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PARA-KÄHLER TANGENT BUNDLES OF CONSTANT PARA-HOLOMORPHIC SECTIONAL CURVATURE

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Abstract. We characterize the natural diagonal almost product (locally product) structures on the tangent bundle of a Riemannian manifold. We obtain the conditions under which the tangent bundle endowed with the determined structure and with a metric of natural diagonal lift type is a Riemannian almost product (locally product) manifold, or an (almost) para-Hermitian manifold. We find the natural diagonal (almost) para-Kählerian structures on the tangent bundle, and we study the conditions under which they have constant para-holomorphic sectional curvature.

1. **Introduction**

The natural fiber bundles over manifolds, and in particular the tangent and cotangent bundles endowed with various structures of natural lift type, were studied in papers such as [1, 10, 11] [16]-[18], [23]-[32], [35].

Roughly speaking, a natural operator is a fibred manifold mapping, which is invariant with respect to the group of local diffeomorphisms of the base manifold. The results from [16]-[18] allowed the extension of the Sasaki metric, which is very rigid in certain senses, to the metrics of natural lift type, leading to interesting geometric structures and to

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interesting relations with some problems in Lagrangian and Hamiltonian mechanics (e.g. see [7]).

On the other hand, authors like Bejan, Cruceanu, Heydari, Ianus, Ishihara, Mihai, Nicolau, Oproiu, Ornea, Papaghiuc, Peyghan, Yano, considered almost product structures and almost para-Hermitian structures (called also almost hyperbolic Hermitian structures) on the tangent and cotangent bundles.

The study of the Riemannian almost product manifolds was initiated in 1965 by K. Yano (see [36]). A classification of these manifolds with respect to the covariant derivative of the almost product structure, was made by Naveira in 1983. In [26] he obtained 36 classes of almost product manifolds. In 1992 Staikova and Gribachev realized a classification of the Riemannian almost product manifolds, for which the trace of the almost product structure vanishes (see [34]). The basic class is that of the almost product manifolds with nonintegrable structure, studied for example in the recent paper [19].

Bejan made a classification of the almost para-Hermitian manifolds. In 1988 she obtained 36 classes, up to duality, and the characterizations of some of them (see [3]). In 1991 Gadea and Muñoz Masqué gave a classification à la Gray-Hervella (see [13]). They obtained 136 classes, up to duality. One of the most studied classes of (almost) para-Hermitian manifolds is the class of (almost) para-Kähler manifolds (e.g., see [2]), characterized by the vanishing condition for the exterior differential of the associated 2-form.

In the present paper we obtain the almost product structures P of natural diagonal lift type on the tangent bundle TM of a Riemannian manifold M. Then we determine the conditions under which the tangent bundle endowed with a natural diagonal almost product structure, and with a natural diagonal lifted metric G is a Riemannian almost product (locally product) manifold, or an (almost) para-Hermitian manifold. We characterize the natural diagonal (almost) para-Kahler structures on the tangent bundle.

The analogue notion for the holomorphic sectional curvature of a Kahler manifold is the para-holomorphic sectional curvature of a para-Kahler manifold, introduced by M. Prvanovic in the paper [33], in 1971, and studied by Gadea and Montesinos Amilibia in 1989 (see [12]). Prvanovic introduced the para-holomorphic projective curvature tensor, or H-projectiv curvature tensor, and she obtained the explicit expression of the curvature tensor field for spaces with constant para-holomorphic sectional curvature.

The final purpose of the present paper is to obtain the conditions under which the determined para-Kahler structures have constant para-holomorphic sectional curvature.

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class C^∞ (i.e., smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r, being always $\{1, \dots, n\}$.

2. **Preliminary results**

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \to M$. The total space TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from the usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates $x^i, y^j,\ i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e., $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^j,\ j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of *M-tensor field* on it (see [22]), called by Miron and his collaborators *distinguished tensor field* or *d-tensor field* (e.g., see [7], [21]).

Denote by $\dot\nabla$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \operatorname{Ker} \tau_*$ and the horizontal distribution HTM defined by $\dot\nabla$ (see [37]). The set of vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ on $\tau^{-1}(U)$ defines a local frame field for VTM, and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k\Gamma^h_{ki}$, and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$ defines a local frame on TM, adapted to the direct sum decomposition (2.1).

Extensive literature for the Finsler geometry, namely the study of some classes of Finsler connections, may be found in a few recent papers, such as [6] and [7].

Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U). \tag{2.2}$$

Obviously, we have $t \in [0,\infty)$ for all $y \in TM$.

We shall use the following lemma, which may be proved easily.

