

MRA PARSEVAL FRAME MULTIWAVELETS IN $L^2(R^d)$

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Communicated by Behzad Djafari Rouhani

ABSTRACT. In this paper, we characterize multiresolution analysis (MRA) Parseval frame multiwavelets in $L^2(R^d)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $d \times d$ matrix with integer coefficients, such that $|\det A| = 2$. We study a class of generalized low pass matrix filters that allow us to define (and construct) the subclass of MRA tight frame multiwavelets. This leads us to an associated class of generalized scaling functions that are not necessarily obtained from a multiresolution analysis. We also investigate several properties of these classes of generalized multiwavelets, scaling functions, matrix filters and give some characterizations about them. Finally, we describe the matrix multipliers classes associated with Parseval frame multiwavelets (PFMWs) in $L^2(R^d)$ and give an example to support our theory.

1. Introduction

Wavelets are mathematical functions that take account into the resolutions and the frequencies simultaneously and the wavelet transform can make up for the Fourier's transform disadvantage[1]-[4]. Since wavelets could cut up data into different frequency components, people can study each component with a resolution matched to its scale.

MSC(2010): Primary: 42A38; Secondary: 42C40.

Keywords: Frame, matrix filter, pseudo-scaling function, MRA Parseval frame multiwavelets, matrix multiwavelets multiplier.

Received: 22 March 2011, Accepted: 24 May 2011.

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The classical MRA scalar wavelets are probably the most important class of orthonormal wavelets. However the scalar wavelets cannot have the orthogonality, compact support, and symmetry at the same time (except the Haar wavelet). It is a disadvantage for signal processing. Multiwavelets have attracted much attention in the research community, since multiwavelets have more desired properties than any scalar wavelet function, such as orthogonality, short compact support, symmetry, high approximation order and so on [5]-[7]. It is natural, therefore, to develop the multiwavelets theory that can produce systems having these properties.

The concept of frames was introduced a long time ago [8] and has received much attention recently due to the development and study of wavelet theory [9, 10]. Frames can be regarded as the generalizations of orthonormal bases of Hilbert spaces. Inspired by these and other applications, many people explored in their articles the theory of frame wavelets. In particular, they are interested in Parseval Frame Wavelets (PFWs) and their construction via general multiresolution analysis because of the existence of fast implementation algorithm [11, 17].

Since there exist useful filters that do not produce orthonormal basis, nevertheless, they do produce systems that have the reconstruction property, as well as many other useful features, then in [19], authors generalized notions of the wavelet multipliers in [18] to the case of wavelet frame and introduced a class of generalized low pass filter that allowed them to define and construct the MRA Parseval frame wavelets. This led them to an associated class of generalized scaling functions that were not necessarily obtained from a multiresolution analysis. But all these results just base on scalar wavelet.

It is well known that multiwavelets have more desired properties than any scalar wavelet and are very different from scalar wavelet. We have to recur to some special ways to solve problem. In this paper, we study some properties of the generalized multiwavelets, scaling functions and matrix filters in $L^2(R^d)$. Our results generalize the results in [19] to the Parseval frame multiwavelets and give a new way to construct MRA PFMWs by matrix $K(\omega)$. We also give some characterizations of the matrix multiplier classes associated with PFMWs in $L^2(R^d)$. Because the matrix multiplier classes are so complex, our results are very different from original ones.

Let us now describe the organization of the material that follows. In Section 2 contains some definitions in this correspondence. Also,

we review some relative notations. In the next Section, we study the generalized low pass matrix filters, which allows us to construct the subclass of MRA Parseval frame multiwavelets. We also show an associated class of pseudo-scaling functions that are not necessarily obtained from a multiresolution analysis. Finally, we describe the multiplier classes associated with Parseval frame multiwavelets in $L^2(\mathbb{R}^d)$. An example is given to support our theory.

2. Preliminaries

Let us now establish some basic notations.

A collection of elements $\{\phi_j : j \in J\}$ in a Hilbert space H is called a frame if there exist constants a and b , $0 < a \leq b < \infty$, such that

$$a\|f\|^2 \leq \sum_{j \in J} |\langle f, \phi_j \rangle|^2 \leq b\|f\|^2, \quad \forall f \in H.$$

Let a_0 be the supremum of all such numbers a and let b_0 be the infimum of all such numbers b , then a_0 and b_0 are called the frame bounds of the frame $\{\phi_j : j \in J\}$. When $a_0 = b_0$ we say that the frame is tight. When $a_0 = b_0 = 1$ we say the frame is a Parseval frame.

We denote the translation operators $T_k : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, $k \in \mathbb{Z}^d$, defined by $(T_k f)(x) = f(x - k)$.

In what follows, the system of all integer translates $\{T_k f : k \in \mathbb{Z}^d\}$ of a function $f \in L^2(\mathbb{R}^d)$ will be denoted by $T(f)$.

Let $E_d^{(2)}$ denote the set of all expanding matrices A such that $|\det A| = 2$. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_d^{(2)}$, we denote a unitary operator D acting on $L^2(\mathbb{R}^d)$ defined by

$$Df(t) = |\det A|^{\frac{1}{2}} f(At), \quad \forall f \in L^2(\mathbb{R}^d), t \in \mathbb{R}^d.$$

The operator D corresponding to a real expansive matrix A is called an A -dilation operator. Let B be the transpose of $A : B = A^t$. It is obvious that $B \in E_d^{(2)}$.

If $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then we define its Fourier transform:

$$\hat{f}(s) = \int_{\mathbb{R}^d} e^{-i2\pi\langle s, t \rangle} f(t) dt,$$

where $\langle s, t \rangle$ denotes the standard inner product in \mathbb{R}^d . So we have the relative notation:

$$\text{supp} f = \{s \in \mathbb{R}^d : \hat{f}(s) \neq 0\}.$$

It is useful to note the following formulae for any $f \in L^2(\mathbb{R}^d)$:

$$T_k \hat{f}(\xi) = e^{-2\pi i \langle k, \xi \rangle} \hat{f}(\xi), \forall k \in \mathbb{Z}^d,$$

$$D^j \hat{f}(\xi) = 2^{-\frac{j}{2}} \hat{f}(B^{-j}\xi), \forall j \in \mathbb{Z}$$

and, in particular,

$$\hat{D}f(\xi) = 2^{-\frac{1}{2}} \hat{f}(B^{-1}\xi).$$

The following elementary lemma [16, Lemma 2.2], provides us with a convenient description of $B\mathbb{Z}^d$ for an arbitrary $A \in E_d^{(2)}$, and it will be used in section 4.

Lemma 2.1. *Let $B \in E_d^{(2)}$ be any integer matrix such that $|\det B| = 2$. Then the group $\mathbb{Z}^d/B\mathbb{Z}^d$ is isomorphic to $B^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ and the order of $\mathbb{Z}^d/B\mathbb{Z}^d$ is equal to 2. In particular, if $\alpha \in \mathbb{Z}^d/B\mathbb{Z}^d$ and $\beta = B^{-1}\alpha$, then $\mathbb{Z}^d = B\mathbb{Z}^d \cup (B\mathbb{Z}^d + \alpha)$ and $B^{-1}\mathbb{Z}^d = \mathbb{Z}^d \cup (\mathbb{Z}^d + \beta)$.*

Our standard example that will be frequently used is the quincunx matrix $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in E_d^{(2)}$. Observe that Q acts on \mathbb{R}^2 as rotation by $\frac{\pi}{4}$ composed with dilation by $\sqrt{2}$. In the quincunx case, our standard choice will be $\alpha = (1, 0), \beta = (\frac{1}{2}, \frac{1}{2})$.

We denote by \mathbf{T}^d the d -dimensional torus. By $L^p(\mathbf{T}^d)$ we denote the space of all \mathbb{Z}^d -periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{\mathbf{T}^d} |f(x)|^p dx < +\infty$. The standard unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$ will be denoted by \mathbb{C} . The subsets of \mathbb{R}^d invariant under \mathbb{Z}^d translations and the subsets of \mathbf{T}^d are often identified.

