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# SOME DIFFERENCE RESULTS ON HAYMAN CONJECTURE AND UNIQUENESS

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ABSTRACT. In this paper, we show that for any finite order entire function f(z), the function of the form  $f(z)^n [f(z+c)-f(z)]^s$  has no nonzero finite Picard exceptional value for all nonnegative integers n, s satisfying  $n \geq 3$ , which can be viewed as a difference result on Hayman conjecture. We also obtain some uniqueness theorems for difference polynomials of entire functions sharing one common value.

## 1. Introduction

A meromorphic function f means meromorphic in the whole complex plane. If no poles occur, then f reduces to an entire function. Recall that  $\alpha(z) \neq 0, \infty$  is a small function with respect to f, if  $T(r, \alpha) = S(r, f)$ , where S(r, f) denotes any quantity satisfying S(r, f) = o(T(r, f)), as  $r \to \infty$  outside of a possible exceptional set of finite logarithmic measure. We say that two meromorphic functions f and g share a small function  $\alpha$ IM (ignoring multiplicities) when  $f - \alpha$  and  $g - \alpha$  have the same zeros. If  $f - \alpha$  and  $g - \alpha$  have the same zeros with the same multiplicities, then we say that f and g share the small function  $\alpha$  CM (counting multiplicities). We assume that the reader is familiar with standard

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notations and fundamental results of Nevanlinna Theory, such as [11, 12, 22]. We use  $\rho(f)$  to denote the order of f(z).

The Hayman conjecture [10] is an important research subject in considering the value distributions of differential polynomials, that is if f is a transcendental meromorphic function and  $n \in \mathbb{N}$ , then  $f^n f'$  takes every finite nonzero value infinitely often. This conjecture has been solved by Hayman [9] for  $n \geq 3$ , by Mues [18] for n = 2, by Bergweiler and Eremenko [1] for n = 1. From above, it shows that the finite Picard exceptional value of  $f^n f'$  may only be zero. Recently, Chiang and Feng [4], Halburd and Korhonen [6, 7, 8] established some difference results of Nevanlinna theory, such as the lemma of difference of logarithmic derivative, the difference Clunie lemma, the difference second main theorem and so on. Bergweiler and Langley [2] also considered the zeros distributions of f(z + c) - f(z) that can be viewed as discrete analogues of the zeros of f'(z) [5].

Hence, it is necessary to consider the subject of Hayman conjecture in difference, which means that f'(z) will be replaced by f(z + c) or  $\Delta_c f = f(z+c) - f(z)$ . Here and in the following, c is a nonzero complex constant. Recently, some papers are devoted to this subject or related subjects, such as [13]-[17], [23]. Laine and Yang [13, Theorem 2] proved the following:

**Theorem A.** Let f be a transcendental entire function with finite order, and  $n \geq 2$ . Then  $f(z)^n f(z+c)$  assumes every nonzero value  $a \in \mathbb{C}$ infinitely often.

About the zeros distributions of  $f(z)^n \Delta_c f$ , a result can be stated as follows:

**Theorem B.** [14, Theorem 1.4] Let f(z) be a transcendental entire function with finite order,  $\Delta_c f \neq 0$  and let  $n \geq 2$  be an integer. Then  $f(z)^n \Delta_c f - p(z)$  has infinitely many zeros, where  $p(z) \neq 0$  is a polynomial in z.

In this paper, we will use the Nevanlinna second main theorem and Hadmard factorization theorem of entire functions to obtain the following result.

**Theorem 1.1.** Let f(z) be a transcendental entire function with finite order and  $\Delta_c f \neq 0$ . If  $n \geq 3$ ,  $n, s \in \mathbb{N}$ , then  $f(z)^n (\Delta_c f)^s - \alpha(z)$  has infinitely many zeros, where  $\alpha(z)$  is a small function with respect to f(z). If  $\alpha(z)$  is an entire function satisfies  $\rho(\alpha) < \rho(f)$ , then n can be reduced to  $n \geq 2$ .