Lemma 2.1. *If $n > 1$ and u, v are smooth functions on TM such that*
$ug_{ij} + vg_{0i}g_{0j} = 0$, *or* $u\delta^j_i + vy^jg_{0i} = 0$,
on the domain of any induced local chart on TM, then $u = 0,\ v = 0$. We used the notation $g_{0i} = y^hg_{hi}$.

3. Natural diagonal almost product structures on the tangent bundle

In the sequel we shall find the almost product structures on the tangent bundle, obtained as natural diagonal lifts of the metric from the base manifold.

An *almost product structure* J on a differentiable manifold M is a $(1,1)$- tensor field on M such that $J^2 = I$. The pair (M,J) is called an *almost product manifold*.

An *almost paracomplex manifold* is an almost product manifold (M,J), such that the two eigenbundles associated to the two eigenvalues $+1$ and -1 of J, respectively, have the same rank. Equivalently, a splitting of the tangent bundle TM into the Whitney sum of two subbundles $T^{\pm}M$ of the same fiber dimension is called *almost paracomplex structure* on M.

Considering a linear connection $\dot{\nabla}$ on the base manifold M, the vertical lift X^V and the horizontal lift X^H of a vector field $X \in \mathcal{T}^1_0(M)$ to the tangent bundle TM, one can define the simplest almost product structures on TM by the relations

$$P(X^H) = -X^H,\ P(X^V) = X^V, \tag{3.1}$$

$$Q(X^H) = X^V,\ Q(X^V) = X^H. \tag{3.2}$$

P is a paracomplex structure if and only if $\dot{\nabla}$ has vanishing curvature, while Q is paracomplex if and only if $\dot{\nabla}$ has both vanishing torsion and curvature. These structures have been extended to the case of a nonlinear connection, and to the specific case of a nonlinear connection defined by a Finsler, Lagrange or Hamilton structure.

Using the adapted frame $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$ to TM, we define a natural diagonal lift of the metric g on the base manifold to the tangent bundle by the relations:

$$P\delta_i = P1_i^j \partial_j,\ P\partial_i = P2_i^j \delta_j, \tag{3.3}$$

where the M-tensor fields involved as coefficients have the forms

$$P\alpha_i^j = a_\alpha(t)\delta_i^j + b_\alpha(t) y^j g_{0i}, \forall \alpha = \overline{1,2}, \tag{3.4}$$

$a_\alpha,\ b_\alpha$, being smooth functions of the energy density t, for $\alpha = \overline{1,2}$.

The invariant expression of the defined structure is

$$\begin{cases} PX_y^H = a_1(t)X_y^V + b_1(t)g_{\tau(y)}(X,y)y_y^V, \\ PX_y^V = a_2(t)X_y^H + b_2(t)g_{\tau(y)}(X,y)y_y^H, \end{cases} \tag{3.5}$$

$\forall X \in \mathcal{T}_0^1(TM),\ \forall y \in TM$.

Example 3.1. *When* $a_1(t) = a_2(t) = 1$, *and* $b_1(t), b_2(t)$ *vanish, we have the structure given by* (3.2).

Next we shall find the conditions under which the above $(1,1)$-tensor field P is an almost product structure on the tangent bundle. Using the relation (3.3), the condition $P^2 = I$, from the definition of the almost product structure, becomes

$$P1_i^j P2_l^i = \delta_l^j,\ P2_i^j P1_l^i = \delta_l^j,$$

which due to (3.4) may be written in the form

$$(a_1 a_2 - 1)\delta_l^j + [b_1(a_2 + 2tb_2) + a_1 b_2] y^j g_{0l} = 0.$$

Taking Lemma 2.1 into account, we have that the coefficients from the above expression must vanish simultanously. The first coefficient vanishes if and only if $a_1 = \frac{1}{a_2}$.

Multiplying the second coefficient by $2t$ and then adding the first coefficient we obtain $(a_1 + 2tb_1)(a_2 + 2tb_2) - 1$, which is equal to zero if and only if $a_1 + 2tb_1 = \frac{1}{a_2 + 2tb_2}$.

Now we may state the following result.

Theorem 3.2. *The natural tensor field P of type $(1,1)$ on TM, defined by the relations* (3.3) *or* (3.5), *is an almost product structure on TM, if and only if the coefficients are related by*

$$a_1 = \frac{1}{a_2}\ , \quad a_1 + 2tb_1 = \frac{1}{a_2 + 2tb_2}. \tag{3.6}$$

Remark 3.3. *When we consider $b_1 = b_2 = 0$ and some particular values of a_1 and a_2 such that $a_1 a_2 = 1$, we obtain the almost product structures studied in* [14], [31] and [32].

In the following theorem we characterize the locally product structures of natural diagonal lift type on the tangent bundle.

