# SOME DIFFERENCE RESULTS ON HAYMAN CONJECTURE AND UNIQUENESS 

K. LIU*, T. B. CAO AND X. L. LIU

Communicated by Javad Mashreghi


#### Abstract

In this paper, we show that for any finite order entire function $f(z)$, the function of the form $f(z)^{n}[f(z+c)-f(z)]^{s}$ has no nonzero finite Picard exceptional value for all nonnegative integers $n, s$ satisfying $n \geq 3$, which can be viewed as a difference result on Hayman conjecture. We also obtain some uniqueness theorems for difference polynomials of entire functions sharing one common value.


## 1. Introduction

A meromorphic function $f$ means meromorphic in the whole complex plane. If no poles occur, then $f$ reduces to an entire function. Recall that $\alpha(z) \not \equiv 0, \infty$ is a small function with respect to $f$, if $T(r, \alpha)=S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f)=o(T(r, f))$, as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure. We say that two meromorphic functions $f$ and $g$ share a small function $\alpha$ IM (ignoring multiplicities) when $f-\alpha$ and $g-\alpha$ have the same zeros. If $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities, then we say that $f$ and $g$ share the small function $\alpha$ CM (counting multiplicities). We assume that the reader is familiar with standard

[^0]notations and fundamental results of Nevanlinna Theory, such as [11, 12,22 . We use $\rho(f)$ to denote the order of $f(z)$.

The Hayman conjecture [10] is an important research subject in considering the value distributions of differential polynomials, that is if $f$ is a transcendental meromorphic function and $n \in \mathbb{N}$, then $f^{n} f^{\prime}$ takes every finite nonzero value infinitely often. This conjecture has been solved by Hayman [9] for $n \geq 3$, by Mues [18] for $n=2$, by Bergweiler and Eremenko [1] for $n=1$. From above, it shows that the finite Picard exceptional value of $f^{n} f^{\prime}$ may only be zero. Recently, Chiang and Feng [4], Halburd and Korhonen [6, 7, 8] established some difference results of Nevanlinna theory, such as the lemma of difference of logarithmic derivative, the difference Clunie lemma, the difference second main theorem and so on. Bergweiler and Langley [2] also considered the zeros distributions of $f(z+c)-f(z)$ that can be viewed as discrete analogues of the zeros of $f^{\prime}(z)$ [5].

Hence, it is necessary to consider the subject of Hayman conjecture in difference, which means that $f^{\prime}(z)$ will be replaced by $f(z+c)$ or $\Delta_{c} f=f(z+c)-f(z)$. Here and in the following, $c$ is a nonzero complex constant. Recently, some papers are devoted to this subject or related subjects, such as [13]-[17], [23]. Laine and Yang [13, Theorem 2] proved the following:
Theorem A. Let $f$ be a transcendental entire function with finite order, and $n \geq 2$. Then $f(z)^{n} f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.

About the zeros distributions of $f(z)^{n} \Delta_{c} f$, a result can be stated as follows:

Theorem B. [14, Theorem 1.4] Let $f(z)$ be a transcendental entire function with finite order, $\Delta_{c} f \not \equiv 0$ and let $n \geq 2$ be an integer. Then $f(z)^{n} \Delta_{c} f-p(z)$ has infinitely many zeros, where $p(z) \not \equiv 0$ is a polynomial in $z$.

In this paper, we will use the Nevanlinna second main theorem and Hadmard factorization theorem of entire functions to obtain the following result.

Theorem 1.1. Let $f(z)$ be a transcendental entire function with finite order and $\Delta_{c} f \not \equiv 0$. If $n \geq 3, n, s \in \mathbb{N}$, then $f(z)^{n}\left(\Delta_{c} f\right)^{s}-\alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small function with respect to $f(z)$.

If $\alpha(z)$ is an entire function satisfies $\rho(\alpha)<\rho(f)$, then $n$ can be reduced to $n \geq 2$.