**Remark:** (1) Obviously, Theorem 1.1 is an improvement of Theorem B. Moreover, it is interesting to find that the result is independent of the number s. In fact, if s = 0, the same conclusion can be obtained using the second main theorem for three small functions [11, Theorem 2.5].

(2) If n = 1, Theorem 1.1 is not true, which can be seen by the function  $f(z) = e^{z} + p(z)$ , where  $c = 2k\pi i$  and p(z) is a nonconstant polynomial. Thus,

$$f(z)[f(z+c) - f(z)]^{s} = [e^{z} + p(z)][p(z+c) - p(z)]^{s}.$$

Let  $\alpha(z) = p(z)[p(z+c) - p(z)]^s$ , then

$$f(z)[f(z+c) - f(z)]^{s} - \alpha(z) = [p(z+c) - p(z)]^{s}e^{z},$$

which has only finitely many zeros. If n = 2, and  $\alpha$  is any small function with respect to f, we have not succeeded in proving that Theorem 1.1 is still valid.

(3) In the condition of finite order can not be removed, which can be seen by the function  $f(z) = \frac{z}{e^{e^z}}$  of infinite order, and

$$f(z)^{n}\Delta_{c}f - z^{n}(z+c) = -\frac{z^{n+1}}{e^{(n+1)e^{z}}}$$

has only one zero with n + 1 multiplicities, where  $e^c = -n$ .

In the following, we will consider the case of meromorphic functions of Theorem 1.1. We also can obtain the following result.

**Theorem 1.2.** Let f be a transcendental meromorphic function with finite order and  $\Delta_c f \neq 0$ . If  $n \geq s + 6$ ,  $n, s \in \mathbb{N}$ , then the difference polynomial  $f(z)^n (\Delta_c f)^s - \alpha(z)$  has infinitely many zeros, where  $\alpha(z)$  is a small function with respect to f(z).

Some results on the existence and growth of solutions of difference equations can be found in [3, 4, 6, 19]. Here, from Theorem 1.1 and Theorem 1.2, we immediately get the following result on some nonlinear difference equations.

**Corollary 1.3.** Let p(z), q(z) be polynomials,  $n, s \in \mathbb{N}$ . Then the difference equation

$$f(z)^n (\Delta_c f)^s - p(z) = q(z)$$

has no transcendental entire solutions with finite order, provided that,  $n \ge 2$ . This equation has no transcendental meromorphic functions with finite order if  $n \ge s+6$ , unless f(z) is a periodic function with period c.

In 1997, Yang and Hua studied the unicity of the differential monomials  $f^n f'$  and proved the uniqueness theorem of meromorphic function, such as [21, Theorem 1]. They also pointed out that the following uniqueness result in the case f is an entire function.

**Theorem D.** Let f and g be two nonconstant entire functions, let a be a nonzero constant, and let  $n \ge 7$  be a positive integer. If  $f^n f'$  and  $g^n g'$  share a CM, then either f = tg for  $t^{n+1} = 1$ , or  $f = c_1 e^{cz}$  and  $g = c_2 e^{-cz}$  for some nonzero constants  $c, c_1$  and  $c_2$  with  $(c_1c_2)^{n+1}c^2 = -a^2$ .

Similarly, we can investigate the uniqueness of difference polynomials sharing one common value. We now obtain the following two theorems.

**Theorem 1.4.** Let f and g be transcendental entire functions with finite order, and let  $n \ge 9$ ,  $n, s \in \mathbb{N}$ . If  $f(z)^n [f(z+c) - f(z)]^s$  and  $g(z)^n [g(z+c) - g(z)]^s$  share the nonzero value d CM, then  $f(z)^n [f(z+c) - f(z)]^s = g(z)^n [g(z+c) - g(z)]^s$  or fg = t for  $t^{n+s} = \frac{d^2}{(a-1)^s(b-1)^s}$ , where a, b are nonzero constants.

