \{G_g | g \in G\}$. We show an element $A$ of $\mathcal{G}$ by $A = A_x$ for every $x \in A$, so $A = A_x = G_x = A^x \cup \{x\}$ if $x \in A \cap G$. Therefore, $\mathcal{G}$ is a set of nonempty disjoint classes $G_g$ and $|\mathcal{G}| = |G|$. So, there exists a bijection $\varphi : \mathcal{G} \to G$ such that $\varphi(G_e) = e$. Define the binary operation $\circ$ on $\mathcal{G}$ by $A_x \circ A_y = \varphi^{-1}(\varphi(A_x)\varphi(A_y))$. Hence, $(\mathcal{G}, \circ)$ is a group and isomorph to $G$. Now, putting $\Gamma = \cup\mathcal{G}$, we have $G \subseteq \Gamma$. Note that $\varphi : \mathcal{G} \to \Gamma$ is a choice function such that $\varphi(A) \in A \cap G$, for every $A \in \mathcal{G}$ and $\varphi(\mathcal{G}) = G$. Therefore, we have class united grouplike $(\Gamma, \centerdot^{\varphi}) = (\Gamma, \centerdot)$ such that

$$x \centerdot y = \varphi(G_x \circ G_y) = \varphi(G_x)\varphi(G_y) = x \cdot y \ \ ; \ \ \forall x, y \in G.$$

So, $\centerdot|_{G\times G} = \cdot$. Also,

$$e \centerdot \Gamma = \{\varphi(G_e \circ A_x)|x \in \Gamma\} = \{\varphi(A_x)|x \in \Gamma\} = \{\varphi(A_g)|g \in G\} = G.$$

In addition, the function $\Psi : \Gamma \to \Gamma$, defined by $\Psi(x) = e \centerdot x$, is a semigroup homomorphism and $\Psi(\Gamma) = G$. Notice that $Iz(\Gamma) = G_e$, $|\Gamma| = \sum_{g\in G} |G_g|$ and we are free to choice every $G_g$ with a given cardinal $\geq 1$. Therefore, we can get the results. For the last part, if we take $A_e$ and the bijection $\varphi$ such that $\varphi(G_e) = \epsilon \neq e$, then $\epsilon$ is the standard identity-like of $\Gamma$. □

Now, we want to state an isomorphism equivalent condition for a grouplike to be class united grouplike. Hence, we introduce grouplike homomorphisms after proving a lemma.

**Lemma 2.13.** *Let $X_1$, $X_2$ be magmas and $\Gamma_1$, $\Gamma_2$ be grouplikes.*
*(i) If $\mu : X_1 \to X_2$ is magma epimorphism, then $\mu(\chi(X_1)) \subseteq \chi(X_2)$, for all the cases $\chi = Z, It, Zt, Nz, Iz, Izt, Sq, Sv$. So, if $\mu$ is bijection (magma isomorphism), then $\mu(\chi(X_1)) = \chi(X_2)$, for all the cases.*
*(ii) If $\mu : (\Gamma_1, \cdot, e_1) \to (\Gamma_2, \cdot, e_2)$ is magma homomorphism, then*

$$\mu(e_1) = e_2 \Leftrightarrow \mu(e_1) \in Iz(\Gamma_2). \tag{2.3}$$

*Moreover, if the hypothesis $(H_2)$ is correct for $\Gamma_2$, then*

$$\mu(e_1) = e_2 \Leftrightarrow \mu(e_1) \in Iz(\Gamma_2) \Leftrightarrow \mu(Iz(\Gamma_1)) \subseteq Iz(\Gamma_2).$$

*Proof.* If $\chi(X_1) = \emptyset$, for some cases of $\chi$, then it is trivial. So, we consider the nonempty cases. Choose $\varepsilon_1 \in Nz(X_1)$ and put $\varepsilon_2 = \mu(\varepsilon_1)$. If $x_2 \in X_2$, then $x_2 = \mu(x_1)$, for some $x_1 \in X_1$ and

$$(\varepsilon_2^2)x_2 = \mu((\varepsilon_1^2)x_1) = \mu(\varepsilon_1(\varepsilon_1 x_1)) = \varepsilon_2(\varepsilon_2 x_2).$$