Theorem 3.4. *The almost product structure P of natural diagonal lift type on the tangent bundle of an $n(> 2)$-dimensional connected Riemannian manifold (M, g) is integrable (i.e., P is a locally product structure on TM) if and only if (M, g) has constant sectional curvature c, and the coefficients $b_1,\ b_2$ have the forms:*

$$b_1 = \frac{a_1 a_1' + c}{a_1 - 2ta_1'},\ \ b_2 = \frac{a_1 a_2' - a_2^2 c}{a_1 + 2cta_2}. \tag{3.7}$$

Proof. The integrability of an almost product structure P on TM is characterized by the vanishing condition for its Nijenhuis tensor field N_P defined by

$$N_P(X, Y) = [PX, PY] - P[PX, Y] - P[X, PY] + P^2[X, Y],$$

for all vector fields X and Y on TM.

When both arguments are vertical generators, the Nijenhuis tensor field has the form

$$N_P(\partial_i, \partial_j) = [P1^h_m(\partial_j P2^m_i - \partial_i P2^m_j) - \mathrm{Rim}^h_{0ml}\, P2^m_i P2^l_j]\partial_h,$$

which after replacing the values of the M-tensor fields $P1^j_i$ and $P2^j_i$ from (3.4) becomes

$$\begin{aligned} &a_1(a_2' - b_2)(\delta^h_i g_{0j} - \delta^h_j g_{0i}) - a_2^2\, \mathrm{Rim}^h_{0ij} \\ &+a_2 b_2(\mathrm{Rim}^h_{0j0}\, g_{0i} - \mathrm{Rim}^h_{0i0}\, g_{0j}) = 0. \end{aligned} \tag{3.8}$$

Since the curvature of the base manifold does not depend on y, we differentiate with respect to y^k in (3.8). Taking the value of this derivative in $y = 0$, we get

$$R^h_{kij} = c(\delta^h_i g_{kj} - \delta^h_j g_{ki}), \tag{3.9}$$

where

$$c = \frac{a_1(0)}{a_2^2(0)}(a_2'(0) - b_2(0)),$$

which is a function depending only on $x^1, ..., x^n$. According to Schur's theorem, c must be a constant when $n > 2$ and M is connected.

Replacing the expression (3.9) of the curvature, the relation (3.8) becomes

$$(a_1a_2' - a_1b_2 - a_2^2c - 2ta_2b_2c)(\delta_i^h g_{kj} - \delta_j^h g_{ki}) = 0. \tag{3.10}$$

By solving the above equation with respect to b_2, we obtain the second relation in (3.7).

Now, from the vanishing conditions of the Nijenhuis tensor field computed for horizontal arguments,

$$N_P(\delta_i, \delta_j) = (P1_i^l \partial_l P1_j^h - P1_j^l \partial_l P1_i^h - \mathrm{Rim}_{0ij}^h)\partial_h,$$

we obtain

$$(a_1a_1' - a_1b_1 + c + 2a_1'b_1t)(\delta_j^h g_{0i} - \delta_i^h g_{0j}) = 0,$$

which is true if and only if b_1 has the expression from (3.7) presented in the theorem.

The components $N_P(\delta_i, \partial_j) = -N_P(\partial_j, \delta_i)$ of the Nijenhuis tensor field have the expression

$$(P1_i^m \partial_m P2_j^h + P2_l^h \partial_j P1_i^l + P2_j^l P2_m^h \,\mathrm{Rim}_{0il}^m)\delta_h,$$

which after the computations becomes

$$(a_1'a_2 + a_1b_2 + a_2^2c + 2a_2b_2ct)\delta_j^h g_{0i} + (a_1a_2' + a_2b_1 - a_2^2c + 2ta_2'b_1)\delta_i^h g_{0j}$$
$$+(a_2b_1' + a_1'b_2 + 3b_1b_2 + a_1b_2' - a_2b_2c + 2tb_1'b_2 + 2tb_1b_2')g_{0i}g_{0j}y^h.$$
$$+(a_2b_1 + a_1b_2 + 2b_1b_2t)g_{ij}y^h,$$

and it is easy to prove that it vanishes if and only if b_1 and b_2 have the expressions (3.7).

Since all the components of the Nijenhuis tensor field vanish under the same conditions, it follows that the almost product structure P on TM is integrable. □

Example 3.5. *If $b_1 = b_2 = 0$, and $a_2 = \frac{1}{a_1} = a(L^2)$, where $L^2 = 2t$, the relation* (3.10) *takes the form $2a' = -ca^3$. Using the notations of A. Heydary and E. Peyghan, and denoting by k the quantity $-c$, the characterization of the first locally product structure constructed in* [14, Theorem 3.3] *is proved. In an analogous way, we can obtain the other* *locally product structures in the mentioned paper, and some structures in* [31] *and* [32].