**Remark:** Let  $f(z) = 2^{\frac{z}{c}} \frac{A}{e^z}$  and  $g(z) = \frac{1}{2}^{\frac{z}{c}} e^z$ , where  $c = 2k\pi i, k \in \mathbb{N}$  and  $A^{n+s} = (-2)^s$ . By calculating, we know that

$$f(z)^{n}[f(z+c) - f(z)]^{s} - 1 = \frac{(2^{\frac{z}{2k\pi i}}A)^{n+s} - e^{z(n+s)}}{e^{z(n+s)}},$$

and

$$g(z)^{n}[g(z+c) - g(z)]^{s} - 1 = \frac{e^{z(n+s)} - (2\frac{z}{2k\pi i}A)^{n+s}}{(2\frac{z}{2k\pi i})^{n+s}(-2)^{s}}$$

Furthermore, all zeros of  $f(z)^n [f(z+c) - f(z)]^s - 1$  and  $g(z)^n [g(z+c) - g(z)]^s - 1$  are simple. Thus,  $f(z)^n [f(z+c) - f(z)]^s$  and  $g(z)^n [g(z+c) - g(z)]^s$  share 1 CM. It shows that the second case of Theorem 1.4 can occur.

**Theorem 1.5.** Let f and g be transcendental entire functions with finite order, and let  $n \ge 15$ ,  $n, s \in \mathbb{N}$ . If  $f(z)^n [f(z+c)-f(z)]^s$  and  $g(z)^n [g(z+c)-g(z)]^s$  share the nonzero value d IM, then  $f(z)^n [f(z+c)-f(z)]^s = g(z)^n [g(z+c)-g(z)]^s$  or fg = t for  $t^{n+s} = \frac{d^2}{(a-1)^s(b-1)^s}$ , where a, b are nonzero constants.

**Remark:** If  $f(z)^n [f(z+c) - f(z)]^s = g(z)^n [g(z+c) - g(z)]^s$ , we believe that f = tg, where  $t^{n+s} = 1$ , or f, g are periodic functions with period c, but we did not succeed in proving that.

### 2. Some Lemmas

true cm The difference logarithmic derivative lemma of functions with finite order, given by Chiang and Feng [4, Corollary 2.5], Halburd and Korhonen [6, Theorem 2.1], plays an important part in considering the difference Nevanlinna theory. Here, we state the following version.

**Lemma 2.1.** [8, Theorem 5.6] Let f be a transcendental meromorphic function with finite order. Then

(2.1) 
$$m\left(r,\frac{f(z+c)}{f(z)}\right) = S(r,f),$$

for all r outside of a set of finite logarithmic measure.

The following result is important when considering the characteristic function of difference operator of entire functions.

**Lemma 2.2.** Let f be a transcendental entire function with finite order. Then

(2.2) 
$$T(r, f(z+c) - f(z)) \le T(r, f) + S(r, f).$$

*Proof.* Since f is an entire function, then

(2.3)  

$$T(r, f(z+c) - f(z)) = m\left(r, f(z)(\frac{f(z+c)}{f(z)} - 1)\right)$$

$$\leq m(r, f) + m\left(r, \frac{f(z+c)}{f(z)}\right) + O(1)$$

$$\leq T(r, f) + S(r, f).$$

But, if f is a transcendental meromorphic function in Lemma 2.2, we only get

$$T(r, f(z+c) - f(z)) \le 2T(r, f) + S(r, f).$$

The proof is trivial using the following lemma.

**Lemma 2.3.** [4, Theorem 2.1] Let f(z) be a transcendental meromorphic function with finite order. Then,

(2.4) 
$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

For the proof of Theorem 1.1 and Theorem 1.2, we need the following lemma.