Analogously, we can show that $x\varepsilon_2 = \varepsilon_2 x = \varepsilon_2^2 x = \varepsilon_2(\varepsilon_2 x) = (x\varepsilon_2)\varepsilon_2 = x\varepsilon_2^2$, for every $x_2 \in X_2$. So, $\mu(\varepsilon_1) \in Nz(X_2)$. The proof of other parts of (i) are similar.
The identity $\mu(e_1) = \mu(e_1)^2$ and Lemma 2.5 imply (2.3). Now, if $\varepsilon_1 \in Iz(\Gamma_1)$, then $\mu(\varepsilon_1)\mu(e_1) = \mu(e_1)$. Hence, if $\mu(e_1) \in Iz(\Gamma_2)$, then $\mu(e_1) = e_2$ and the hypothesis $(H_2)$ implies $\mu(\varepsilon_1) \in Inv(e_2) = Iz(\Gamma_2)$. □

Considering Lemma 2.13, we state the following definition of *grouplike homomorphism* and *sub-grouplikes.*

**Definition 2.14.** *Let $\Gamma_1$, $\Gamma_2$ be grouplikes. We call $\mu : \Gamma_1 \to \Gamma_2$* grouplike homomorphism, *if it is magma homomorphism and $\mu(Iz(\Gamma_1)) \subseteq Iz(\Gamma_2)$. A grouplike homomorphism is a grouplike isomorphism, if it is*

*a bijection. Also, we say* $\Gamma_1$ *is isomorphic to* $\Gamma_2$ *and denote by* $\Gamma_1 \cong \Gamma_2$*, if there exists a grouplike isomorphism (equivalently, magma isomorphism) between them. We call a subset* $H$ *of* $\Gamma$ sub-grouplike *and denote by* $H \sqsubseteq \Gamma$*, if it satisfies the following properties*:
*(a)* $H \dot{\leq} \Gamma$ ; *(b)* $h' \in H$*, for all* $h \in H$ *(i.e.,* $\bigcup_{h\in H} Inv(h) \subseteq H$*).*

Note that if $H \sqsubseteq \Gamma$, then $Iz(\Gamma) \subseteq H$. For, if $h \in H$, then $h' \in H$, thus $e = hh' \in H$, and so $Iz(\Gamma) \subseteq Inv(e) \subseteq H$. Therefore, if $\Gamma$ satisfies $(H_2)$, then $Iz(\Gamma)$ is the smallest sub-grouplike of $\Gamma$. In fact,

$$\Gamma \text{ satisfies the inverse-like hypothesis } \Leftrightarrow\ Iz(\Gamma) \sqsubseteq \Gamma\ \Leftrightarrow\ Iz(\Gamma) \sqsubseteq_{\min} \Gamma.$$

Now, let $\mu : \Gamma_1 \to \Gamma_2$ be grouplike homomorphism and put

$$\ker(\mu) := \{x \in \Gamma_1 | \mu(x) \in Iz(\Gamma_2)\}\ ,\ Sker(\mu) := \{x \in \Gamma_1 | \mu(x) = e_2\},$$

and call $Sker(\mu)$ *standard kernel of* $\mu$. Since $\mu(e_1) = e_2$, then these are nonempty. Also, if $x, y \in \ker(\mu)$, then $\mu(xy) = e_2$. So

$$Sker(\mu) \dot{\leq} \ker(\mu) \dot{\leq} \Gamma_1\ ,\ e_1 \ker(\mu) \dot{\leq} \ker^2(\mu) \dot{\leq} Sker(\mu) \dot{\leq} \Gamma_1.$$