Remark: (1) Obviously, Theorem 1.1 is an improvement of Theorem B. Moreover, it is interesting to find that the result is independent of the number $s$. In fact, if $s=0$, the same conclusion can be obtained using the second main theorem for three small functions [11, Theorem 2.5].
(2) If $n=1$, Theorem 1.1 is not true, which can be seen by the function $f(z)=e^{z}+p(z)$, where $c=2 k \pi i$ and $p(z)$ is a nonconstant polynomial. Thus,

$$
f(z)[f(z+c)-f(z)]^{s}=\left[e^{z}+p(z)\right][p(z+c)-p(z)]^{s} .
$$

Let $\alpha(z)=p(z)[p(z+c)-p(z)]^{s}$, then

$$
f(z)[f(z+c)-f(z)]^{s}-\alpha(z)=[p(z+c)-p(z)]^{s} e^{z},
$$

which has only finitely many zeros. If $n=2$, and $\alpha$ is any small function with respect to $f$, we have not succeeded in proving that Theorem 1.1 is still valid.
(3) In the condition of finite order can not be removed, which can be seen by the function $f(z)=\frac{z}{e^{e^{z}}}$ of infinite order, and

$$
f(z)^{n} \Delta_{c} f-z^{n}(z+c)=-\frac{z^{n+1}}{e^{(n+1) e^{z}}}
$$

has only one zero with $n+1$ multiplicities, where $e^{c}=-n$.
In the following, we will consider the case of meromorphic functions of Theorem 1.1. We also can obtain the following result.

Theorem 1.2. Let $f$ be a transcendental meromorphic function with finite order and $\Delta_{c} f \not \equiv 0$. If $n \geq s+6, n, s \in \mathbb{N}$, then the difference polynomial $f(z)^{n}\left(\Delta_{c} f\right)^{s}-\alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a small function with respect to $f(z)$.

Some results on the existence and growth of solutions of difference equations can be found in $[3,4,6,19]$. Here, from Theorem 1.1 and Theorem 1.2 , we immediately get the following result on some nonlinear difference equations.

Corollary 1.3. Let $p(z), q(z)$ be polynomials, $n, s \in \mathbb{N}$. Then the difference equation

$$
f(z)^{n}\left(\Delta_{c} f\right)^{s}-p(z)=q(z)
$$

has no transcendental entire solutions with finite order, provided that, $n \geq 2$. This equation has no transcendental meromorphic functions with finite order if $n \geq s+6$, unless $f(z)$ is a periodic function with period $c$.

In 1997, Yang and Hua studied the unicity of the differential monomials $f^{n} f^{\prime}$ and proved the uniqueness theorem of meromorphic function, such as [21, Theorem 1]. They also pointed out that the following uniqueness result in the case $f$ is an entire function.
Theorem D. Let $f$ and $g$ be two nonconstant entire functions, let $a$ be a nonzero constant, and let $n \geq 7$ be a positive integer. If $f^{n} f^{\prime}$ and $g^{n} g^{\prime}$ share $a \mathrm{CM}$, then either $f=t g$ for $t^{n+1}=1$, or $f=c_{1} e^{c z}$ and $g=c_{2} e^{-c z}$ for some nonzero constants $c, c_{1}$ and $c_{2}$ with $\left(c_{1} c_{2}\right)^{n+1} c^{2}=-a^{2}$.

Similarly, we can investigate the uniqueness of difference polynomials sharing one common value. We now obtain the following two theorems.

Theorem 1.4. Let $f$ and $g$ be transcendental entire functions with finite order, and let $n \geq 9, n, s \in \mathbb{N}$. If $f(z)^{n}[f(z+c)-f(z)]^{s}$ and $g(z)^{n}[g(z+$ $c)-g(z)]^{s}$ share the nonzero value $d C M$, then $f(z)^{n}[f(z+c)-f(z)]^{s}=$ $g(z)^{n}[g(z+c)-g(z)]^{s}$ or $f g=t$ for $t^{n+s}=\frac{d^{2}}{(a-1)^{s}(b-1)^{s}}$, where $a, b$ are nonzero constants.
Remark: Let $f(z)=2^{\frac{z}{c}} \frac{A}{e^{z}}$ and $g(z)=\frac{1}{2}^{\frac{z}{c}} e^{z}$, where $c=2 k \pi i, k \in \mathbb{N}$ and $A^{n+s}=(-2)^{s}$. By calculating, we know that

$$
f(z)^{n}[f(z+c)-f(z)]^{s}-1=\frac{\left(2^{\frac{z}{2 k \pi i}} A\right)^{n+s}-e^{z(n+s)}}{e^{z(n+s)}}
$$

and

$$
g(z)^{n}[g(z+c)-g(z)]^{s}-1=\frac{e^{z(n+s)}-\left(2^{\frac{z}{2 k \pi i}} A\right)^{n+s}}{\left(2^{\frac{z}{2 k \pi i}}\right)^{n+s}(-2)^{s}}
$$

Furthermore, all zeros of $f(z)^{n}[f(z+c)-f(z)]^{s}-1$ and $g(z)^{n}[g(z+c)-$ $g(z)]^{s}-1$ are simple. Thus, $f(z)^{n}[f(z+c)-f(z)]^{s}$ and $g(z)^{n}[g(z+c)-$ $g(z)]^{s}$ share 1 CM. It shows that the second case of Theorem 1.4 can occur.