**Lemma 2.4.** Let f(z) be a transcendental meromorphic function with finite order. Then,

(2.5) 
$$T(r, f(z)^n [f(z+c) - f(z)]^s) \ge (n-s)T(r, f) + S(r, f).$$

If f(z) is a transcendental entire function with finite order. Then, (2.6)

$$(n+s)T(r,f) + S(r,f) \ge T(r,f(z)^n [f(z+c) - f(z)]^s) \ge nT(r,f) + S(r,f).$$

*Proof.* Assume that  $G(z) = f(z)^n [f(z+c) - f(z)]^s$ , then

(2.7) 
$$\frac{1}{f(z)^{n+s}} = \frac{1}{G} \left[ \frac{f(z+c) - f(z)}{f(z)} \right]^s.$$

Combining the first main theorem of Nevanlinna, Lemma 2.1 and (2.4), we get

$$(n+s)T(r,f) \leq T(r,G(z)) + T(r,\left[\frac{f(z+c) - f(z)}{f(z)}\right]^{s}) + O(1)$$
  
$$\leq T(r,G(z)) + sm\left(r,\frac{f(z+c) - f(z)}{f(z)}\right)$$
  
$$+ sN\left(r,\frac{f(z+c) - f(z)}{f(z)}\right) + O(1)$$
  
$$\leq T(r,G(z)) + sN\left(r,\frac{f(z+c)}{f(z)}\right) + S(r,f)$$
  
$$\leq T(r,G(z)) + 2sT(r,f) + S(r,f),$$

thus (2.5) follows.

If f(z) is a transcendental entire function with finite order, we get  $T(r, f(z)^n [f(z+c) - f(z)]^s) \leq (n+s)T(r, f) + S(r, f)$  from Lemma 2.2. Using similar method as above, it is easy to obtain (2.6).

**Lemma 2.5.** [21, Lemma 3] Let F and G be two nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:

(i)  $\max\{T(r, F), T(r, G)\} \le N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) + S(r, F) + S(r, G),$ (ii) F = G,(iii)  $F \cdot G = 1,$ 

where  $N_2(r, \frac{1}{F})$  denotes the counting function of zeros of F such that simple zeros are counted once and multiple zeros are counted twice.

For the proof of Theorem 1.5, we need the following lemma.

**Lemma 2.6.** [20, Lemma 2.3] Let F and G be two nonconstant meromorphic functions sharing the value 1 IM. Let

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}.$$

If  $H \not\equiv 0$ , then

$$T(r,F) + T(r,G) \leq 2\left(N_2(r,\frac{1}{F}) + N_2(r,F) + N_2(r,\frac{1}{G}) + N_2(r,G)\right) + 3\left(\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G})\right) (2.9) + S(r,F) + S(r,G).$$

## 3. Proofs of Theorem 1.1 and 1.2

Let  $G(z) = f(z)^n [f(z+c) - f(z)]^s$ . From (2.6), we know that G(z) is not a constant, we also get S(r, G) = S(r, f). Since f is an entire function, from the Nevanlinna second main theorem for three small functions [11, Theorem 2.5], Lemma 2.2 and (2.6), we obtain

$$nT(r,f) \leq T(r,G) + S(r,f)$$

$$\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-\alpha}) + S(r,f)$$

$$\leq \overline{N}(r,\frac{1}{f}) + \overline{N}\left(r,\frac{1}{f(z+c)-f(z)}\right)$$

$$+\overline{N}(r,\frac{1}{G-\alpha}) + S(r,f)$$

$$\leq 2T(r,f) + \overline{N}(r,\frac{1}{G-\alpha}) + S(r,f).$$
(3.1)

Since  $n \ge 3$ , then (3.1) implies that  $G - \alpha$  has infinitely many zeros.

If n = 2,  $\alpha$  is an entire function with respect to f and  $\rho(\alpha) < \rho(f) = \rho$ , that is,  $T(r, \alpha) = O(r^{\rho-\varepsilon})$ ,  $0 < \varepsilon < 1$ . Suppose on contrary to the assertion that  $f(z)^2 [f(z + c) - f(z)]^s - \alpha(z)$  has finitely many zeros. Then

(3.2) 
$$f(z)^{2}[f(z+c) - f(z)]^{s} - \alpha(z) = P(z)e^{Q(z)},$$

where P(z) is nonzero polynomials and Q(z) is nonconstant polynomials. Because otherwise, we get

$$T(f(z)^{2}[f(z+c) - f(z)]^{s}) = O(r^{\rho-\varepsilon}) + S(r, f),$$

which is a contradiction to (2.6).