Remark 3.6. *Taking* $a_1 = \frac{1}{a_2} = \sqrt{2t},\ b_1 = b_2 = 0$, *the relation* (3.8) *becomes*

$$-\frac{1}{2t}[(\delta_i^h g_{kj} - \delta_j^h g_{ki}) + R_{kij}^h]y^k = 0,$$

which is satisfied if and only if the base manifold has constant sectional curvature -1, *and since all the other components of the Nijenhuis tensor vanish,* [31, Theorem 12] *is proved.*

4. Riemannian almost product and almost para-Hermitian structures of natural diagonal lift type on TM

A lot of papers were dedicated to the almost product and almost para-Hermitian structures on the tangent and cotangent bundles (see [4, 5, 8, 9, 14, 15, 20, 31, 32, 36]).

A Riemannian manifold (M, g), endowed with an almost product structure J, satisfying the relation

$$g(JX, JY) = \varepsilon g(X, Y),\ \forall X, Y \in \mathcal{T}_0^1(M), \tag{4.1}$$

is called *Riemannian almost product manifold* if $\varepsilon = 1$, or *almost para-Hermitian manifold*(called also *almost hyperbolic Hermitian structures*) if $\varepsilon = -1$.

In this section we shall find the conditions under which the tangent bundle TM, endowed with an almost product structure P and with a metric G, both of them being natural diagonal lifts of the metric from the base manifold, is a Riemannian almost product manifold, or a para-Hermitian manifold.

In [27] V. Oproiu defined the semi-Riemannian metric G of natural diagonal lift type on TM by the relations:

$$\begin{cases} G(X_y^H, Y_y^H) = c_1(t)g_{\tau(y)}(X, Y) + d_1(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^V) = c_2(t)g_{\tau(y)}(X, Y) + d_2(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ G(X_y^V, Y_y^H) = 0, \end{cases} \tag{4.2}$$

$\forall X, Y \in \mathcal{T}_0^1(TM),\ \forall y \in TM$, where $c_\alpha,\ d_\alpha,\ \alpha = \overline{1,2}$ are four smooth functions of the energy density on TM.

The conditions for G to be nondegenerate are assured if

$$c_1c_2 \neq 0,\ (c_1 + 2td_1)(c_2 + 2td_2) \neq 0.$$

The metric G is positively defined if

$$c_1 + 2td_1 > 0, \quad c_2 + 2td_2 > 0.$$

The symmetric matrix of type $2n \times 2n$

$$\begin{pmatrix} G^{(1)}_{ij} & 0 \\ 0 & G^{(2)}_{ij} \end{pmatrix} = \begin{pmatrix} c_1(t)g_{ij} + d_1(t)g_{0i}g_{0j} & 0 \\ 0 & c_2(t)g_{ij} + d_2(t)g_{0i}g_{0j} \end{pmatrix}, \tag{4.3}$$

associated to the metric G in the adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$, has the inverse

$$\begin{pmatrix} H^{kl}_{(1)} & 0 \\ 0 & H^{kl}_{(2)} \end{pmatrix} = \begin{pmatrix} p_1(t)g^{kl} + q_1(t)y^k y^l & 0 \\ 0 & p_2(t)g^{kl} + q_2(t)y^k y^l \end{pmatrix},$$

where g^{kl} are the entries of the inverse matrix of $(g_{ij})_{i,j=\overline{1,n}}$, and $p_1,\ q_1,\ p_2,\ q_2$, are some real smooth functions of the energy density. More precisely, they may be expressed as rational functions of $c_1,\ d_1,\ c_2,\ d_2$:

$$p_1 = \frac{1}{c_1},\ p_2 = \frac{1}{c_2},\ q_1 = -\frac{d_1}{c_1(c_1 + 2td_1)},\ q_2 = -\frac{d_2}{c_2(c_2 + 2td_2)}. \tag{4.4}$$

Now we may prove the characterization theorem for the Riemannian almost product (locally product), or (almost) para-Hermitian tangent bundles of natural diagonal lift type.

Theorem 4.1. *The tangent bundle of a Riemannian manifold M, endowed with the natural diagonal metric G and with the almost product structure P characterized in* Theorem 3.2, *is a Riemannian almost product manifold, or an almost para-Hermitian manifold if and only if the coefficients of G and P satisfy the following proportionality relations*

$$\frac{c_1}{a_1} = \varepsilon \frac{c_2}{a_2} = \lambda,\ \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \varepsilon \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu, \tag{4.5}$$

where ε takes the corresponding values from definition (4.1)*, and the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are some functions depending on the energy density t.*

If moreover, the conditions in the statements of Theorem 3.4 *hold, then (TM, G, P) is a Riemannian locally product manifold for $\varepsilon = 1$, or a para-Hermitian manifold for $\varepsilon = -1$.*