Theorem 1.5. Let $f$ and $g$ be transcendental entire functions with finite order, and let $n \geq 15, n, s \in \mathbb{N}$. If $f(z)^{n}[f(z+c)-f(z)]^{s}$ and $g(z)^{n}[g(z+$ $c)-g(z)]^{s}$ share the nonzero value $d$ IM, then $f(z)^{n}[f(z+c)-f(z)]^{s}=$ $g(z)^{n}[g(z+c)-g(z)]^{s}$ or $f g=t$ for $t^{n+s}=\frac{d^{2}}{(a-1)^{s}(b-1)^{s}}$, where $a, b$ are nonzero constants.

Remark: If $f(z)^{n}[f(z+c)-f(z)]^{s}=g(z)^{n}[g(z+c)-g(z)]^{s}$, we believe that $f=t g$, where $t^{n+s}=1$, or $f, g$ are periodic functions with period $c$, but we did not succeed in proving that.

## 2. Some Lemmas

true cm The difference logarithmic derivative lemma of functions with finite order, given by Chiang and Feng [4, Corollary 2.5], Halburd and Korhonen [6, Theorem 2.1], plays an important part in considering the difference Nevanlinna theory. Here, we state the following version.

Lemma 2.1. [8, Theoem 5.6] Let $f$ be a transcendental meromorphic function with finite order. Then

$$
\begin{equation*}
m\left(r, \frac{f(z+c)}{f(z)}\right)=S(r, f) \tag{2.1}
\end{equation*}
$$

for all $r$ outside of a set of finite logarithmic measure.
The following result is important when considering the characteristic function of difference operator of entire functions.

Lemma 2.2. Let $f$ be a transcendental entire function with finite order. Then

$$
\begin{equation*}
T(r, f(z+c)-f(z)) \leq T(r, f)+S(r, f) \tag{2.2}
\end{equation*}
$$

Proof. Since $f$ is an entire function, then

$$
\begin{align*}
T(r, f(z+c)-f(z)) & =m\left(r, f(z)\left(\frac{f(z+c)}{f(z)}-1\right)\right) \\
& \leq m(r, f)+m\left(r, \frac{f(z+c)}{f(z)}\right)+O(1) \\
& \leq T(r, f)+S(r, f) . \tag{2.3}
\end{align*}
$$

But, if $f$ is a transcendental meromorphic function in Lemma 2.2, we only get

$$
T(r, f(z+c)-f(z)) \leq 2 T(r, f)+S(r, f)
$$

The proof is trivial using the following lemma.
Lemma 2.3. [4, Theorem 2.1] Let $f(z)$ be a transcendental meromorphic function with finite order. Then,

$$
\begin{equation*}
T(r, f(z+c))=T(r, f)+S(r, f) \tag{2.4}
\end{equation*}
$$

For the proof of Theorem 1.1 and Theorem 1.2, we need the following lemma.

Lemma 2.4. Let $f(z)$ be a transcendental meromorphic function with finite order. Then,

$$
\begin{equation*}
T\left(r, f(z)^{n}[f(z+c)-f(z)]^{s}\right) \geq(n-s) T(r, f)+S(r, f) \tag{2.5}
\end{equation*}
$$

If $f(z)$ is a transcendental entire function with finite order. Then, (2.6)
$(n+s) T(r, f)+S(r, f) \geq T\left(r, f(z)^{n}[f(z+c)-f(z)]^{s}\right) \geq n T(r, f)+S(r, f)$.
Proof. Assume that $G(z)=f(z)^{n}[f(z+c)-f(z)]^{s}$, then

$$
\begin{equation*}
\frac{1}{f(z)^{n+s}}=\frac{1}{G}\left[\frac{f(z+c)-f(z)}{f(z)}\right]^{s} \tag{2.7}
\end{equation*}
$$

Combining the first main theorem of Nevanlinna, Lemma 2.1 and (2.4), we get

$$
\begin{align*}
(n+s) T(r, f) \leq & T(r, G(z))+T\left(r,\left[\frac{f(z+c)-f(z)}{f(z)}\right]^{s}\right)+O(1) \\
\leq & T(r, G(z))+s m\left(r, \frac{f(z+c)-f(z)}{f(z)}\right) \\
& +s N\left(r, \frac{f(z+c)-f(z)}{f(z)}\right)+O(1) \\
\leq & T(r, G(z))+s N\left(r, \frac{f(z+c)}{f(z)}\right)+S(r, f) \\
\leq & T(r, G(z))+2 s T(r, f)+S(r, f) \tag{2.8}
\end{align*}
$$

thus (2.5) follows.
If $f(z)$ is a transcendental entire function with finite order, we get $T\left(r, f(z)^{n}[f(z+c)-f(z)]^{s}\right) \leq(n+s) T(r, f)+S(r, f)$ from Lemma 2.2. Using similar method as above, it is easy to obtain (2.6).