The Lebesgue measure of a set $S \subseteq \mathbb{R}^d$ will be denoted by $|S|$. When measurable sets X and Y are equal up to a set of measure zero, we write $X \doteq Y$.

In the following, we introduce some notations and definitions about frame wavelet and shift invariant subspace.

Let us now fix an arbitrary matrix $A \in E_d^{(2)}$. For $\Psi = \{\{\psi_1, \dots, \psi_r\} \subset L^2(\mathbb{R}^d)$, we will consider the affine system $X(\Psi)$ defined by

$$(2.1) \quad X(\Psi) = \left\{ \psi_{j,k}^l(x) \mid \psi_{j,k}^l(x) = 2^{\frac{j}{2}} \psi^l(A^j x - k) : \right. \\ \left. j \in \mathbb{Z}; k \in \mathbb{Z}^d; l = 1, \dots, r \right\}.$$

Then we define the multiwavelet frame and the multiwavelet Parseval frame.

Definition 2.2. We say that $X(\Psi) \subset L^2(\mathbb{R}^d)$ is a multiwavelet frame if the system (2.1) is a frame for $L^2(\mathbb{R}^d)$.

Definition 2.3. We say that $X(\Psi) \subset L^2(\mathbb{R}^d)$ is a multiwavelet Parseval frame if the system (2.1) is a Parseval frame for $L^2(\mathbb{R}^d)$.

Definition 2.4. A closed subspace $V \subset L^2(\mathbb{R}^d)$ is shift invariant if $\forall f \in V$ implies $T_k f \in V$ for any $k \in \mathbb{Z}^d$.

We recall a result from [20] that characterizes Parseval frame wavelets associated with more general matrix dilations. We state the special case of that theorem appropriate to the discussion in this paper.

Lemma 2.5. [20, Theorem 6.12] Let A be an arbitrary matrix in $E_d^{(2)}$, $B = A^t$ and $\Psi := \{\psi_1, \psi_2, \dots, \psi_r\} \in L^2(\mathbb{R}^d)$. Then the system (2.1) is a PFMW if and only if both the equality

$$(2.2) \quad \sum_{i=1}^r \sum_{j \in \mathbb{Z}} |\hat{\psi}_i(B^j \omega)|^2 = 1, \text{ a.e.}$$

and the equality

$$(2.3) \quad \sum_{i=1}^r \sum_{j=0}^{+\infty} \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + Bk + \alpha))} = 0, \\ \text{a.e., } \forall k \in \mathbb{Z}^d, \alpha \in \mathbb{Z}^d / B\mathbb{Z}^d$$

are satisfied.

In the following, we will give some definitions which will be used in this paper. Given a collection of functions $M = \{m_0, m_1, \dots, m_r\} \subset L^\infty([0, 1]^d)$, we have the following definition:

Definition 2.6. A measurable \mathbb{Z}^d -periodic function $M := \{m_0, m_1, \dots, m_r\}$ on \mathbb{R}^d is called a generalized matrix filter if it satisfies the equation

$$(2.4) \quad \sum_{i=0}^r |m_i(\omega)|^2 = 1 \text{ a.e. } \omega,$$

$$(2.5) \quad m_0(\omega) \overline{m_0(\omega + \beta)} - \left(\sum_{i=1}^r m_i(\omega) \overline{m_i(\omega + \beta)} \right) = 0 \text{ a.e. } \omega,$$

where β is defined in Lemma 2.1.

we shall denote by \tilde{F} the set of generalized Mattix filters and put $\tilde{F}^+ = \{M \in \tilde{F} : m_0 \geq 0, m_0 \in M\}$. Observe that $M \in \tilde{F} \Rightarrow M_{|m_0|} := \{|m_0|, m_1, \dots, m_r\} \in \tilde{F}^+$.

Definition 2.7. A function $\varphi \in L^2(\mathbb{R}^d)$ is called a pseudo-scaling function if there exists a generalized matrix filter $M = \{m_0, m_1, \dots, m_r\} \in \tilde{F}$ such that

$$(2.6) \quad \hat{\varphi}(B\omega) = m_0(\omega)\hat{\varphi}(\omega), \text{ a.e. } \omega.$$

Notice that M is not uniquely determined by the pseudo-scaling function φ . Therefore, we shall denote by \tilde{F}_φ the set of all $M \in \tilde{F}$ such that M satisfies (2.6) for φ . For example, if $\varphi = 0$, then, $\tilde{F}_\varphi = \tilde{F}$.

Suppose that $M \in \tilde{F}^+$. Since $0 \leq m_0(\omega) \leq 1$, a.e. ω , the function

$$\widehat{\varphi_{m_0}}(\omega) =: \prod_{j=1}^{+\infty} m_0(B^{-j}\omega)$$

is well defined a.e., moreover, we have

$$(2.7) \quad \widehat{\varphi_{m_0}}(B\omega) = m_0(\omega)\widehat{\varphi_{m_0}}(\omega), \text{ a.e. } \omega.$$

Following [16], the function $\widehat{\varphi_{m_0}}$ defined by (2.7) belongs to $L^2(\mathbb{R}^d)$ and the function $\widehat{\varphi_{m_0}}$ is a pseudo-scaling function such that $M \in \tilde{F}_{\varphi_{m_0}}$.

Consequently, if $M \in \tilde{F}$ is an arbitrary generalized matrix filter, then the function $\widehat{\varphi_{|m_0|}}$ is a pseudo-scaling function and $M_{|m_0|} \in \tilde{F}_{\varphi_{|m_0|}}$.

Definition 2.8. For $M := \{m_0, m_1, \dots, m_r\} \in \tilde{F}_{\varphi_{m_0}}$, define

$$(2.8) \quad N_0(m_0) = \{\omega \in \mathbb{R}^d : \lim_{j \rightarrow +\infty} \widehat{\varphi_{m_0}}(B^{-j}\omega) = 0\}.$$

We say that M is a generalized low-pass matrix filter if $|N_0(|m_0|)| = 0$. The set of all generalized matrix filters satisfying (2.8) is denoted by \tilde{F}_0 .

Let $e_i = \{0^0, 0^1, \dots, 1^i, \dots, 0^r\}^T$, we will give the definition of MRA PFMW.

Definition 2.9. A PFMW $\Psi := \{\psi_1, \psi_2, \dots, \psi_r\}$ is an MRA PFMW if there exists a pseudo-scaling function φ and $M \in \tilde{F}_\varphi$ and unimodular functions $s_i \in L^2(\mathbb{T}^d)$, $1 \leq i \leq r$ such that

$$(2.9) \quad \hat{\psi}_i(B\omega) = e^{2\pi i\omega} s_i(B\omega) \overline{M(\omega)} \hat{\varphi}(\omega) e_i, \text{ a.e. } \omega.$$

In the following, we give the definitions of the multiplier classes associated with PFMWs in $L^2(\mathbb{R}^d)$. They will be used in Section 4.

Definition 2.10. (1) A PFMW multiplier Γ is a function such that $\tilde{\Psi} = (\hat{\Psi}\Gamma)^\vee$ is a PFMW whenever Ψ is a PFMW; (2) An MRA PFMW multiplier is a function Γ such that $\tilde{\Psi} = (\hat{\Psi}\Gamma)^\vee$ is an MRA PFMW whenever Ψ is an MRA PFMW; (3) A pseudo-scaling function multiplier is a function ν such that $\tilde{\varphi} = (\hat{\varphi}\nu)^\vee$ is a pseudo-scaling function associated with an MRA PFMW whenever φ has the same property; (4) A generalized low pass matrix filter multiplier is a function Γ such that $\tilde{M} = M\Gamma$ is a generalized low pass matrix filter whenever M is a generalized low pass matrix filter.

3. MRA Parseval frame multiwavelets

This section is devoted to characterization the pseudo-scaling functions, the generalized filters and the MRA PFMWs in $L^2(\mathbb{R}^d)$.

Firstly, we give a Lemmas in order to prove our main results.