On the other hand, $e_2 = \mu(x)\mu(x') = \mu(x')\mu(x)$, for every $x$, and so

$$\mu(Inv_{e_1}(x)) \subseteq Inv_{e_2}(\mu(x)).$$

Therefore, if $x \in \ker(\mu)$, then $\mu(Inv_{e_1}(x)) \subseteq Inv_{e_2}(\mu(x)) = Inv_{e_2}(e_2)$. Now, if the hypothesis $(H_2)$ holds for $\Gamma_2$, then $\mu(Inv(x)) \subseteq Iz(\Gamma_2)$, and so $Inv(x) \subseteq \ker(\mu)$. Therefore, if the hypothesis $(H_2)$ holds for $\Gamma_2$, then $Sker(\mu) \dot{\leq} \ker(\mu) \sqsubseteq \Gamma_1$.
For a nice example of grouplike isomorphism, consider two constant non-zero real numbers $b$ and $\beta$ and define $\mu : (\mathbb{R}, +_b) \to (\mathbb{R}, +_\beta)$, by $\mu(x) = \frac{\beta}{b}x$. It is easy to check that

$$\mu(x +_b y) = \mu(x) +_\beta \mu(y) = \beta(\frac{x+y}{b})_1 \quad : \quad \forall x, y \in \mathbb{R},$$

$$\mu(Iz(\mathbb{R}, +_b)) = Iz(\mathbb{R}, +_\beta)\ ,\ \ker(\mu) = Iz(\mathbb{R}, +_b) = b\mathbb{Z}\ ,\ Sker(\mu) = \{0\}.$$

So, every real $b$-grouplike is isomorphic to the real grouplike.

**Lemma 2.15.** *A grouplike* $(\Gamma, \centerdot, e)$ *is class united if and only if there exists a* $\varphi$*-class united grouplike* $(\cup\mathcal{G}, \centerdot^\varphi)$ *such that* $(\Gamma, \centerdot) \cong (\cup\mathcal{G}, \centerdot^\varphi)$.

*Proof.* Let $\mu : (\cup\mathcal{G}, \centerdot^\varphi) \to (\Gamma, \centerdot)$ be an isomorphism and $x, y \in \Gamma$, $x = \mu(z)$, $y = \mu(w)$, for some $z, w \in \cup\mathcal{G}$. Then

$$e \centerdot x \centerdot y = \mu(\varphi(E) \centerdot^\varphi z \centerdot^\varphi w) = \mu(z \centerdot^\varphi w) = x \centerdot y.$$

Hence, Theorem 2.11 implies that $(\Gamma, \centerdot)$ is class united grouplike. The converse is clear. □

**Remark 2.16.** *Let $\Gamma$ be a class united grouplike. Recall that all the equivalent conditions of Lemma* 2.9 *and the following important properties hold:*

*(a) $e$ is the unique idempotent element of $\Gamma$.*
*(b) $\Gamma$ is zero grouplike if and only if it contains the zero element if and only if it is null grouplike.*
*(c) $\Gamma$ contains a left or right identity if and only if it is monoid if and only if $\Gamma$ is group.*
*(d) Every left and right coset of $\Gamma$ is its subgroup ($x\Gamma \leq \Gamma$ and $\Gamma x \leq \Gamma$, for every $x \in \Gamma$).*
*Figure 3 shows the inclusion relation for the mentioned subsets of class united grouplikes.*

## 3. Some other related topics for semigroups

Here, we show that the introduced concepts and subsets in the topic grouplikes can be used for studying semigroups (even magmas and groups) in general. In fact, the concepts neutral-likes, identity-likes and solvable elements and also the basic subsets $Z(S)$, $It(S)$, $Nz(S), Sv(S)$, and $Sq(S)$ play important roles and can help us in studying such algebraic structures. Of course, the other mentioned subsets $Zt(S)$, $Iz(S)$, $Izt(S)$ and $Sz(S)$, are obtained by their products or intersections. The followings are some of their basic properties:

$$\begin{gathered} Nz(S)Nz(S) = Zt(S) \subseteq Nz(S)\ ,\ Nz(S)It(S) \subseteq It(S), \\ Iz(S)Iz(S) = Izt(S) \subseteq Iz(S), \\ Sv^2(S) \subseteq Sv(S)\ ,\ Sv(S)Z(S) \subseteq Sv(S), \\ Sq(S)Z(S) \subseteq Sq(S)\ ,\ Sv(S) \subseteq Sz(S). \end{gathered} \tag{3.1}$$