Proof. The relation (4.1) has the following forms in the adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$:

$$G(P\delta_i, P\delta_j) = \varepsilon G(\delta_i, \delta_j),\ G(P\partial_i, P\partial_j) = \varepsilon G(\partial_i, \partial_j),\ G(P\partial_i, P\delta_j) = 0. \tag{4.6}$$

Taking (3.3) and (4.3) into account, the relations (4.6) become

$$(-\varepsilon c_1+a_1^2c_2)g_{ij}+[-\varepsilon d_1+a_1^2d_2+2b_1c_2(a_1+tb_1)+4tb_1d_2(a_1+tb_1)]g_{0i}g_{0j}=0,$$

$$(a_2^2c_1-\varepsilon c_2)g_{ij}+[-\varepsilon d_2+a_2^2d_1+2b_2c_1(a_2+tb_2)+4tb_2d_1(a_2+tb_2)]g_{0i}g_{0j}=0.$$

Using Lemma 2.1, we have that the coefficients of g_{ij} and $g_{0i}g_{0j}$ from the above expressions are equal to zero. Since the first relation in (3.6) must be satisfied, we get, by imposing the vanishing conditions for the coefficients of g_{ij}, the first relation in (4.5).

Then, multiplying by $2t$ the coefficients of $g_{0i}g_{0j}$ and adding the corresponding coefficients of g_{ij} we obtain the relations

$$\begin{aligned} &-\varepsilon(c_1+2td_1)+(a_1+2tb_1)^2(c_2+2td_2)=0,\\ &(a_2+2tb_2)^2(c_1+2tb_1)-\varepsilon(c_2+2td_2)=0, \end{aligned} \tag{4.7}$$

which due to the second relation in (3.6) lead to the second relation in (4.5) presented in the theorem. □

Example 4.2. *When $\varepsilon=-1$, $\lambda=1$, and $a_1=a_2=1$, it follows from* (4.5) *that $c_1=-c_2=1$. If the other parameters from the definitions of P and G are zero, we get the almost para-Hermitian structure studied in* [8].

Remark 4.3. *In the case when $\varepsilon=-1$, $\lambda=1$, $a_1=\frac{1}{a_2}=\sqrt{2t}$, $c_1=-2$, $c_2=\frac{1}{t}$, and the other coefficients involved in the definitions* (3.5) *and* (4.2) *vanish,* Theorem 4.1 *reduces to* [31, Theorem 16]. *Next, by using* Remark (3.6), *the results in* [31, Theorem 18] *are proved.*

5. Natural diagonal almost para-Kähler structures on the tangent bundle

In this section we shall study a special class of almost para-Hermitian structures on the tangent bundles, characterized by the vanishing condition of the exterior differential of the associated 2-form Ω. This structures are called almost para-Kähler structures.

The 2-form Ω associated to the almost para-Hermitian structure (G,P) on the tangent bundle is given by the relation

$$\Omega(X,Y)=G(X,PY),\forall X,Y\in\mathcal{T}_0^1(TM).$$

Computing the exterior differential of Ω, we may prove the following characterization theorem:

Theorem 5.1. *The natural diagonal almost para-Hermitian structure (G,P) on TM is almost para-Kählerian if and only if*

$$\mu = \lambda'.$$

Proof. Expressing Ω with respect to the local adapted frame on TM, we have

$$\Omega(\delta_i,\delta_j)=\Omega(\partial_i,\partial_j)=0,\quad \Omega(\delta_i,\partial_j)=G^{(1)}_{ih}P2^h_j,\quad \Omega(\partial_j,\delta_i)=G^{(2)}_{jh}P1^h_i.$$

Moreover, replacing the involved M-tensor fields by their values given in (3.4) and (4.3), and taking into account that all the conditions for (TM,G,P) to be an almost para-Hermitian structure (see Theorem 4.1), we obtain

$$\Omega(\delta_i,\partial_j)=-\Omega(\partial_j,\delta_i)=\lambda g_{ij}+\mu g_{0i}g_{0j},$$

so the associated 2-form Ω has the expression

$$\Omega=(\lambda g_{ij}+\mu g_{0i}g_{0j})dx^i\wedge Dy^j, \tag{5.1}$$

where $Dy^i=dy^i+\Gamma^i_{0h}dx^h$ is the absolute differential of y^i.