Lemma 2.5. [21, Lemma 3] Let $F$ and $G$ be two nonconstant meromorphic functions. If $F$ and $G$ share $1 C M$, then one of the following three cases holds:
(i) $\max \{T(r, F), T(r, G)\} \leq N_{2}\left(r, \frac{1}{F}\right)+N_{2}(r, F)+N_{2}\left(r, \frac{1}{G}\right)+N_{2}(r, G)+$ $S(r, F)+S(r, G)$,
(ii) $F=G$,
(iii) $F \cdot G=1$,
where $N_{2}\left(r, \frac{1}{F}\right)$ denotes the counting function of zeros of $F$ such that simple zeros are counted once and multiple zeros are counted twice.

For the proof of Theorem 1.5, we need the following lemma.
Lemma 2.6. [20, Lemma 2.3] Let $F$ and $G$ be two nonconstant meromorphic functions sharing the value 1 IM. Let

$$
H=\frac{F^{\prime \prime}}{F^{\prime}}-2 \frac{F^{\prime}}{F-1}-\frac{G^{\prime \prime}}{G^{\prime}}+2 \frac{G^{\prime}}{G-1} .
$$

If $H \not \equiv 0$, then

$$
\begin{aligned}
T(r, F)+T(r, G) & \leq 2\left(N_{2}\left(r, \frac{1}{F}\right)+N_{2}(r, F)+N_{2}\left(r, \frac{1}{G}\right)+N_{2}(r, G)\right) \\
& +3\left(\bar{N}(r, F)+\bar{N}(r, G)+\bar{N}\left(r, \frac{1}{F}\right)+\bar{N}\left(r, \frac{1}{G}\right)\right) \\
& +S(r, F)+S(r, G) .
\end{aligned}
$$

## 3. Proofs of Theorem 1.1 and 1.2

Let $G(z)=f(z)^{n}[f(z+c)-f(z)]^{s}$. From (2.6), we know that $G(z)$ is not a constant, we also get $S(r, G)=S(r, f)$. Since $f$ is an entire function, from the Nevanlinna second main theorem for three small functions [11, Theorem 2.5], Lemma 2.2 and (2.6), we obtain

$$
\begin{align*}
n T(r, f) \leq & T(r, G)+S(r, f) \\
\leq & \bar{N}(r, G)+\bar{N}\left(r, \frac{1}{G}\right)+\bar{N}\left(r, \frac{1}{G-\alpha}\right)+S(r, f) \\
\leq & \bar{N}\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{f(z+c)-f(z)}\right) \\
& +\bar{N}\left(r, \frac{1}{G-\alpha}\right)+S(r, f) \\
\leq & 2 T(r, f)+\bar{N}\left(r, \frac{1}{G-\alpha}\right)+S(r, f) . \tag{3.1}
\end{align*}
$$

Since $n \geq 3$, then (3.1) implies that $G-\alpha$ has infinitely many zeros.
If $n=2, \alpha$ is an entire function with respect to $f$ and $\rho(\alpha)<\rho(f)=\rho$, that is, $T(r, \alpha)=O\left(r^{\rho-\varepsilon}\right), 0<\varepsilon<1$. Suppose on contrary to the assertion that $f(z)^{2}[f(z+c)-f(z)]^{s}-\alpha(z)$ has finitely many zeros. Then

$$
\begin{equation*}
f(z)^{2}[f(z+c)-f(z)]^{s}-\alpha(z)=P(z) e^{Q(z)}, \tag{3.2}
\end{equation*}
$$

where $P(z)$ is nonzero polynomials and $Q(z)$ is nonconstant polynomials. Because otherwise, we get

$$
T\left(f(z)^{2}[f(z+c)-f(z)]^{s}\right)=O\left(r^{\rho-\varepsilon}\right)+S(r, f),
$$

which is a contradiction to (2.6).
Differentiating (3.2) and eliminating $e^{Q(z)}$, we obtain