Lemma 3.1. *If $f \in L^2(\mathbb{R}^d)$, then, for a.e. $\omega \in \mathbb{R}^d$, $\lim_{j \rightarrow +\infty} |f(B^j\omega)| = 0$.*

Proof. Assuming that $f \in L^2(\mathbb{R}^d)$ and applying the monotone convergence theorem we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \sum_{j \in \mathbb{N}} |f(B^j\omega)|^2 d\omega &= \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^d} |f(B^j\omega)|^2 d\omega \\ &= \sum_{j \in \mathbb{N}} 2^{-j} \int_{\mathbb{R}^d} |f(\xi)|^2 d\xi = \|f\|^2 < +\infty. \end{aligned}$$

It follows that for a.e. $\omega \in \mathbb{R}^d$, $\sum_{j \in \mathbb{N}} |f(B^j\omega)|^2$ is finite. Therefore, for a.e. $\omega \in \mathbb{R}^d$, $\lim_{j \rightarrow +\infty} |f(B^j\omega)|^2 = 0$. \square

The following theorem gives a characterization of the generalized low pass filter.

Theorem 3.2. *Suppose $\Psi := \{\psi_1, \psi_2, \dots, \psi_r\}$ is an MRA PFMW and φ is a pseudo-scaling function satisfying (2.9). If M is defined by (2.9), then $M \in \tilde{F}_0$.*

Proof. Notice that Ψ is an MRA PFMW, by (2.2), (2.4), (2.5) and (2.9), we have

$$\begin{aligned}
1 &= \sum_{i=1}^r \sum_{j \in Z} |\hat{\psi}_i(B^j \omega)|^2 \\
&= \sum_{i=1}^r \sum_{j \in Z} |e^{2\pi i \omega} s_i(B^j \omega) \overline{M(B^{j-1} \omega)} e_i|^2 |\hat{\varphi}(B^{j-1} \omega)|^2 \\
&= \sum_{i=1}^r \sum_{j \in Z} |m_i(B^{j-1} \omega)|^2 |\hat{\varphi}(B^{j-1} \omega)|^2 \\
&= \lim_{n \rightarrow +\infty} \sum_{j=-n}^n \left(\sum_{i=1}^r |m_i(B^{j-1} \omega)|^2 \right) |\hat{\varphi}(B^{j-1} \omega)|^2 \\
&= \lim_{n \rightarrow +\infty} \sum_{j=-n}^n (1 - |m_0(B^{j-1} \omega)|^2) |\hat{\varphi}(B^{j-1} \omega)|^2 \\
&= \lim_{n \rightarrow +\infty} \{ |\hat{\varphi}(B^{-n-1} \omega)|^2 - |\hat{\varphi}(B^n \omega)|^2 \}.
\end{aligned}$$

Using $\varphi \in L^2(\mathbb{R}^d)$, Lemma 3.1 implies $\lim_{n \rightarrow +\infty} |\hat{\varphi}(B^n \omega)|^2 = 0$ for a.e. ω .

Then for a.e. ω , $\lim_{n \rightarrow +\infty} |\hat{\varphi}(B^{-n} \omega)| = 1$ holds.

From (2.6), we have

$$|\hat{\varphi}(\omega)| = \left| \prod_{j=1}^n M(B^{-j} \omega) \right| |\hat{\varphi}(B^{-j} \omega)| e_0, \text{ a.e. } \omega.$$

Using (2.7), we obtain that $|\hat{\varphi}(\omega)| = \widehat{\varphi}_{|M|}$ and $|N_0(|M|)| = 0$ is clearly satisfied. Thus, by Definition 2.7, we have $M \in \tilde{F}_0$. \square

In order to obtain the main result in this part, we firstly introduce a result in [16, Lemma 2.8].

Lemma 3.3. *Let $B \in M_d(\mathbb{Z})$ be an expanding matrix. Let μ be a \mathbb{Z}^d -periodic, unimodular function. Then there exists a unimodular function ν that satisfies*

$$(3.1) \quad \mu(\omega) = \nu(B\omega) \overline{\nu(\omega)}, \text{ a.e. } \omega.$$

Theorem 3.4. *Suppose that $M \in \tilde{F}_0$ is a generalized matrix filter. Then, there exist a pseudo-scaling function φ and an MRA PFMW $\Psi := \{\psi_1, \psi_2, \dots, \psi_r\}$ such that they satisfy (2.9).*

Proof. Assume that $M \in \tilde{F}_0$ is a generalized matrix filter. Define the signum function μ for $m_0 \in M$:

$$\mu(\omega) = \begin{cases} \frac{m_0(\omega)}{|m_0(\omega)|}, & m_0(\omega) \neq 0; \\ 1, & m_0(\omega) = 0. \end{cases}$$

Obviously, μ is a measurable unimodular function and the following equation

$$m_0(\omega) = \mu(\omega)|m_0(\omega)|, \text{ a.e. } \omega$$

holds. By Lemma 3.3, there exists a unimodular measurable function ν such that

$$\nu(B\omega)\overline{\nu(\omega)} = \mu(\omega), \text{ a.e. } \omega.$$

By (2.7), we constructed the function $\widehat{\varphi}_{|m_0|}(\omega)$ from $|m_0|$. Let

$$(3.2) \quad \hat{\varphi}(\omega) := \nu(\omega)\widehat{\varphi}_{|m_0|}(\omega),$$

then, $\varphi \in L^2(\mathbb{R}^d)$. Using (2.7), (3.1), (3.2) and the definition of the signum function μ , we have

$$\begin{aligned} \hat{\varphi}(B\omega) &= \nu(B\omega)\widehat{\varphi}_{|m_0|}(B\omega) \\ &= \nu(B\omega)|m_0(\omega)|\widehat{\varphi}_{|m_0|}(\omega) \\ &= \nu(B\omega)|m_0(\omega)|\nu(\omega)\hat{\varphi}(\omega) \\ &= \mu(\omega)|m_0(\omega)|\hat{\varphi}(\omega) \\ &= m_0(\omega)\hat{\varphi}(\omega). \end{aligned}$$

Hence, φ is a pseudo-scaling function.

For $\hat{\Psi} := \{\psi_1, \dots, \psi_r\}$, let

$$\hat{\psi}_i(B\omega) = e^{2\pi\omega i} s_i(B\omega)\overline{M(\omega)}\hat{\varphi}(\omega)e_i, 1 \leq i \leq r, \text{ a.e. } \omega,$$

where $s_i(\omega) \in L^2(\mathbb{T}^d)$, $m_i \in M$. By Lemma 2.5, if $\hat{\Psi}$ satisfies (2.2) and (2.3), then $\Psi(x)$ is an MRA PFMW.

Notice that m_0 is a generalized low pass filter, hence we can get that $\lim_{j \rightarrow +\infty} |\hat{\varphi}(B^{-j}\omega)| = 1$, a.e. ω holds. Using Lemma 3.1, hence we have

$$\begin{aligned}
 & \sum_{i=1}^r \sum_{j \in Z} |\hat{\psi}_i(B^j \omega)|^2 \\
 &= \sum_{i=1}^r \sum_{j \in Z} |m_i(B^{j-1} \omega)|^2 |\hat{\varphi}(B^{j-1} \omega)|^2 \\
 &= \lim_{n \rightarrow +\infty} \sum_{j=-n}^n \left(\sum_{i=1}^r |m_i(B^{j-1} \omega)|^2 \right) |\hat{\varphi}(B^{j-1} \omega)|^2 \\
 &= \lim_{n \rightarrow +\infty} \sum_{j=-n}^n (1 - |m_0(B^{j-1} \omega)|^2) |\hat{\varphi}(B^{j-1} \omega)|^2 \\
 &= \lim_{n \rightarrow +\infty} \{ |\hat{\varphi}(B^{-n-1} \omega)|^2 - |\hat{\varphi}(B^n \omega)|^2 \} \\
 &= \lim_{n \rightarrow +\infty} |\hat{\varphi}(B^{-n-1} \omega)|^2.
 \end{aligned}$$

The above equation shows that the first characterizing condition (2.2) in Lemma 2.5 is satisfied precisely when $\lim_{j \rightarrow +\infty} |\hat{\varphi}(B^{-j} \omega)| = 1$, *a.e.* ω holds.