Now, let us compute the differential of Ω:

$$d\Omega=(d\lambda g_{ij}+\lambda dg_{ij}+d\mu g_{0i}g_{0j}+\mu dg_{0i}g_{0j}+\mu g_{0i}dg_{0j})\wedge dx^i\wedge Dy^j$$
$$-(\lambda g_{ij}+\mu g_{0i}g_{0j})dx^i\wedge dDy^j.$$

We first calculate the expressions of $d\lambda,\ d\mu,\ dg_{0i}$ and dDy^i:

$$d\lambda=\lambda' g_{0h}Dy^h,\quad d\mu=\mu' g_{0h}Dy^h,\quad dg_{0i}=g_{hi}Dy^h+g_{0h}\Gamma^h_{ik}dx^k,$$
$$dDy^h=\frac{1}{2}R^h_{0ik}dx^i\wedge dx^k+\Gamma^h_{ik}Dy^i\wedge dx^k.$$

Replacing these relations in the expression of $d\Omega$, using the properties of the external product, the symmetry of $g_{ij},\ \Gamma^h_{ik}$ and the Bianchi identities, we get

$$d\Omega=\frac{1}{2}(\mu-\lambda')(g_{ij}g_{0k}-g_{0i}g_{jk})Dy^k\wedge Dy^i\wedge dx^j.$$

Now we can conclude that $d\Omega=0$ if and only if $\mu=\lambda'$, hence the theorem is proved. □

Remark 5.2. *The almost para-Kählerian structures of natural diagonal lift type on TM depend on three essential coefficients $a_1,\ b_1,\lambda$, which must satisfy the supplementary conditions $a_1>0,\ a_1+2tb_1>0,\ \lambda>0,\ \lambda+2t\mu>0$.*

Taking theorems 3.2, 3.4 and 5.1 into account, the following result is proved.

Theorem 5.3. *A natural diagonal almost para-Hermitian structure* (G, P) *on* TM *is para-Kählerian if and only if the almost product structure* P *is integrable (see* Theorem 3.4*) and* $\mu = \lambda'$.

Remark 5.4. *The natural diagonal para-Kählerian structures on* TM *depend on two essential coefficients* a_1, λ, *which must satisfy the supplementary conditions* $a_1 > 0, a_1 + 2tb_1 > 0, \lambda > 0, \ \lambda + 2t\lambda' > 0$, *with* b_1 *given by* (3.7).

6. Natural diagonal para-Kähler tangent bundles of constant para-holomorphic sectional curvature

The aim of this section is to obtain the conditions under which the para-Kahler structures determined in the previous section have constant para-holomorphic sectional curvature.

Let us recall some results from [27], using the notations from the present paper.

Theorem 6.1. ([27]) *The Levi-Civita connection* ∇ *of the natural diagonal lifted metric* G *on the tangent bundle of the Riemannian manifold* (M, g) *has the following expression with respect to the local adapted frame* $\{\delta_i, \partial_j\}_{i,j=1,\dots,n}$

$$\begin{cases} \nabla_{\partial_i}\partial_j = Q^h_{ij}\partial_h, \ \nabla_{\delta}i\partial_j = \Gamma^h_{ij}\partial_h + P^h_{ji}\delta_h, \\ \nabla_{\partial_i}\delta_j = P^h_{ij}\delta_h, \ \nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + S^h_{ij}\partial_h, \end{cases}$$

where Γ^h_{ij} *are the Christoffel symbols of the Levi-Civita connection* $\dot{\nabla}$ *on the base manifold, and the* M*-tensor fields appearing as coefficients in the above expressions are given as*

$$\begin{cases} Q^h_{ij} = \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij}) H^{kh}_{(2)}, \\ P^h_{ij} = \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk} G^{(2)}_{li}) H^{kh}_{(1)}, \\ S^h_{ij} = -\frac{1}{2}(\partial_k G^{(2)}_{ij} + R^l_{0ij} G^{(2)}_{lk}) H^{kh}_{(2)}, \end{cases} \tag{6.1}$$

R^h_{kij} *being the components of the curvature tensor field of* $\dot{\nabla}$.

The components of the curvature tensor field K of the connection ∇ with respect to the adapted frame $\{\delta_i, \partial_j\}_{i,j=1,\dots,n}$ are

$$\begin{aligned}
K(\delta_i,\delta_j)\delta_k &= (P^h_{li}S^l_{jk} - P^h_{lj}S^l_{ik} + R^h_{kij} + R^l_{0ij}P^h_{lk})\delta_h,\\
K(\delta_i,\delta_j)\partial_k &= (P^l_{kj}S^h_{il} - P^l_{ki}S^h_{jl} + R^l_{0ij}Q^h_{lk} + R^h_{kij})\partial_h,\\
K(\partial_i,\partial_j)\delta_k &= (\partial_i P^h_{jk} - \partial_j P^h_{ik} + P^l_{jk}P^h_{il} - P^l_{ik}P^h_{jl})\delta_h,\\
K(\partial_i,\partial_j)\partial_k &= (\partial_i Q^h_{jk} - \partial_j Q^h_{ik} + Q^l_{jk}Q^h_{il} - Q^l_{ik}Q^h_{jl})\partial_h,\\
K(\partial_i,\delta_j)\delta_k &= (\partial_i S^h_{jk} + S^l_{jk}Q^h_{il} - P^l_{ik}S^h_{jl} - \dot\nabla_j R^r_{0ik}G^{(2)}_{rl}H^{(1)}_{hl})\partial_h,\\
K(\partial_i,\delta_j)\partial_k &= (\partial_i P^h_{kj} + P^l_{kj}P^h_{il} - Q^l_{ik}P^h_{lj})\delta_h.
\end{aligned}$$