$$
\begin{equation*}
f(z)^{2+s} F(z, f)=q^{*}(z) \tag{3.3}
\end{equation*}
$$

where

$$
\begin{aligned}
F(z, f)= & 2 \frac{f^{\prime}(z)}{f(z)}\left[\frac{f(z+c)-f(z)}{f(z)}\right]^{s}+s\left[\frac{f(z+c)-f(z)}{f(z)}\right]^{s-1} \\
& \frac{f^{\prime}(z+c)-f^{\prime}(z)}{f(z)}-p^{*}(z)\left[\frac{f(z+c)-f(z)}{f(z)}\right]^{s},
\end{aligned}
$$

and

$$
p^{*}(z)=\frac{P^{\prime}(z)+P(z) Q^{\prime}(z)}{P(z)}, \quad q^{*}(z)=\alpha^{\prime}(z)-p^{*}(z) \alpha(z) .
$$

We now prove that $F(z, f)$ cannot vanish identically. Indeed, if $F(z, f) \equiv 0$, then

$$
\alpha^{\prime}(z)-p^{*}(z) \alpha(z) \equiv 0 .
$$

By integrating above equation, we have

$$
\alpha(z)=A P(z) e^{Q(z)},
$$

where $A$ is a nonzero constant. Substitute the above into (3.2), we get

$$
\begin{equation*}
f(z)^{2}[f(z+c)-f(z)]^{s}=(A+1) P(z) e^{Q(z)} . \tag{3.4}
\end{equation*}
$$

Thus, the zeros of $f$ must be the zeros of $P(z)$. Then $f(z)=a(z) e^{b(z)}$, $a(z), b(z)$ are nonzero polynomials. Since the zeros of $f(z+c)-f(z)$ also must be the zeros of $P(z)$, then $b(z+c)-b(z) \equiv b$, where $b$ is a nonzero constant. Thus, $b(z)$ must be a linear polynomial. It implies that the order of $f$ must be one. Hence, from (3.4), we know that $Q(z)$ is also a linear polynomial, which is a contradiction to $\rho(\alpha)<\rho(f)=\rho$. Thus, we get $F(z, f) \not \equiv 0$.

By Lemma 2.1 and the lemma of logarithmic derivative, we get

$$
\begin{equation*}
m(r, F(z, f))=S(r, f) \tag{3.5}
\end{equation*}
$$

From the Clunie lemma [12, Lemma 2.4.2], we obtain

$$
\begin{equation*}
m(r, f F(z, f))=O\left(r^{\rho-\varepsilon}\right)+S(r, f) \tag{3.6}
\end{equation*}
$$

We know that the poles of $F(z, f)$ may be located only at the zeros of $f(z)$ and $P(z)$. And the zero of $f(z)$ with multiplicity $l$ should be a pole of $F(z, f)$ with multiplicity at most $s l+1$. Thus the zero multiplicity of the left hand side of (3.3) is at least $2 l-1$. Hence these zeros must be the zeros of $q^{*}(z)$. Thus, we obtain that

$$
\begin{equation*}
N\left(r, \frac{1}{f}\right)=O\left(r^{\rho-\varepsilon}\right)+S(r, f) \tag{3.7}
\end{equation*}
$$

So

$$
\begin{equation*}
N(r, F(z, f))=O\left(r^{\rho-\varepsilon}\right) \tag{3.8}
\end{equation*}
$$

and

$$
\begin{equation*}
N(r, f F(z, f))=O\left(r^{\rho-\varepsilon}\right) \tag{3.9}
\end{equation*}
$$

Hence from (3.5) and (3.8),

$$
T(r, F(z, f))=O\left(r^{\rho-\varepsilon}\right)+S(r, f)
$$

and from (3.6) and (3.9),

$$
T(r, f F(z, f))=O\left(r^{\rho-\varepsilon}\right)+S(r, f) .
$$

Therefore

$$
T(r, f(z))=O\left(r^{\rho-\varepsilon}\right)+S(r, f)
$$

which contradicts the assumption that $f(z)$ is a transcendental function of finite order $\rho$. Thus, we have completed the proof of Theorem 1.1.