Now we prove that the function Ψ given above satisfies (2.3).

Let us fix $\omega \in \mathbb{R}^d$ and $q = Bk + \alpha, k \in Z^d, \alpha \in Z^d/BZ^d$, and write

$$\begin{aligned}
 & \sum_{i=1}^r \sum_{j=0}^{+\infty} \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + q))} \\
 (3.3) \quad &= \sum_{i=1}^r \hat{\psi}_i(\omega) \overline{\hat{\psi}_i(\omega + q)} + \sum_{i=1}^r \sum_{j=1}^{+\infty} \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + q))}.
 \end{aligned}$$

From the fact that $e^{2\pi i B^{-1} \alpha} = e^{2\pi i \beta} = -1$, by (2.5), (2.6) and Lemma 3.1, the first term on the right-hand side of (3.3) have the following equation.

$$\begin{aligned}
 & \sum_{i=1}^r \hat{\psi}_i(\omega) \overline{\hat{\psi}_i(\omega + q)} \\
 &= \sum_{i=1}^r e^{2\pi i B^{-1} \omega} s_i(\omega) m_i(B^{-1} \omega) \hat{\varphi}(B^{-1} \omega) \\
 & \quad \times \overline{e^{-2\pi i B^{-1}(\omega + q)} s_i(\omega + q) m_i(B^{-1}(\omega + q)) \hat{\varphi}(B^{-1}(\omega + q))} \\
 &= e^{-2\pi i B^{-1} q} \left(\sum_{i=1}^r |s_i(\omega)|^2 m_i(B^{-1} \omega) \overline{m_i(B^{-1}(\omega + q))} \right) \\
 & \quad \times \hat{\varphi}(B^{-1} \omega) \overline{\hat{\varphi}(B^{-1}(\omega + q))} \\
 &= e^{-2\pi i B^{-1} \alpha} m_0(B^{-1} \omega) \overline{m_0(B^{-1} \omega + \beta)} \hat{\varphi}(B^{-1} \omega) \overline{\hat{\varphi}(B^{-1}(\omega + q))} \\
 &= -\hat{\varphi}(\omega) \overline{\hat{\varphi}(\omega + q)}.
 \end{aligned}$$

By (2.4), (2.6), (2.9) and Lemma 3.1, for the second term on the right-hand side of (3.3), we have

$$\begin{aligned}
 & \sum_{i=1}^r \sum_{j=1}^{+\infty} \widehat{\psi}_i(B^j \omega) \overline{\widehat{\psi}_i(B^j(\omega + q))} \\
 &= \sum_{i=1}^r \sum_{j=1}^{+\infty} e^{2\pi i B^{j-1} \omega} s_i(B^j \omega) m_i(B^{j-1} \omega) \widehat{\varphi}(B^{j-1} \omega) \\
 & \quad \times \overline{e^{2\pi i B^{j-1}(\omega+q)} s_i(B^j(\omega + q)) m_i(B^{j-1}(\omega + q)) \widehat{\varphi}(B^{j-1}(\omega + q))} \\
 &= \sum_{j=1}^{+\infty} e^{2\pi i B^{j-1} q} \left(\sum_{i=1}^r m_i(B^{j-1} \omega) \overline{m_i(B^{j-1}(\omega + q))} \right) \\
 & \quad \times \widehat{\varphi}(B^{j-1} \omega) \overline{\widehat{\varphi}(B^{j-1}(\omega + q))} \\
 &= \sum_{j=1}^{+\infty} (1 - |m_0(B^{j-1} \omega)|^2) \widehat{\varphi}(B^{j-1} \omega) \overline{\widehat{\varphi}(B^{j-1}(\omega + q))} \\
 &= \sum_{j=1}^{+\infty} \{ \widehat{\varphi}(B^{j-1} \omega) \overline{\widehat{\varphi}(B^{j-1}(\omega + q))} - \widehat{\varphi}(B^j \omega) \overline{\widehat{\varphi}(B^j(\omega + q))} \} \\
 &= \widehat{\varphi}(\omega) \overline{\widehat{\varphi}(\omega + q)} - \lim_{p \rightarrow +\infty} \widehat{\varphi}(B^p \omega) \overline{\widehat{\varphi}(B^p(\omega + q))} \\
 &= \widehat{\varphi}(\omega) \overline{\widehat{\varphi}(\omega + q)}.
 \end{aligned}$$

From the above results, we have that the expression in (3.3) is equal to 0. Hence, by Lemma 2.5, Ψ is a PFMW. \square

Theorem 3.5. *Suppose $K(\omega)$ is a $l \times 1$ matrix with $B^{-1}Z^d$ -periodic entries $e^{2\pi i \omega} a_t(\omega)$ such that $\sum_{t=1}^l |e^{2\pi i \omega} a_t(\omega)|^2 = 1$ a.e. ω . Let $M := \{m_0, \dots, m_r\} \in \tilde{F}_\varphi$. Define new filters via*

$$\begin{pmatrix} n_{1,1}(\xi) \\ \vdots \\ n_{l,1}(\xi) \\ \vdots \\ n_{1,r}(\xi) \\ \vdots \\ n_{l,r}(\xi) \end{pmatrix} = \begin{pmatrix} K(\xi) m_1(\xi) \\ \vdots \\ K(\xi) m_r(\xi) \end{pmatrix}.$$

If there exist MRA PFMWs associated with M , then the affine systems generated by $\Psi = \{\psi_{j,t} : 1 \leq j \leq r, 1 \leq t \leq l\}$ obtained via

$$\widehat{\psi}_{j,t}(B\xi) = n_{j,t}(\xi) \widehat{\varphi}(\xi)$$

are Parseval frames.

Proof. Suppose that $M \in \tilde{F}_0$ is a generalized matrix filter.

Notice that $m_0 \in M$ is a generalized low pass filter, hence we can get that $\lim_{j \rightarrow +\infty} |\hat{\varphi}(B^{-j}\omega)| = 1$, *a.e.* ω holds. Using Lemma 3.1, we have

$$\begin{aligned} & \sum_{t=1}^l \sum_{j=1}^r \sum_{k \in \mathbb{Z}} |\hat{\psi}_{j,t}(B^k\omega)|^2 \\ &= \sum_{t=1}^l \sum_{j=1}^r \sum_{k \in \mathbb{Z}} |e^{2\pi i B^{k-1}\omega} a_t(B^{k-1}\omega)|^2 |m_j(B^{k-1}\omega)|^2 |\hat{\varphi}(B^{k-1}\omega)|^2 \\ &= \sum_{j=1}^r \sum_{k \in \mathbb{Z}} |m_j(B^{k-1}\omega)|^2 |\hat{\varphi}(B^{k-1}\omega)|^2 \left(\sum_{t=1}^l |e^{2\pi i B^{k-1}\omega} a_t(B^{k-1}\omega)|^2 \right) \\ &= \lim_{n \rightarrow +\infty} \sum_{k=-n}^n \left(\sum_{j=1}^r |m_j(B^{k-1}\omega)|^2 \right) |\hat{\varphi}(B^{k-1}\omega)|^2 \\ &= \lim_{n \rightarrow +\infty} \sum_{k=-n}^n (1 - |m_0(B^{k-1}\omega)|^2) |\hat{\varphi}(B^{k-1}\omega)|^2 \\ &= \lim_{n \rightarrow +\infty} \{ |\hat{\varphi}(B^{-n-1}\omega)|^2 - |\hat{\varphi}(B^n\omega)|^2 \} \\ &= \lim_{n \rightarrow +\infty} |\hat{\varphi}(B^{-n-1}\omega)|^2. \end{aligned}$$

The above equation shows that the first characterizing condition (2.2) in Lemma 2.5 is satisfied precisely when $\lim_{j \rightarrow +\infty} |\hat{\varphi}(B^{-j}\omega)| = 1$, *a.e.* ω holds.