Notice that, in the case of the obtained para-Kähler manifold (TM, G, P), due to the integrability condition of P, the base manifold (M, g) has constant sectional curvature, so $\dot\nabla R^h_{kij} = 0$, and the above formulas become simpler.

Replacing the M-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ by their values given in Theorem 6.1, and taking into account that all the conditions for (TM, G, P) to be a para-Kahler manifold (see Theorem 5.3), the components of the corresponding curvature tensor field will have some expressions of the form

$$\begin{aligned}
(6.2)\quad \alpha_1\delta^h_i g_{jk} + \alpha_2\delta^h_j g_{ik} + \alpha_3\delta^h_k g_{ij} + \alpha_4\delta^h_k g_{0i}g_{0j} + \alpha_5\delta^h_j g_{0i}g_{0k} + \alpha_6\delta^h_i g_{0j}g_{0k}\\
+ \alpha_7 g_{jk}g_{0i}y^h + \alpha_8 g_{ik}g_{0j}y^h + \alpha_9 g_{ij}g_{0k}y^h + \alpha_{10}g_{0i}g_{0j}g_{0k}y^h = 0,
\end{aligned}$$

where the coefficients $\alpha_1,\dots,\alpha_{10}$ are some quite complicated smooth functions on TM, depending on the parameters a_1, λ, their derivatives of orders $1, 2$ and 3, the energy density, and the constant sectional curvature c of the base manifold. We mention that in a few components of K some of this ten coefficients vanish.

The curvature tensor field corresponding to a para-Kälerian manifold of constant para-holomorphic sectional curvature, k, was independently defined in [12] and [33]. In the case of the para-Käler manifold (TM, G, P) it is given by the formula:

$$\begin{aligned}
K_0(X,Y)Z = \frac{k}{4}[G(Y,Z)X - G(X,Z)Y + G(Y,PZ)PX\\
- G(X,PZ)PY - 2G(X,PY)PZ], \quad \forall X,Y,Z \in \mathcal{T}^1_0(TM).
\end{aligned}$$

The curvature of the natural diagonal para-Kähler manifold (TM, G, P) of constant para-holomorphic sectional curvature has the following components with respect to the adapted frame $\{\delta_i, \partial_j\}_{i,j=1,\dots,n}$:

$$K_0(\delta_i,\delta_j)\delta_k = \frac{k}{4}(G^{(1)}_{jk}\delta^h_i - G^{(1)}_{ik}\delta^h_j)\delta_h,$$
$$K_0(\delta_i,\delta_j)\partial_k = \frac{k}{4}(G^{(1)}_{jl}P1^h_i - G^{(1)}_{il}P1^h_j)P2^l_k\partial_h,$$
$$K_0(\partial_i,\partial_j)\delta_k = \frac{k}{4}(G^{(2)}_{jl}P2^h_i - G^{(2)}_{il}P2^h_j)P1^l_k\delta_h,$$
$$K_0(\partial_i,\partial_j)\partial_k = \frac{k}{4}(G^{(2)}_{jk}\delta^h_i - G^{(2)}_{ik}\delta^h_j)\partial_h,$$
$$K_0(\partial_i,\delta_j)\delta_k = \frac{k}{4}[G^{(1)}_{jk}\delta^h_i - G^{(2)}_{il}(P1^l_kP1^h_j - 2P1^l_jP1^h_k)]\partial_h,$$
$$K_0(\partial_i,\delta_j)\partial_k = \frac{k}{4}(-G^{(2)}_{ik}\delta^h_j + G^{(1)}_{jl}P2^l_kP2^h_i - 2G^{(2)}_{il}P1^l_jP2^h_k)\delta_h.$$

Next we have to study the vanishing conditions for the components of the difference $K - K_0$. In this study the following generic result similar to Lemma 2.1 is useful.