If $f$ is a meromorphic function, from (2.5) and Lemma 2.3, we know that $G(z)$ is not a constant. Similar to the above,

$$
\begin{align*}
(n-s) T(r, f) \leq & T(r, G)+S(r, f) \\
\leq & \bar{N}(r, G)+\bar{N}\left(r, \frac{1}{G}\right)+\bar{N}\left(r, \frac{1}{G-\alpha}\right)+S(r, G) \\
\leq & 2 T(r, f)+\bar{N}\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{f(z+c)-f(z)}\right) \\
& +\bar{N}\left(r, \frac{1}{G-\alpha}\right)+S(r, f) \\
\leq & 5 T(r, f)+\bar{N}\left(r, \frac{1}{G-\alpha}\right)+S(r, f) . \tag{3.10}
\end{align*}
$$

Since $n \geq s+6$, then (3.10) implies that $G-\alpha$ has infinitely many zeros. Thus, we have competed the proof of Theorem 1.2.

## 4. Proof of Theorem 1.4

Let $F=\frac{f(z)^{n}[f(z+c)-f(z)]^{s}}{d}$ and $G=\frac{g(z)^{n}[g(z+c)-g(z)]^{s}}{d}$. Since $f(z)^{n}[f(z+$ $c)-f(z)]^{s}$ and $g(z)^{n}[g(z+c)-g(z)]^{s}$ share the value $d$ CM, then $F$ and $G$ share the value 1 CM. Hence, Lemma 2.2 and Lemma 2.5 (i) imply that

$$
\begin{aligned}
T(r, F) \leq & N_{2}\left(r, \frac{1}{F}\right)+N_{2}\left(r, \frac{1}{G}\right)+S(r, F)+S(r, G) \\
\leq & 2 N\left(r, \frac{1}{f}\right)+2 N\left(r, \frac{1}{f(z+c)-f(z)}\right)+2 N\left(r, \frac{1}{g(z+c)-g(z)}\right) \\
& +2 N\left(r, \frac{1}{g}\right)+S(r, f)+S(r, g) \\
\leq & 2 N\left(r, \frac{1}{f}\right)+2 T(r, f(z+c)-f(z))+2 T(r, g(z+c)-g(z)) \\
& +2 N\left(r, \frac{1}{g}\right)+S(r, f)+S(r, g) \\
\leq & 4 T(r, f)+4 T(r, g)+S(r, f)+S(r, g) .
\end{aligned}
$$

Thus, from (2.6), we get

$$
\begin{equation*}
n T(r, f) \leq 4 T(r, f)+4 T(r, g)+S(r, f)+S(r, g) \tag{4.1}
\end{equation*}
$$

Similarly, we obtain

$$
\begin{equation*}
n T(r, g) \leq 4 T(r, f)+4 T(r, g)+S(r, f)+S(r, g) \tag{4.2}
\end{equation*}
$$

Thus, combining (4.1) with (4.2), we get that

$$
(4.3) n[T(r, f)+T(r, g)] \leq 8[T(r, f)+T(r, g)]+S(r, f)+S(r, g),
$$

which is a contradiction with $n \geq 9$. Thus, from Lemma 2.5, we get $F=G$ or $F \cdot G=1$. If $F \cdot G=1$, it follows that

$$
\begin{equation*}
f(z)^{n}[f(z+c)-f(z)]^{s} \cdot g(z)^{n}[g(z+c)-g(z)]^{s}=d^{2} \tag{4.4}
\end{equation*}
$$

From (4.4) and the fact that $g$ is an entire function, we know that finite order entire functions $f(z)$ and $f(z+c)-f(z)$ has no zeros. Assume that $f(z)=e^{A(z)}$, where $A(z)$ is a nonconstant polynomial, then

$$
e^{A(z+c)}-e^{A(z)}=e^{A(z)}\left(e^{A(z+c)-A(z)}-1\right)
$$

has no zeros, which implies that $A(z+c)-A(z)$ must be a constant. Thus, we obtain that

$$
\begin{equation*}
\frac{f(z+c)}{f(z)}=a \tag{4.5}
\end{equation*}
$$

where $a$ is a nonzero constant. Thus, the solutions of the equation (4.5) can be written in the form of $f(z)=a^{\frac{z}{c}} \Pi_{1}(z)$, where $\Pi_{1}(z)$ is a periodic function with period $c$. Similarly, $g(z)=b^{\frac{z}{c}} \Pi_{2}(z)$, where $\frac{g(z+c)}{g(z)}=b$ is a nonzero constant and $\Pi_{2}(z)$ is also a periodic function with period $c$. Hence, (4.4) changes into

$$
f(z)^{n+s} g(z)^{n+s}=\frac{d^{2}}{(a-1)^{s}(b-1)^{s}} .
$$

Thus, $f(z) g(z)=t$, where $t$ is a $(n+s)$ th root of $\frac{d^{2}}{(a-1)^{s}(b-1)^{s}}$.
Remark: In a special case, if $s=1$ in Theorem 1.4, then we know that

$$
N_{2}\left(r, \frac{1}{F}\right) \leq 2 N\left(r \cdot \frac{1}{f}\right)+N\left(r, \frac{1}{f(z+c)-f(z)}\right) .
$$

Thus, when $s=1$, from the above proof, we can assume that $n \geq 7$.