Note that if $\delta_1, \delta_2 \in Nz(S)$, then $(\delta_1\delta_2)^2 = \delta_1^2\delta_2^2 = \delta_1\delta_2^2 = \delta_1\delta_2$ belongs to $Zt(S) = Zt^{(2)}(S) = Zt^2(S) \subseteq Nz^2(S)$.
Now, considering the above relations and the fact "$H^2 \subseteq H$ if and only if $H \dot{\leq} S$ or $H = \emptyset$", we are led to the next important lemma.

**Lemma 3.1.** *For every semigroup, we have*:
*(i)* $Nz(S)Zt(S) = Zt(S) = Nz^2(S) = Nz^{(2)}(S) = Zt^2(S) = Zt^{(2)}(S)$.
*(ii)* $Nz(S) \neq \emptyset \Leftrightarrow Zt(S) \neq \emptyset \Leftrightarrow Zt(S) \vartriangle Nz(S) \dot{\trianglelefteq} Z(S) \dot{\trianglelefteq} S$.
*(iii)*

$$\begin{aligned} Nz(S)Iz(S) &= Nz(S)Izt(S) = Izt(S) = Iz(S)Zt(S) \\ &= Iz(S)Izt(S) = Iz^2(S) = Iz^{(2)}(S) = Izt^2(S) = Izt^{(2)}(S). \end{aligned}$$

*(iv)* $Iz(S) \neq \emptyset \Leftrightarrow Izt(S) \neq \emptyset \Leftrightarrow Izt(S) \vartriangle Iz(S) \vartriangle Nz(S) \dot{\trianglelefteq} Z(S) \dot{\trianglelefteq} S.$
*(v)*

$$Sv(S) \neq \emptyset \Leftrightarrow Sz(S) \neq \emptyset \Leftrightarrow Sq(S) \neq \emptyset$$
$$\Leftrightarrow Sv(S) \vartriangle Sz(S) \vartriangle Z(S)\ ,\ Sv(S) \vartriangle Z(S).$$

*So, if $S$ is commutative, then $Sq(S) \vartriangle S$ and $Sv(S) \vartriangle S$.*

*Proof.* We have

$$\begin{aligned} Nz(S)Zt(S) &= Nz(S)(Z(S) \cap It(S)) \subseteq Nz(S)Z(S) \cap Nz(S)It(S) \\ &\subseteq Z(S) \cap It(S) \\ &= Zt(S) = Zt^2(S) \subseteq Nz(S)Zt(S). \end{aligned}$$

Also,

$$\begin{aligned} Nz(S)Iz(S) &= Nz(S)(Nz(S) \cap Sq(S)) \subseteq Nz^2(S) \cap Nz(S)Sq(S) \\ &\subseteq Zt(S) \cap Sq(S) = Izt(S) \subseteq Nz(S)Iz(S). \end{aligned}$$