Now we prove that the function Ψ given above satisfies (2.3).

Let us fix ω and $q = Bk + \alpha, k \in \mathbb{Z}^d, \alpha \in \mathbb{Z}^d / B\mathbb{Z}^d$, and write

$$\begin{aligned} & \sum_{t=1}^l \sum_{j=1}^r \sum_{k=0}^{+\infty} \hat{\psi}_{j,t}(B^k\omega) \overline{\hat{\psi}_{j,t}(B^k(\omega + q))} \\ (3.4) \quad &= \sum_{t=1}^l \sum_{j=1}^r \hat{\psi}_{j,t}(\omega) \overline{\hat{\psi}_{j,t}(\omega + q)} \\ & \quad + \sum_{t=1}^l \sum_{j=1}^r \sum_{k=1}^{+\infty} \hat{\psi}_{j,t}(B^k\omega) \overline{\hat{\psi}_{j,t}(B^k(\omega + q))}. \end{aligned}$$

From the fact that $e^{2\pi i B^{-1}\alpha} = e^{2\pi i \beta} = -1$, by (2.5), (2.6) and Lemma 3.1, the first term on the right-hand side of (3.4) has the following equation:

$$\begin{aligned}
& \sum_{t=1}^l \sum_{j=1}^r \psi_{j,t}(\omega) \overline{\psi_{j,t}(\omega + q)} \\
&= \sum_{t=1}^l \sum_{j=1}^r e^{2\pi i B^{-1}\omega} a_t(B^{-1}\omega) m_j(B^{-1}\omega) \hat{\varphi}(B^{-1}\omega) \\
&\quad \times \overline{e^{2\pi i B^{-1}(\omega+q)} a_t(B^{-1}(\omega+q)) m_j(B^{-1}(\omega+q)) \hat{\varphi}(B^{-1}(\omega+q))} \\
&= \left(\sum_{t=1}^l e^{2\pi i B^{-1}\omega} |a_t(B^{-1}\omega)|^2 \right) \left(\sum_{j=1}^r m_j(B^{-1}\omega) \overline{m_j(B^{-1}(\omega+q))} \right) \\
&\quad \times \hat{\varphi}(B^{-1}\omega) \overline{\hat{\varphi}(B^{-1}(\omega+q))} \\
&= -m_0(B^{-1}\omega) \overline{m_0(B^{-1}(\omega+q))} \hat{\varphi}(B^{-1}\omega) \overline{\hat{\varphi}(B^{-1}(\omega+q))} \\
&= -\hat{\varphi}(\omega) \overline{\hat{\varphi}(\omega + q)}.
\end{aligned}$$

By (2.4), (2.6), (2.9) and Lemma 3.1, for the second term on the right-hand side of (3.4), we have

$$\begin{aligned}
& \sum_{t=1}^l \sum_{j=1}^r \sum_{k=1}^{+\infty} \psi_{j,t}(B^k\omega) \overline{\psi_{j,t}(B^k(\omega + q))} \\
&= \sum_{t=1}^l \sum_{j=1}^r \sum_{k=1}^{+\infty} e^{2\pi i B^{k-1}\omega} a_t(B^{k-1}\omega) m_i(B^{k-1}\omega) \hat{\varphi}(B^{k-1}\omega) \\
&\quad \times \overline{e^{2\pi i B^{k-1}(\omega+q)} a_t(B^{k-1}(\omega+q)) m_i(B^{k-1}(\omega+q)) \hat{\varphi}(B^{k-1}(\omega+q))} \\
&= \sum_{j=1}^{+\infty} \left(\sum_{t=1}^l e^{-2\pi i B^{k-1}q} a_t(B^{k-1}\omega) \overline{a_t(B^{k-1}(\omega+q))} \right) \\
&\quad \times \left(\sum_{j=1}^r m_j(B^{k-1}\omega) \overline{m_j(B^{k-1}(\omega+q))} \right) \hat{\varphi}(B^{k-1}\omega) \overline{\hat{\varphi}(B^{k-1}(\omega+q))} \\
&= \sum_{j=1}^{+\infty} (1 - |m_0(B^{k-1}\omega)|^2) \hat{\varphi}(B^{k-1}\omega) \overline{\hat{\varphi}(B^{k-1}(\omega+q))} \\
&= \sum_{j=1}^{+\infty} \{ \hat{\varphi}(B^{k-1}\omega) \overline{\hat{\varphi}(B^{k-1}(\omega+q))} - \hat{\varphi}(B^k\omega) \overline{\hat{\varphi}(B^k(\omega+q))} \} \\
&= \hat{\varphi}(\omega) \overline{\hat{\varphi}(\omega + q)} - \lim_{p \rightarrow +\infty} \hat{\varphi}(B^p\omega) \overline{\hat{\varphi}(B^p(\omega + q))} \\
&= \hat{\varphi}(\omega) \overline{\hat{\varphi}(\omega + q)}.
\end{aligned}$$

From the above results, we have that the expression in (3.4) is equal to 0. Hence, by Lemma 2.5, Ψ is a PFMW. \square

4. The Multiplier Classes Associated with PFMWs

In this section, we will describe the multiplier classes associated with PFMWs in $L^2(\mathbb{R}^d)$.

At first, we obtain some properties of PFMW multiplier.

Proposition 4.1. *If a measurable function $\Gamma := \{\nu_1, \dots, \nu_r\}$ is a PFMW multiplier, then, $\nu_i, 1 \leq i \leq r$ is unimodular.*

Proof. Let Γ be a PFMW multiplier, then there exist PFMWs Ψ and $\tilde{\Psi}$ such that $\hat{\Psi} = \hat{\Psi}\Gamma$.

By assumption, notice that if $\Psi := \{\psi_1, \dots, \psi_r\} \subset L^2(\mathbb{R}^d)$ is any multiwavelet satisfying $|\hat{\psi}_i(\omega)| > 0, 1 \leq i \leq r$ for a.e. ω , then for every $l \geq 1$, we have that $(\hat{\Psi}\Gamma^l)^\vee$ is a PFMW and satisfies (2.2):

$$\sum_{i=1}^r \sum_{j \in \mathbb{Z}} |\nu_i(B^j \omega)|^{2l} |\hat{\psi}_i(B^j \omega)|^2 = 1, \text{ a.e. } \omega \in \mathbb{R}^d.$$

Moreover, notice $|\nu_i(\omega)| \leq 1$ a.e., hence we easily get

$$|\nu_i(\omega)|^l |\hat{\psi}_i(\omega)| \leq 1, \text{ a.e. } \omega \in \mathbb{R}^d, \forall l \in \mathbb{N}.$$

By (2.2) and $\hat{\tilde{\Psi}} = \hat{\Psi}\Gamma$, notice that $\hat{\tilde{\Psi}}$ and $\hat{\Psi}$ are PFMWs, thus it is easy to get

$$\sum_{i=1}^r \sum_{j \in \mathbb{Z}} |\hat{\psi}_i(B^j \omega)|^2 (1 - |\nu_i(B^j \omega)|^2) = 0, \text{ a.e. } \omega \in \mathbb{R}^d,$$

which is only possible if all terms vanish. Hence, $|\nu_i(\omega)|^2 = 1, 1 \leq i \leq r$ a.e.. The proof is completed. \square

The next theorems give characterization of PFMW multipliers in $L^2(\mathbb{R}^d)$.

Theorem 4.2. *For measurable functions $\Gamma := \{\nu_1, \dots, \nu_r\}^T$, if $\nu_i = \nu_j, 1 \leq i, j \leq r$ is unimodular and $\nu_i(B\omega)\overline{\nu_i(\omega)}$ is \mathbb{Z}^d -periodic, then Γ is a PFMW multiplier.*

Proof. Suppose that Ψ is a PFMW. Let Γ satisfy the assumption in Theorem 4.2. Define $\hat{\tilde{\Psi}} = \hat{\Psi}\Gamma$. If $\hat{\tilde{\Psi}}$ satisfies (2.2) and (2.3), then by Lemma 2.5, $\hat{\tilde{\Psi}}$ is a PFMW.