## 5. Proof of Theorem 1.5

Let $F=\frac{f(z)^{n}[f(z+c)-f(z)]^{s}}{d}, G=\frac{g(z)^{n}[g(z+c)-g(z)]^{s}}{d}$, and let $H$ be defined in Lemma 2.6. If $H \not \equiv 0$, from (2.9), we get

$$
\begin{align*}
n[T(r, f)+T(r, g)] \leq & 2 N_{2}\left(r, \frac{1}{F}\right)+2 N_{2}\left(r, \frac{1}{G}\right)+3 \bar{N}\left(r, \frac{1}{F}\right) \\
& +3 \bar{N}\left(r, \frac{1}{G}\right)+S(r, f)+S(r, g) \\
\leq & \left.7 \bar{N}\left(r, \frac{1}{f}\right)+7 \bar{N} r, \frac{1}{f(z+c)-f(z)}\right)+7 \bar{N}\left(r, \frac{1}{g}\right) \\
& +7 \bar{N}\left(r, \frac{1}{g(z+c)-g(z)}\right)+S(r, f)+S(r, g) \\
\leq & 14[T(r, f)+T(r, g)]+S(r, f)+S(r, g), \tag{5.1}
\end{align*}
$$

which is a contradiction with $n \geq 15$. As above, we remark that if $s=1$, $n \geq 13$ is sufficient.

Thus, we get $H \equiv 0$. The following proof is trivial, which can be seen in many literatures. The original idea is due to Yang and Yi [22]. Here, we give a complete proof. By integrating $H$ twice, we obtain

$$
\begin{equation*}
F=\frac{(b+1) G+(a-b-1)}{b G+(a-b)}, G=\frac{(a-b) F-(a-b-1)}{(b+1)-F b} \tag{5.2}
\end{equation*}
$$

which implies $T(r, F)=T(r, G)+O(1)$. We will show that $F=G$ or $F \cdot G=1$.

Case 1. $b \neq 0,-1$. If $a-b-1 \neq 0$, then by (5.2), we get

$$
\begin{equation*}
\bar{N}\left(r, \frac{1}{F}\right)=\bar{N}\left(r, \frac{1}{G-\frac{a-b-1}{b+1}}\right) \tag{5.3}
\end{equation*}
$$

By the Nevanlinna second main theorem and Lemma 2.4 and (2.6), we have

$$
\begin{align*}
T(r, G) \leq & \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}(r, G)+\bar{N}\left(r, \frac{1}{G-\frac{a-b-1}{b+1}}\right)+S(r, G) \\
\leq & \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}(r, G)+\bar{N}\left(r, \frac{1}{F}\right)+S(r, G) \\
\leq & \bar{N}\left(r, \frac{1}{g}\right)+\bar{N}\left(r, \frac{1}{g(z+c)-g(z)}\right) \\
& +\bar{N}\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{f(z+c)-f(z)}\right)+S(r, g) \\
\leq & 2 T(r, f)+2 T(r, g)+S(r, g) . \tag{5.4}
\end{align*}
$$

Using the same method as above, we have

$$
\begin{equation*}
T(r, F) \leq 2 T(r, f)+2 T(r, g)+S(r, f) \tag{5.5}
\end{equation*}
$$

Thus, from (5.4) and (5.5), we get
(5.6) $n[T(r, f)+T(r, g)] \leq 4[T(r, f)+T(r, g)]+S(r, f)+S(r, g)$.
which contradicts the assumption $n \geq 15$. Thus, $a-b-1=0$, and hence

$$
\begin{equation*}
F=\frac{(b+1) G}{b G+1} \tag{5.7}
\end{equation*}
$$

Using the same method as above,

$$
\begin{align*}
T(r, G) & \leq \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}(r, G)+\bar{N}\left(r, \frac{1}{G+\frac{1}{b}}\right)+S(r, G) \\
& \leq \bar{N}\left(r, \frac{1}{G}\right)+\bar{N}(r, G)+\bar{N}(r, F)+S(r, G) \\
& \leq 2 T(r, g)+S(r, G) \tag{5.8}
\end{align*}
$$

which is also a contradiction.
Case 2. $b=0, a \neq 1$. From (5.2), we have

$$
\begin{equation*}
F=\frac{G+a-1}{a} \tag{5.9}
\end{equation*}
$$

Similarly, we also can get a contradiction. Thus, $a=1$ follows which implies that $F=G$.