Other parts of the proof are similar to the above parts or are easy. □

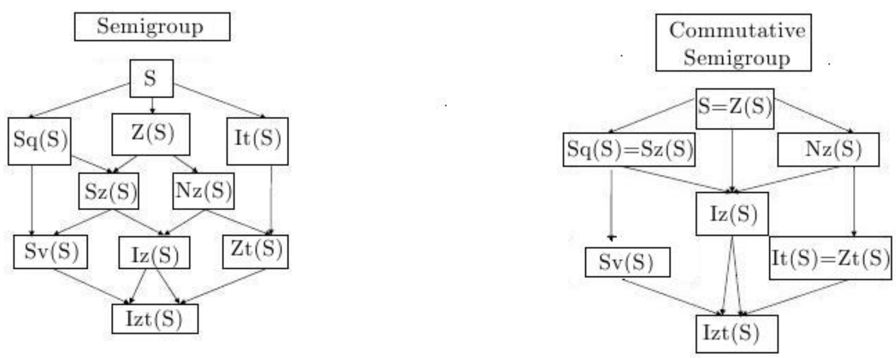


Figure 1. Semigroup and Commutative Semigroup

**Note.** One can see the interesting comprisal relations for semigroups, grouplikes, class united grouplikes and groups in figures 1-4, separately. In the diagrams, wherever two arrows meet togethers, then the resulting set is the intersection, exactly. If $Izt(S) \neq \emptyset$ (e.g., $0 \in S$), then all the above normal sub-semigroups and ideals exist. In view of the above lemma, we see that if $Zt(S) \neq \emptyset$ [respectively $Izt(S) \neq \emptyset$], then it has a property stronger than $Zt(S) \vartriangle Nz(S)$ [respectively $Izt(S) \vartriangle Iz(S)$, $Izt(S) \vartriangle Nz(S)$] which is $Nz(S)$-periodic [respectively $Iz(S)$-periodic and $Nz(S)$-periodic]. Also, $Z(S)Sq(S) = Sq(S)Z(S) \subseteq Sq(S)$

means that $Sq(S)$ is upper $Z(S)$-periodic (note that, in general, neither $Sq(S) \subseteq Z(S)$, nor $Z(S) \subseteq Sq(S)$). We have introduced and studied such subsets as periodic and upper periodic subsets of semigroups in [4]. This way, a new topic about a large class of subsets of semigroups containing all sub-semigroups, ideals and periodic subsets, is introduced. In fact, the *upper periodic subsets* can be considered as a generalization of the conception ideals. Also, a unique direct representation for upper and lower $T$-periodic subsets, when $T$ is an invertible element of $S$, is proved and applying it all sub-semigroups of $S$ containing the fixed element $T$ are classified to three classes.

Now, a corollary of the above lemma and some previous results follows here.

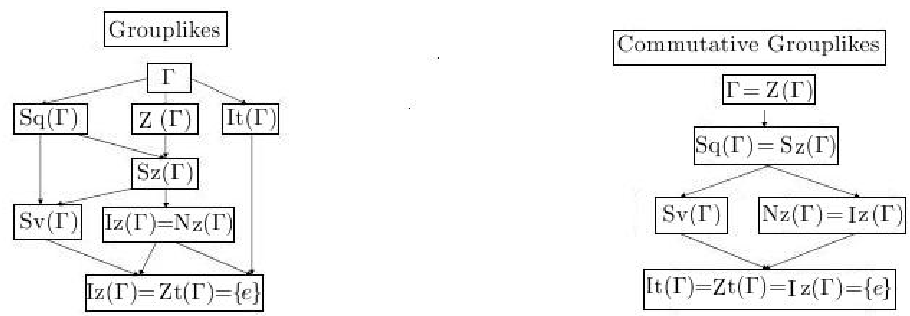


FIGURE 2. Grouplikes and Commutative Grouplikes

**Corollary 3.2.** *For every semigroup $S$, the following statements are equivalent.*
*(i) $S$ is grouplike;*
*(ii) $\emptyset \neq Nz(S) \subseteq Sq(S)$;*
*(iii) $Iz(S) = Nz(S)$ and one [all] of the subsets $Izt(S)$, $Zt(S)$, $Iz(S)$ $Nz(S)$ is [are] nonempty;*
*(iv) $\emptyset \neq Zt(S) \subseteq Sv(S)$;*
*(v) $Zt(S)$ and $Izt(S)$ are singletons;*
*(vi) $Zt(S)$ is a singleton subset of $Sv(S)$;*
*(vii) $S$ has an ideal subgroup containing all its central idempotents;*
*(viii) $S$ contains a <u>minimum</u> ideal which is also its <u>maximal</u> subgroup and $Zt(S)$ is a singleton.*

Note that if $Izt(S)$ is a singleton, then it dose not imply $S$ is a grouplike. For instance, consider $([0,+\infty),\cdot)$ or the following example.