Differentiating (3.2) and eliminating  $e^{Q(z)}$ , we obtain

(3.3) 
$$f(z)^{2+s}F(z,f) = q^*(z),$$

where

$$F(z,f) = 2\frac{f'(z)}{f(z)} \left[\frac{f(z+c) - f(z)}{f(z)}\right]^s + s \left[\frac{f(z+c) - f(z)}{f(z)}\right]^{s-1}$$
$$\frac{f'(z+c) - f'(z)}{f(z)} - p^*(z) \left[\frac{f(z+c) - f(z)}{f(z)}\right]^s,$$

and

$$p^{*}(z) = \frac{P'(z) + P(z)Q'(z)}{P(z)}, \quad q^{*}(z) = \alpha'(z) - p^{*}(z)\alpha(z).$$

We now prove that F(z, f) cannot vanish identically. Indeed, if  $F(z, f) \equiv 0$ , then

$$\alpha'(z) - p^*(z)\alpha(z) \equiv 0.$$

By integrating above equation, we have

$$\alpha(z) = AP(z)e^{Q(z)},$$

where A is a nonzero constant. Substitute the above into (3.2), we get

(3.4) 
$$f(z)^{2}[f(z+c) - f(z)]^{s} = (A+1)P(z)e^{Q(z)}$$

Thus, the zeros of f must be the zeros of P(z). Then  $f(z) = a(z)e^{b(z)}$ , a(z), b(z) are nonzero polynomials. Since the zeros of f(z + c) - f(z) also must be the zeros of P(z), then  $b(z + c) - b(z) \equiv b$ , where b is a nonzero constant. Thus, b(z) must be a linear polynomial. It implies that the order of f must be one. Hence, from (3.4), we know that Q(z) is also a linear polynomial, which is a contradiction to  $\rho(\alpha) < \rho(f) = \rho$ . Thus, we get  $F(z, f) \neq 0$ .

By Lemma 2.1 and the lemma of logarithmic derivative, we get

(3.5) 
$$m(r, F(z, f)) = S(r, f).$$

From the Clunie lemma [12, Lemma 2.4.2], we obtain

(3.6)  $m(r, fF(z, f)) = O(r^{\rho - \varepsilon}) + S(r, f).$ 

We know that the poles of F(z, f) may be located only at the zeros of f(z) and P(z). And the zero of f(z) with multiplicity l should be a pole of F(z, f) with multiplicity at most sl + 1. Thus the zero multiplicity of the left hand side of (3.3) is at least 2l - 1. Hence these zeros must be the zeros of  $q^*(z)$ . Thus, we obtain that

(3.7) 
$$N(r, \frac{1}{f}) = O(r^{\rho-\varepsilon}) + S(r, f).$$

So

(3.8) 
$$N(r, F(z, f)) = O(r^{\rho - \varepsilon})$$

and

(3.9) 
$$N(r, fF(z, f)) = O(r^{\rho - \varepsilon}).$$

Hence from (3.5) and (3.8),

$$T(r, F(z, f)) = O(r^{\rho - \varepsilon}) + S(r, f)$$

and from (3.6) and (3.9),

$$T(r,fF(z,f))=O(r^{\rho-\varepsilon})+S(r,f).$$

Therefore

$$T(r, f(z)) = O(r^{\rho - \varepsilon}) + S(r, f),$$

which contradicts the assumption that f(z) is a transcendental function of finite order  $\rho$ . Thus, we have completed the proof of Theorem 1.1.