We firstly prove the function $\hat{\tilde{\Psi}}$ satisfying (2.2).

Since $\forall \nu_i \in \Gamma$ is a unimodular function, for the function $\tilde{\Psi}$, the equation (2.2) holds.

In the following, let us consider (2.3).

Let $k \in \mathbb{Z}^d$, $\alpha \in \mathbb{Z}^d/B\mathbb{Z}^d$, $j > 0$. For $1 \leq i \leq r$, we have

$$(4.1) \quad \begin{aligned} & \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + Bk + \alpha))} \\ &= \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + Bk + \alpha))} \nu_i(B^j \omega) \overline{\nu_i(B^j(\omega + Bk + \alpha))}. \end{aligned}$$

If $j \geq 1$, by Lemma 3.3, there exists a unimodular function μ_i that satisfies

$$\mu_i(\omega) = \nu_i(B\omega) \overline{\nu_i(\omega)}, \text{ a.e. } \omega.$$

According to the \mathbb{Z}^d -periodicity and unimodularity of μ_i , we have

$$\begin{aligned} & \nu_i(B^j \omega) \overline{\nu_i(B^j(\omega + Bk + \alpha))} \\ &= \frac{\mu_i(B^{j-1} \omega) \overline{\mu_i(B^{j-1}(\omega + Bk + \alpha))}}{\nu_i(B^{j-1} \omega) \overline{\nu_i(B^{j-1}(\omega + Bk + \alpha))}} \\ &= \mu_i(B^{j-1} \omega) \overline{\mu_i(B^{j-1}(\omega + Bk + \alpha))} \\ & \quad \times \nu_i(B^{j-1} \omega) \overline{\nu_i(B^{j-1}(\omega + Bk + \alpha))} \\ &= \mu_i(B^{j-1} \omega) \overline{\mu_i(B^{j-1}(\omega))} \nu_i(B^{j-1} \omega) \overline{\nu_i(B^{j-1}(\omega + Bk + \alpha))} \\ &= \nu_i(B^{j-1} \omega) \overline{\nu_i(B^{j-1}(\omega + Bk + \alpha))}. \end{aligned}$$

Then, from the above argument, we obtain

$$(4.2) \quad \begin{aligned} & \nu_i(B^{j-1} \omega) \overline{\nu_i(B^{j-1}(\omega + Bk + \alpha))} \\ &= \nu_i(\omega) \overline{\nu_i(\omega + Bk + \alpha)}, \text{ for } j \geq 2. \end{aligned}$$

Using (4.1), (4.2), $\nu_i = \nu_j$, $1 \leq i, j \leq r$ and summing over $j \geq 0$, we have

$$(4.3) \quad \begin{aligned} & \sum_{i=1}^r \sum_{j=0}^{+\infty} \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + Bk + \alpha))} \\ &= \sum_{i=1}^r \nu_i(\omega) \overline{\nu_i(\omega + Bk + \alpha)} \\ & \quad \times \sum_{j=0}^{+\infty} \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + Bk + \alpha))} \\ &= \nu_i(\omega) \overline{\nu_i(\omega + Bk + \alpha)} \\ & \quad \times \sum_{i=1}^r \sum_{j=0}^{+\infty} \hat{\psi}_i(B^j \omega) \overline{\hat{\psi}_i(B^j(\omega + Bk + \alpha))}. \end{aligned}$$

Hence, by Lemma 2.5, $\tilde{\Psi}$ also satisfies (2.3).

From the above results, we have that $\tilde{\Psi}$ is a PFMW. Thus, Γ is a PFMW multiplier. \square

Theorem 4.3. *For measurable functions $\Gamma := \{\nu_1, \dots, \nu_r\}^T$, if $\forall \nu_i \in \Gamma$ is unimodular and Z^d -periodic, then Γ is a PFMW multiplier.*

Proof. Similar to the proof of Theorems 4.2, we just need to prove that $\nu_i(B\omega)\overline{\nu_i(\omega)}$ is Z^d -periodic and (4.3) is equal to 0.

For $\forall k \in Z^d$, by the Z^d -periodicity of functions $\nu_i, 1 \leq i \leq r$, we have

$$\nu_i(B(\omega + k))\overline{\nu_i(\omega + k)} = \nu_i(B\omega + Bk)\overline{\nu_i(\omega)} = \nu_i(B\omega)\overline{\nu_i(\omega)},$$

then $\nu_i(B\omega)\overline{\nu_i(\omega)}$ is Z^d -periodic.

In the following, let us consider the equation (4.3). By the Z^d -periodic and unimodular property of functions $\nu_i, 1 \leq i \leq r$, it is easy to get

$$\nu_i(\omega)\overline{\nu_i(\omega + Bk + \alpha)} = 1.$$

Hence, the equation (4.3) is equal to 0. \square

The next result gives a characterization of the MRA PFMW multipliers.

Theorem 4.4. *A measurable function $\Gamma := \{\nu_1, \dots, \nu_r\}^T$ satisfying*

$$(4.4) \quad \nu_i(B\omega)\overline{\nu_i(B(\omega + \beta))} = \overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta)), 1 \leq i \leq r$$

is an MRA PFMW multiplier if and only if $\forall \nu_i \in \Gamma$ is unimodular and Z^d -periodic.

Proof. (if) For a measurable function $\Gamma := \{\nu_1, \dots, \nu_r\}$ satisfying

$$\nu_i(B\omega)\overline{\nu_i(B(\omega + \beta))} = \overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta)), 1 \leq i \leq r,$$

assume that $\forall \nu_i \in \Gamma$ is unimodular and Z^d -periodic. Suppose that $M(\omega)$ is a generalized low pass filter associated with an MRA PFMW Ψ , where $\forall \psi_i \in \Psi, |\psi_i| > 0$ for a.e. ω . Then, it follows that $\forall m_i \in M, |m_i| > 0$ a.e. $\omega \in R^d$. Define $\hat{\Psi}$ by $\hat{\Psi}\Gamma$, clearly, for $1 \leq i \leq r$, we have

$$\begin{aligned} \hat{\psi}_i(B\omega) &= \hat{\psi}_i(B\omega)\nu_i(B\omega) \\ &= e^{2\pi i\omega} \overline{s(B\omega)\overline{m_i(\omega)}}\nu_i(B\omega)\hat{\varphi}(\omega). \end{aligned}$$

Let $\tilde{m}_i(\omega) = m_i(\omega)\overline{\nu_i(B\omega)}$, then the following equations hold:

$$\begin{aligned}
\sum_{i=0}^r |\tilde{m}_i(\omega)|^2 &= |m_0(\omega)|^2 + \sum_{i=1}^r |\overline{\nu_i(B\omega)} m_i(\omega)|^2 = \sum_{i=0}^r |m_i(\omega)|^2 = 1, \\
\tilde{m}_0(\omega) \overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega) \overline{\tilde{m}_i(\omega + \beta)} \\
&= m_0(\omega) \overline{m_0(\omega + \beta)} \\
&\quad - \sum_{i=1}^r \nu_i(B\omega) \overline{\nu_i(B(\omega + \beta))} m_i(\omega) \overline{m_i(\omega + \beta)} \\
&= m_0(\omega) \overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega) \overline{m_i(\omega + \beta)} \\
&= 0.
\end{aligned}$$

Hence, by Theorem 3.4, $\tilde{\Psi}$ is an MRA PFMW. (The only if part) Suppose that Ψ and $\tilde{\Psi}$ are MRA PFMWs, and $\Gamma := \{\nu_1, \dots, \nu_r\}$ satisfying (4.4) is an MRA PFMW multiplier associated with Ψ and $\tilde{\Psi}$. Then we have $\hat{\psi}_i = \hat{\psi}_i \nu_i$, $1 \leq i \leq r$. Let $M(\omega)$ be the generalized low pass filter associated with Ψ , we obtain

$$\begin{aligned}
\hat{\psi}_i(B\omega) &= \hat{\psi}_i(B\omega) \nu_i(B\omega) \\
&= e^{2\pi i \omega} s(B\omega) \overline{m_i(\omega)} \nu_i(B\omega) \hat{\varphi}(\omega).
\end{aligned}$$