**Example 3.3.** *This example show that none of the following conditions (alone) implies $S$ is a grouplike.*
*(a) $Iz(S) = Nz(S)$ (equivalently, $Nz(S) \subseteq Sq(S)$), e.g., $Iz((0,+\infty),+) = Nz((0,+\infty),+) = \emptyset$.*
*(b) $Izt(S)$ is a singleton, e.g., $Izt(\{0\} \cup [1,+\infty),\cdot) = \{0\}$ (because $Zt(\{0\} \cup [1,+\infty),\cdot) = \{0,1\}$).*
*(c) $Zt(S)$ is a singleton, e.g., $Zt([1,+\infty),\cdot) = \{1\}$.*
*(d) $Iz(S) \neq \emptyset$ (or $Iz(S)$ is singleton), e.g., the example (b).*

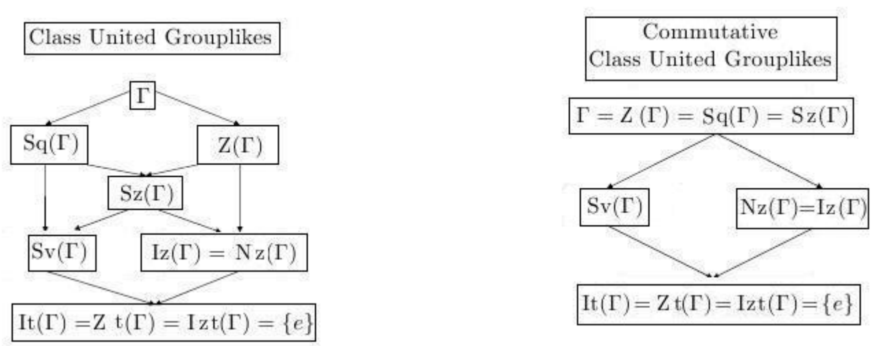


FIGURE 3. Class United Grouplikes and Commutative Class United Grouplikes

The following lemma and its corollary state some minimal conditions for a semigroup to be a grouplike.

**Lemma 3.4.** *If a semigroup $(S,\cdot)$ satisfies the following conditions, then it is a grouplike:*
(1) *There exists $\varepsilon \in S$ such that*

$$\varepsilon^2 x = \varepsilon x = x\varepsilon \;\; [\text{analogously, } x\varepsilon^2 = x\varepsilon = x\varepsilon] \;\; : \;\; \forall x \in S.$$

(2) *If $e$ is an idempotent element satisfying* (1) *and $x \in S$, then there exists $y \in S$ such that $yx = e$ [analogously, $xy = e$].*

*Proof.* Let $\varepsilon$ be an arbitrary element satisfying (1) and put $e = \varepsilon^2$. Then, $e \in Zt(S)$ and so for a given $x \in S$, there exists $y \in S$ such that $yx = e$.

Also, there exists $z \in S$ such that $zy = e$. Now, we have $(ey)x = e^2 = e$ and

$$x(ey) = e(xy) = (zy)(xy) = z(yx)y = zey = ezy = e^2 = e.$$

Therefore, $(S, \cdot)$ is a grouplike. □

**Corollary 3.5.** *(a) A monoidlike $(S, \cdot)$ is a grouplike if and only if for every $e \in Zt(S)$ and $x \in S$ there exists $y \in S$ such that $yx = e$ [analogously, $xy = e$].*
*(b) A standard monoidlike $(S, \cdot, e)$ is grouplike if and only if for every $x \in S$ there exists $y \in S$ such that $yx = e$ [analogously, $xy = e$].*

Now, we like to see what happens if a grouplike $(\Gamma, \cdot, e)$ has the zero. It implies $0 = e$, because $0 \in Zt(\Gamma)$. Therefore, a grouplike is a zero grouplike if and only if it contains the zero. In this case, $\Gamma$ satisfies the hypothesis $(H_2)$ if and only if $\Gamma$ is a null grouplike (because $\Gamma = Inv(0) = Iz(\Gamma)$).
Also, what happens if $(\Gamma, \cdot, e)$ has a left [respectively right] identity $\ell$ [respectively $r$].