If f is a meromorphic function, from (2.5) and Lemma 2.3, we know that G(z) is not a constant. Similar to the above,

$$(n-s)T(r,f) \leq T(r,G) + S(r,f)$$

$$\leq \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-\alpha}) + S(r,G)$$

$$\leq 2T(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f(z+c)-f(z)})$$

$$+\overline{N}(r,\frac{1}{G-\alpha}) + S(r,f)$$

$$(3.10) \leq 5T(r,f) + \overline{N}(r,\frac{1}{G-\alpha}) + S(r,f).$$

Since  $n \ge s+6$ , then (3.10) implies that  $G-\alpha$  has infinitely many zeros. Thus, we have competed the proof of Theorem 1.2.

## 4. Proof of Theorem 1.4

Let  $F = \frac{f(z)^n [f(z+c)-f(z)]^s}{d}$  and  $G = \frac{g(z)^n [g(z+c)-g(z)]^s}{d}$ . Since  $f(z)^n [f(z+c)-f(z)]^s$  and  $g(z)^n [g(z+c)-g(z)]^s$  share the value d CM, then F and G share the value 1 CM. Hence, Lemma 2.2 and Lemma 2.5 (i) imply that

$$\begin{split} T(r,F) &\leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + S(r,F) + S(r,G) \\ &\leq 2N(r,\frac{1}{f}) + 2N\left(r,\frac{1}{f(z+c) - f(z)}\right) + 2N\left(r,\frac{1}{g(z+c) - g(z)}\right) \\ &\quad + 2N(r,\frac{1}{g}) + S(r,f) + S(r,g) \\ &\leq 2N(r,\frac{1}{f}) + 2T\left(r,f(z+c) - f(z)\right) + 2T\left(r,g(z+c) - g(z)\right) \\ &\quad + 2N(r,\frac{1}{g}) + S(r,f) + S(r,g) \\ &\leq 4T(r,f) + 4T(r,g) + S(r,f) + S(r,g). \end{split}$$

Thus, from (2.6), we get

(4.1) 
$$nT(r,f) \le 4T(r,f) + 4T(r,g) + S(r,f) + S(r,g).$$

Similarly, we obtain

(4.2) 
$$nT(r,g) \le 4T(r,f) + 4T(r,g) + S(r,f) + S(r,g).$$

Thus, combining (4.1) with (4.2), we get that

$$(4.3)n[T(r,f) + T(r,g)] \le 8[T(r,f) + T(r,g)] + S(r,f) + S(r,g),$$

which is a contradiction with  $n \ge 9$ . Thus, from Lemma 2.5, we get F = G or  $F \cdot G = 1$ . If  $F \cdot G = 1$ , it follows that

(4.4) 
$$f(z)^n [f(z+c) - f(z)]^s \cdot g(z)^n [g(z+c) - g(z)]^s = d^2.$$

From (4.4) and the fact that g is an entire function, we know that finite order entire functions f(z) and f(z+c) - f(z) has no zeros. Assume that  $f(z) = e^{A(z)}$ , where A(z) is a nonconstant polynomial, then

$$e^{A(z+c)} - e^{A(z)} = e^{A(z)}(e^{A(z+c)-A(z)} - 1)$$

has no zeros, which implies that A(z + c) - A(z) must be a constant. Thus, we obtain that

(4.5) 
$$\frac{f(z+c)}{f(z)} = a_z$$

where *a* is a nonzero constant. Thus, the solutions of the equation (4.5) can be written in the form of  $f(z) = a^{\frac{z}{c}} \Pi_1(z)$ , where  $\Pi_1(z)$  is a periodic function with period *c*. Similarly,  $g(z) = b^{\frac{z}{c}} \Pi_2(z)$ , where  $\frac{g(z+c)}{g(z)} = b$  is a nonzero constant and  $\Pi_2(z)$  is also a periodic function with period *c*. Hence, (4.4) changes into

$$f(z)^{n+s}g(z)^{n+s} = \frac{d^2}{(a-1)^s(b-1)^s}.$$

Thus, f(z)g(z) = t, where t is a (n + s)th root of  $\frac{d^2}{(a-1)^s(b-1)^s}$ . **Remark:** In a special case, if s = 1 in Theorem 1.4, then we know that

$$N_2(r, \frac{1}{F}) \le 2N(r, \frac{1}{f}) + N(r, \frac{1}{f(z+c) - f(z)}).$$

Thus, when s = 1, from the above proof, we can assume that  $n \ge 7$ .