Hence, Ψ and $\tilde{\Psi}$ can be generated by the same scaling function φ . Let $\tilde{m}_0 = m_0$, notice that Ψ and $\tilde{\Psi}$ are MRA PFMWs, we have the following results:

$$(4.5) \quad \sum_{i=0}^r |\tilde{m}_i(\omega)|^2 = \sum_{i=0}^r |m_i(\omega)|^2 = 1,$$

$$\begin{aligned}
\tilde{m}_0(\omega) \overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega) \overline{\tilde{m}_i(\omega + \beta)} \\
(4.6) \quad &= m_0(\omega) \overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega) \overline{m_i(\omega + \beta)} \\
&= 0.
\end{aligned}$$

From (4.5),

$$\sum_{i=1}^r (1 - |\overline{\nu_i(B\omega)}|^2) |m_i(\omega)|^2 = 0.$$

By the definition of Ψ , clearly,

$$(4.7) \quad |\overline{\nu_i(B\omega)}|^2 = 1, \quad 1 \leq i \leq r, \text{ a.e. } \omega.$$

By (4.6), we have

$$\sum_{i=1}^r (1 - \overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta)))m_i(\omega)\overline{m_i(\omega + \beta)} = 0.$$

Using (4.7) and $\nu_i(B\omega)\overline{\nu_i(B(\omega + \beta))} = \overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta))$, we obtain

$$\begin{aligned} & \overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta))\nu_i(B\omega)\overline{\nu_i(B(\omega + \beta))} \\ &= \overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta))\nu_i(B\omega)\nu_i(B(\omega + \beta)) \\ &= 1, \end{aligned}$$

so,

$$\overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta)) = \frac{1}{\nu_i(B\omega)\nu_i(B(\omega + \beta))}.$$

Notice that $|\overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta))| \leq 1$, from the above results, clearly $\overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta)) = 1$, then $\overline{\nu_i(B\omega)}\nu_i(B(\omega + \beta)) = 1 = \overline{\nu_i(B\omega)}\nu_i(B\omega)$. Hence, $\forall \nu_i \in \Gamma$ is unimodular and Z^d -periodic.

Thus, we complete the proof. □

The next theorems characterize the class of the generalized low pass filter multipliers.

Theorem 4.5. *For a measurable function $\Gamma := \{\nu_0, \nu_1, \dots, \nu_r\}^T, \nu_i = \nu_j, 1 \leq i, j \leq r$, then Γ is a generalized low pass matrix filter multiplier if ν_i is unimodular and Z^d -periodic.*

Proof. (if) Assume that $\forall \nu_i \in \Gamma$ is unimodular and Z^d -periodic and $M(\omega)$ is a generalized low pass matrix filter.

Define

$$\tilde{M}(\omega) = M(\omega)\Gamma(\omega).$$

Then, by the definition of the generalized low pass matrix filter, we have $m_0 \in \tilde{F}_{\varphi_{m_0}}$ and $|N_0(m_0)| = 0$. Again by the function ν_0 being unimodular and Z^d -periodic, it is clear that these properties are also true for $\tilde{m}_0(\omega)$.

Moreover, by the assumption in Theorem 4.5, we can get the following result:

$$\sum_{i=0}^r |\tilde{m}_i(\omega)|^2 = \sum_{i=0}^r |\nu_i(\omega)m_i(\omega)|^2 = \sum_{i=0}^r |m_i(\omega)|^2 = 1,$$

$$\begin{aligned}
& \tilde{m}_0(\omega)\overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega)\overline{\tilde{m}_i(\omega + \beta)} \\
&= \nu_i(\omega)\overline{\nu_i(\omega + \beta)}(m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)}) \\
&= 0.
\end{aligned}$$

Hence, $\tilde{M}(\omega)$ is a generalized low pass matrix filter. \square

Theorem 4.6. *A measurable function $\Gamma := \{1, \nu_1, \dots, \nu_r\}$ satisfying $\nu_i(\omega)\overline{\nu_i(\omega + \beta)} = \overline{\nu_i(\omega)}\nu_i(\omega + \beta)$, $1 \leq i \leq r$ is a generalized low pass matrix filter multiplier if and only if ν_i , $1 \leq i \leq r$, is unimodular and $B^{-1}\mathbb{Z}^d$ -periodic.*

Proof. (if) For $\Gamma := \{1, \nu_1, \dots, \nu_r\}$ satisfying $\nu_i(\omega)\overline{\nu_i(\omega + \beta)} = \overline{\nu_i(\omega)}\nu_i(\omega + \beta)$, $1 \leq i \leq r$, assume that ν_i , $1 \leq i \leq r$, is unimodular and $B^{-1}\mathbb{Z}^d$ -periodic and $M(\omega)$ is a generalized low pass matrix filter.

Let

$$\tilde{M}(\omega) = M(\omega)\Gamma(\omega),$$

then, we have the following result:

$$\begin{aligned}
\sum_{i=0}^r |\tilde{m}_i(\omega)|^2 &= |m_0(\omega)|^2 + \sum_{i=1}^r |\nu_i(\omega)m_i(\omega)|^2 = \sum_{i=0}^r |m_i(\omega)|^2 = 1, \\
& \tilde{m}_0(\omega)\overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega)\overline{\tilde{m}_i(\omega + \beta)} \\
&= m_0(\omega)\overline{m_0(\omega + \beta)} \\
& \quad - \sum_{i=1}^r \nu_i(\omega)\overline{\nu_i(\omega + \beta)}m_i(\omega)\overline{m_i(\omega + \beta)} \\
&= m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)} \\
&= 0.
\end{aligned}$$

Then, $\tilde{M}(\omega)$ is a generalized low pass matrix filter.

(The only if part) Suppose that $M(\omega)$ and $\tilde{M}(\omega)$ are the generalized low pass matrix filters, where $\forall m_i(\omega) \in M(\omega)$, $|m_i(\omega)| > 0$ for a.e. $\omega \in \mathbb{R}^d$, $\tilde{M}(\omega) = M(\omega)\Gamma(\omega)$ and $\Gamma = \{1, \nu_1, \dots, \nu_r\}$ satisfying $\nu_i(\omega)\overline{\nu_i(\omega + \beta)} = \overline{\nu_i(\omega)}\nu_i(\omega + \beta)$, $1 \leq i \leq r$. Then, from the definition of the generalized low pass matrix filter, we have $\sum_{i=0}^r |\tilde{m}_i(\omega)|^2 = 1$ and

$\sum_{i=0}^r |m_i(\omega)|^2 = 1$. Thus, it is clear that:

$$\begin{aligned}
 0 &= \sum_{i=0}^r |\tilde{m}_i(\omega)|^2 - \sum_{i=0}^r |m_i(\omega)|^2 \\
 (4.8) \quad &= \sum_{i=0}^r (|\tilde{m}_i(\omega)|^2 - |m_i(\omega)|^2) \\
 &= \sum_{i=1}^r (|\nu_i(\omega)|^2 - 1)|m_i(\omega)|^2, \\
 0 &= \tilde{m}_0(\omega)\overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega)\overline{\tilde{m}_i(\omega + \beta)} \\
 &\quad - (m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)}) \\
 &= \sum_{i=1}^r (1 - \nu_i(\omega)\overline{\nu_i(\omega + \beta)})m_i(\omega)\overline{m_i(\omega + \beta)}.
 \end{aligned}$$

From (4.8), by assumption of $M(\omega)$, clearly,

$$(4.9) \quad |\overline{\nu_i(\omega)}|^2 = 1, \quad 1 \leq i \leq r, \text{ a.e. } \omega.$$