Lemma 6.2. *If $\alpha_1,\dots,\alpha_{10}$ are smooth functions on TM such that*

$$\alpha_1\delta^h_i g_{jk} + \alpha_2\delta^h_j g_{ik} + \alpha_3\delta^h_k g_{ij} + \alpha_4\delta^h_k g_{0i}g_{0j} + \alpha_5\delta^h_j g_{0i}g_{0k} + \alpha_6\delta^h_i g_{0j}g_{0k} + \alpha_7 g_{jk}g_{0i}y^h + \alpha_8 g_{ik}g_{0j}y^h + \alpha_9 g_{ij}g_{0k}y^h + \alpha_{10}g_{0i}g_{0j}g_{0k}y^h = 0,$$

then $\alpha_1 = \dots = \alpha_{10} = 0$.

Now we may prove the characterization theorem for the para-Kähler tangent bundles of natural diagonal lift type, which have constant para-holomorphic sectional curvature.

Theorem 6.3. *The natural diagonal para-Kählerian manifold (TM, G, P) is of constant para-holomorphic sectional curvature k, if and only if the base manifold M is flat, the coefficient c_1 involved in the definition of G is a constant $\mathcal{C}$, the coefficient a_1 of P satisfies the differential equation*

$$a_1'' = \frac{4{a_1'}^2(a_1 - a_1't)}{a_1(a_1 - 2a_1't)}, \tag{6.3}$$

and the proportionality factor λ is expressed by

$$\lambda = \frac{\mathcal{C}}{a_1}. \tag{6.4}$$

Moreover, the para-Kählerian manifold (TM, G, P) *cannot have nonzero constant para-holomorphic sectional curvature.*

Proof. Computing the difference $K - K_0$ with respect to the adapted frame $\{\delta_i, \partial_j\}_{i,j=1,\dots,n}$, we remark that all its components are combinations of the form (6.2).

Analyzing the vanishing problem of the mentioned difference, we choose the component $(K - K_0)(\partial_i, \delta_j)\partial_k$. Since in its final expression there appear two terms with shorter expressions, using Lemma (6.2), we have that all the coefficients, which are some quite complicated functions depending on a_1, λ, their first three order derivatives, the energy density, and the constant sectional curvature c of the base manifold, must vanish. The coefficient of the term containing $g_{ik}\delta_j^h$ is zero if and only if the derivative of the proportionality factor λ has the expression

$$\lambda' = \lambda\frac{a_1k(a_1 - 2ta_1')\lambda - a_1^2a_1' - 2c(a_1 - ta_1')}{a_1(a_1^2 + 2ct)}. \tag{6.5}$$

Taking (6.5) into account and imposing the vanishing conditions for the coefficients of $g_{jk}\delta_i^h$ and $g_{ik}\delta_j^h$, we obtain the equations

$$-4a_1c + a_1^2k\lambda - 2ck\lambda t = 0,$$
$$4a_1c - 3a_1^2k\lambda + 6ck\lambda t - 4a_1k^2\lambda^2t = 0.$$

Replacing the value $k = \frac{4a_1c}{\lambda(a_1^2-2ct)}$, obtained from the first equation above, into the second one, we have that

$$-\frac{8a_1c(a_1^2 + 2ct)^2}{(a_1^2 - 2ct)^2} = 0. \tag{6.6}$$

There are three cases under which this relation is true.

If $a_1 = 0$ the structure is no more an almost product one, so this case is not valid. Moreover, the case $a_1^2 = -2ct$ may be discussed only when the base manifold is of negative constant sectional curvature. This situation reduces to $a_1c = 0$. Hence, the equality (6.6) is true only in the case when the base manifold is flat, i.e., $c = 0$, and the constant para-holomorphic sectional curvature of the para-Kähler manifold (TM, G, P) also becomes zero.

Replacing the values $c = k = 0$ into the expression (6.5), we obtain the differential equation

$$\frac{\lambda'}{\lambda} = -\frac{a_1'}{a_1},$$

which has the solution given by (6.4).

Substituting the value (6.4) of λ into the condition (4.5) for the manifold (TM, G, P) to be para-Hermitian, we obtain that $\frac{c_1}{a_1} = \frac{\mathcal{C}}{a_1}$, so the coefficient c_1 involved in the definition of the metric G is a constant.

Next, replacing the expression (6.4) of λ, and taking into account that $c = k = 0$, by using the RICCI package from Mathematica, we get that all the components of $K - K_0$ become zero, except $(K - K_0)(\partial_i, \partial_j)\partial_k$, which has the final form

$$2\frac{a_1 a_1''(a_1 - 2a_1' t) + 4{a_1'}^2(a_1' t - a_1)}{a_1 - 2a_1' t}\Big[\frac{1}{a_1^2}(\delta_i^h g_{0j} - \delta_j^h g_{0i})g_{0k}$$

$$+\frac{1}{(a_1 - 2a_1' t)^2}(g_{jk}g_{0i} - g_{ik}g_{0j})y^h\Big]\partial_h.$$

The above expression vanishes if and only if the relation (6.3) presented in the theorem is satisfied. □

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