## 5. Proof of Theorem 1.5

Let  $F = \frac{f(z)^n [f(z+c)-f(z)]^s}{d}$ ,  $G = \frac{g(z)^n [g(z+c)-g(z)]^s}{d}$ , and let H be defined in Lemma 2.6. If  $H \neq 0$ , from (2.9), we get

$$n[T(r,f) + T(r,g)] \leq 2N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + 3\overline{N}(r,\frac{1}{F}) + 3\overline{N}(r,\frac{1}{G}) + S(r,f) + S(r,g) \leq 7\overline{N}(r,\frac{1}{f}) + 7\overline{N}r, \frac{1}{f(z+c) - f(z)}) + 7\overline{N}(r,\frac{1}{g}) + 7\overline{N}(r,\frac{1}{g(z+c) - g(z)}) + S(r,f) + S(r,g) (5.1) \leq 14[T(r,f) + T(r,g)] + S(r,f) + S(r,g),$$

which is a contradiction with  $n \ge 15$ . As above, we remark that if s = 1,  $n \ge 13$  is sufficient.

Thus, we get  $H \equiv 0$ . The following proof is trivial, which can be seen in many literatures. The original idea is due to Yang and Yi [22]. Here, we give a complete proof. By integrating H twice, we obtain

(5.2) 
$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, G = \frac{(a-b)F - (a-b-1)}{(b+1) - Fb},$$

which implies T(r, F) = T(r, G) + O(1). We will show that F = G or  $F \cdot G = 1$ .

**Case 1**.  $b \neq 0, -1$ . If  $a - b - 1 \neq 0$ , then by (5.2), we get

(5.3) 
$$\overline{N}(r,\frac{1}{F}) = \overline{N}\left(r,\frac{1}{G-\frac{a-b-1}{b+1}}\right).$$

By the Nevanlinna second main theorem and Lemma 2.4 and (2.6), we have

$$T(r,G) \leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G-\frac{a-b-1}{b+1}}\right) + S(r,G)$$

$$\leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{F}) + S(r,G)$$

$$\leq \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g(z+c)-g(z)})$$

$$+ \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f(z+c)-f(z)}) + S(r,g)$$

$$\leq 2T(r,f) + 2T(r,g) + S(r,g).$$

Using the same method as above, we have

(5.5) 
$$T(r,F) \le 2T(r,f) + 2T(r,g) + S(r,f).$$

Thus, from (5.4) and (5.5), we get

$$(5.6)n[T(r,f) + T(r,g)] \le 4[T(r,f) + T(r,g)] + S(r,f) + S(r,g).$$

which contradicts the assumption  $n \ge 15$ . Thus, a - b - 1 = 0, and hence

(5.7) 
$$F = \frac{(b+1)G}{bG+1}.$$

Using the same method as above,

$$T(r,G) \leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right) + S(r,G)$$
$$\leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}(r,F) + S(r,G)$$
$$\leq 2T(r,g) + S(r,G),$$

which is also a contradiction.

**Case 2**.  $b = 0, a \neq 1$ . From (5.2), we have

(5.9) 
$$F = \frac{G+a-1}{a}.$$

Similarly, we also can get a contradiction. Thus, a = 1 follows which implies that F = G.

Case 3. b = -1,  $a \neq -1$ . From (5.2), we obtain

(5.10) 
$$F = \frac{a}{a+1-G}$$

Similarly, we get a contradiction, and a = -1 follows. Therefore, we get  $F \cdot G = 1$ .

Therefore, using the same method as Theorem 1.4, we can get the proof of Theorem 1.5.

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