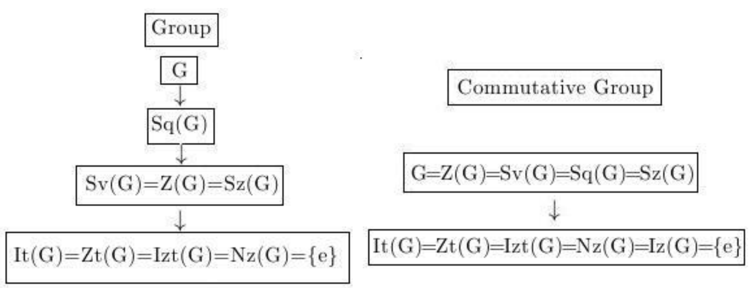


FIGURE 4. Group and Commutative Group

In this case, we have $\ell \in Inv(e)$ [respectively $r \in Inv(e)$] (because it is idempotent). Hence, if the hypothesis $(H_2)$ holds, then $\ell$ [respectively $r$] is the same (unique) idempotent identity-like, and so $\Gamma = e\Gamma$ is a group. Therefore, $(\Gamma, \cdot)$ is both grouplike and monoid if and only if it is a group (because $1 \in Zt(\Gamma)$).

In further studies of semigroup theory and grouplikes, considering some of the following classes of semigroups may be useful:
(A) The semigroups $S$ such that $Iz(S) \neq \emptyset$ (e.g., all homogroups that are

semigroups containing an ideal subgroup, and specially all grouplikes). In this case, all the normal sub-semigroups and ideals exist and

$$Izt(S) \vartriangle Iz(S) \vartriangle Nz(S) \dot{\trianglelefteq} Z(S) \dot{\trianglelefteq} S.$$

(B) All grouplikes $(\Gamma, \cdot, e)$ satisfying one (some) of the following conditions:
- $Inv(e) = Iz(\Gamma)$ (e.g., all class united grouplikes).
- $Inv(e) = Iz(\Gamma) \cup It(\Gamma)$ (e.g., all grouplikes with the hypothesis $(H_2)$).
- $Inv(e) = It(\Gamma)$ (e.g., all bands $B$ with zero such that $Zt(B) = \{0\}$).
- $Inv(e) = \{e\}$ (e.g., all groups).
- $It(\Gamma) = Zt(\Gamma)$ (i.e., standard grouplikes).
- $|Iz(\Gamma)| = 1$ (e.g., all groups).
- $|Iz(\Gamma)| = 1$ and $Sv(\Gamma) = Sq(\Gamma) = \Gamma$ (e.g., all abelian groups).

We close by an example illustrating some of the mentioned subsets.

**Example 3.6.** *Consider the semigroup $S$ and its power set $P(S) = 2^S$ and $P^*(S) = 2^S \setminus \{\emptyset\}$. Then, $(P(S), \cdot)$ is the semigroup of all subsets of $S$ with the same product of subsets of $S$ and $P^*(S) \dot{\leq} P(S)$. Clearly, $\emptyset$ is the zero of $P(S)$. If $S = M$ is monoid, then $M$ is the zero of $P^*(S)$ and*

$$\{N | N \dot{\leq}_1 M\} \subseteq It(P^*(M)) \subseteq \{H | H \dot{\leq} M\},$$
$$Izt(P^*(M)) = Sv(P^*(M)) = \{M\},$$

*where $N \dot{\leq}_1 M$ means $N$ is a sub-monoid of $M$.*

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