Case 3. $b=-1, a \neq-1$. From (5.2), we obtain

$$
\begin{equation*}
F=\frac{a}{a+1-G} . \tag{5.10}
\end{equation*}
$$

Similarly, we get a contradiction, and $a=-1$ follows. Therefore, we get $F \cdot G=1$.

Therefore, using the same method as Theorem 1.4, we can get the proof of Theorem 1.5.

## Acknowledgments

The authors thank the referee for his/her valuable suggestions to improve the present paper. This work was partially supported by the NNSF (No.11026110), the NSF of Jiangxi (No.2010GQS0144, No.2010GQS0139) and the YFED of Jiangxi (No.GJJ11043, No.GJJ10050) of China.

## References

[1] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana 11 (1995), no. 2, 355-373.
[2] W. Bergweiler and J. K. Langley, Zeros of difference of meromorphic functions, Math. Proc. Cambridge Philos. Soc. 142 (2007), no. 1, 133-147.
[3] Z. X. Chen, Growth and zeros of meromorphic solution of some linear difference equations, J. Math. Anal. Appl. 373 (2011), no. 1, 235-241.
[4] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic $f(z+\eta)$ and difference equations in the complex plane, Ramanujan. J. 16 (2008), no. 1, 105129.
[5] A. Eremenko, J. K. Langley and J. Rossi, On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_{k}}{z-z_{k}}$, J. Anal. Math. 62 (1994) 271-286.
[6] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with application to difference equations, J. Math. Anal. Appl. 314 (2006), no. 2, 477-487.
[7] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math. 31 (2006), no. 2, 463-478.
[8] R. G. Halburd and R. J. Korhonen, Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations, J. Phys. A. 40 (2007), no. 6, 1-38.
[9] W. K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. Math. 70 (1959) 9-42.
[10] W. K. Hayman, Research Problems in Function Theory, The Athlone Press, University of London, London, 1967.
[11] W. K. Hayman, Meromorphic Functions, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
[12] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter \& Co., Berlin, 1993.
[13] I. Laine and C. C. Yang, Value distribution of difference polynomials, Proc. Japan Acad. Ser. A Math. Sci. 83 (2007), no. 8, 148-151.
[14] K. Liu and L. Z. Yang, Value distribution of the difference operator, Arch. Math. (Basel) 92 (2009), no. 3, 270-278.
[15] K. Liu and I. Laine, A note on value distribution of difference polynomials, Bull. Aust. Math. Soc. 81 (2010), no. 3, 353-360.
[16] K. Liu, Zeros of difference polynomials of meromorphic functions, Results Math. 57 (2010), no. 3-4, 365-376.
[17] Y. Liu and H. X. Yi, On zeros of differences of meromorphic functions, Ann. Polon. Math. 100 (2011), no. 2, 167-178.
[18] E. Mues, Über ein Problem von Hayman (German), Math. Z. 164 (1979), no. 3, 239-259.
[19] J. Wang, Growth and poles of meromorphic solutions of some difference equations, J. Math. Anal. Appl. 379 (2011), no. 1, 367-377.
[20] J. F. Xu and H. X. Yi, Uniqueness of entire functions and differential polynomials, Bull. Korean Math. Soc. 44 (2007), no. 4, 623-629.
[21] C. C. Yang and X. H. Hua, Uniqueness and value sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math. 22 (1997), no. 2, 395-406.
[22] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Acad. Publ., Dordrecht, 2000.
[23] J. L. Zhang, Value distribution and shared sets of differences of meromorphic functions, J. Math. Anal. Appl. 367 (2010), no. 2, 401-408.

## Kai Liu

Department of Mathematics, Nanchang University, Nanchang, Jiangxi, P. R. China
Email: liukai418@126.com

## Ting-Bin Cao

Department of Mathematics, Nanchang University, Nanchang, Jiangxi, P. R. China
Email: tbcao@ncu.edu.cn

## Xin-Ling Liu

Department of Mathematics, Nanchang University, Nanchang, Jiangxi, P. R. China Email: sdliuxinling@hotmail.com


[^0]:    MSC(2010): Primary: 30D35; Secondary: 39A05.
    Keywords: Entire functions; difference; finite order; uniqueness; value sharing.
    Received: 6 April 2011, Accepted: 31 May 2011.
    *Corresponding author
    (C) 2012 Iranian Mathematical Society.