Using (4.9) and $\nu_i(\omega)\overline{\nu_i(\omega + \beta)} = \overline{\nu_i(\omega)}\nu_i(\omega + \beta), 1 \leq i \leq r$, we have

$$\overline{\nu_i(\omega)}\nu_i(\omega + \beta)\nu_i(\omega)\overline{\nu_i(\omega + \beta)} = \overline{\nu_i(\omega)}\nu_i(\omega + \beta)\overline{\nu_i(\omega)}\nu_i(\omega + \beta) = 1,$$

so,

$$\overline{\nu_i(\omega)}\nu_i(\omega + \beta) = \frac{1}{\nu_i(\omega)\nu_i(\omega + \beta)}.$$

Notice that $|\overline{\nu_i(\omega)}\nu_i(\omega + \beta)| \leq 1$, from the above results, we have $\overline{\nu_i(\omega)}\nu_i(\omega + \beta) = 1$. Then

$$\overline{\nu_i(\omega)}\nu_i(\omega + \beta) = 1 = \overline{\nu_i(\omega)}\nu_i(\omega),$$

Hence, $\nu_i \in \Gamma, 1 \leq i \leq r$ is unimodular and $B^{-1}Z^d$ -periodic. □

For the next theorem let us introduce a notation. For a measurable function ν , let $\nu(\omega) \neq 0, \text{ a.e. } \omega$ and

$$(4.10) \quad \mu(\omega) = \frac{\nu(B\omega)}{\nu(\omega)}, \quad \text{a.e. } \omega.$$

The following result shows that the situation for pseudo-scaling function multipliers is completely different from the others.

Theorem 4.7. *A function ν is a pseudo-scaling multiplier if and only if*

- (1). $\lim_{j \rightarrow \infty} |\nu(B^{-j}\omega)| = 1$ a.e.;
- (2). $|\mu(\omega)|^2 = 1$ is a $B^{-1}Z^d$ -periodic function.

Proof. (if part) Assume that φ is a pseudo-scaling function satisfying (2.4) and (2.5), and suppose that ν satisfies (1)-(2).

Define

$$(4.11) \quad \hat{\varphi}(\omega) = \varphi(\omega)\nu(\omega).$$

By condition (1), we obtain

$$\lim_{j \rightarrow \infty} |\hat{\varphi}(B^{-j}\omega)| = \lim_{j \rightarrow \infty} |\nu(B^{-j}\omega)| \lim_{j \rightarrow \infty} |\varphi(B^{-j}\omega)| = 1.$$

Suppose that the scale equation $\hat{\varphi}(B\omega) = \tilde{m}_0(\omega)\hat{\varphi}(\omega)$ holds. Using (4.11), we have

$$\hat{\varphi}(\omega)\nu(B\omega)m_0(\omega) = \nu(\omega)\tilde{m}_0(\omega)\hat{\varphi}(\omega).$$

Hence,

$$(4.12) \quad \tilde{m}_0(\omega) = \frac{\nu(B\omega)}{\nu(\omega)}m_0(\omega), \quad a.e. \ \omega.$$

By (4.10), according to the condition (2), $\tilde{m}_0(\omega)$ defined by (4.12), is Z^d -periodic.

Define $\tilde{M} := \{\tilde{m}_0, \tilde{m}_1, \dots, \tilde{m}_r\} = \{\tilde{m}_0, m_1, \dots, m_r\}$. By condition (2), the following equations hold:

$$\begin{aligned} \sum_{i=0}^r |\tilde{m}_i(\omega)|^2 &= |\mu(\omega)m_0(\omega)|^2 + |m_1(\omega)|^2 + \dots + |m_r(\omega)|^2 \\ &= |m_0(\omega)|^2 + |m_1(\omega)|^2 + \dots + |m_r(\omega)|^2 \\ &= 1, \end{aligned}$$

$$\begin{aligned} &\tilde{m}_0(\omega)\overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega)\overline{\tilde{m}_i(\omega + \beta)} \\ &= \mu(\omega)\overline{\mu(\omega + \beta)}m_0(\omega)\overline{m_0(\omega + \beta)} \\ &\quad - \sum_{i=1}^r \tilde{m}_i(\omega)\overline{\tilde{m}_i(\omega + \beta)} \\ &= m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)} \\ &= 0. \end{aligned}$$

Thus, $\tilde{M} \in \tilde{F}$. From the above results, we have that $\hat{\varphi}$ is a pseudo-scaling function. Hence, ν is a pseudo-scaling function multiplier.

(only if) Let $\hat{\varphi}$ and $\hat{\varphi}(\omega) = \hat{\varphi}(\omega)\nu(\omega) = \nu(\omega)$ be the pseudo-scaling functions in $L^2(\mathbb{R}^d)$, where the function ν is a pseudo-scaling function multiplier and $|\hat{\varphi}| > 0$ for *a.e.* $\omega \in \mathbb{R}^d$. By the definition of pseudo-scaling function, it is easy to get

$$1 = \lim_{j \rightarrow \infty} |\hat{\varphi}(B^{-j}\omega)| = \lim_{j \rightarrow \infty} |\nu(B^{-j}\omega)| \lim_{j \rightarrow \infty} |\hat{\varphi}(B^{-j}\omega)|, \text{ a.e. } \omega.$$

Using $\lim_{j \rightarrow \infty} |\hat{\varphi}(B^{-j}\omega)| = 1$, we get

$$\lim_{j \rightarrow \infty} |\nu(B^{-j}\omega)| = 1, \text{ a.e. } \omega.$$

The condition (1) is proved.

For $M \in \tilde{F}_\varphi$ and $\tilde{M} \in \tilde{F}_{\hat{\varphi}}$, we have

$$\begin{aligned} & \tilde{m}_0(\omega)\overline{\tilde{m}_0(\omega + \beta)} - \sum_{i=1}^r \tilde{m}_i(\omega)\overline{\tilde{m}_i(\omega + \beta)} \\ (4.13) \quad & = m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)} \\ & = 0, \end{aligned}$$

$$(4.14) \quad \sum_{i=0}^r |\tilde{m}_i(\omega)|^2 = \sum_{i=0}^r |m_i(\omega)|^2 = 1.$$

Let $M := \{m_0, m_1, \dots, m_r\}$ and $\tilde{M} := \{\mu m_0, m_1, \dots, m_r\}$, notice that $\hat{\varphi}$ and φ are pseudo-scaling functions and $\nu(\omega) \neq 0$, *a.e.* ω is a pseudo-scaling function multiplier, similar to the above calculation, we have $\mu(\omega) = \frac{\nu(B\omega)}{\nu(\omega)}$ *a.e.* ω . From (4.13) and (4.14), it is easy to know that

$$\begin{aligned} & \mu(\omega)\overline{\mu(\omega + \beta)}m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)} \\ & = m_0(\omega)\overline{m_0(\omega + \beta)} - \sum_{i=1}^r m_i(\omega)\overline{m_i(\omega + \beta)}, \\ & (|\mu(\omega)|^2 - 1)^2|m_0(\omega)|^2 = 0, \end{aligned}$$

then

$$\mu(\omega)\overline{\mu(\omega + \beta)}m_0(\omega)\overline{m_0(\omega + \beta)} = m_0(\omega)\overline{m_0(\omega + \beta)}.$$

Notice $\mu(\omega)\overline{\mu(\omega + \beta)} = 1 = \mu(\omega)\overline{\mu(\omega)}$, and clearly, $\overline{\mu(\omega + \beta)} = \overline{\mu(\omega)}$. This establishes (2) and we get the results. \square

Now, we will provide an example to support our results.

Example 4.8. Let A be the quincunx matrix $Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in E_2^{(2)}$,

then we get $B = Q^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in E_2^{(2)}$. Furthermore, we have $B^{-1}C \subseteq C$ and $B^jC \subseteq B^{j+1}C, \forall j \in \mathbb{N}$, where C is the standard unit square in \mathbb{R}^2 . Let us define

$$\hat{\varphi}(\omega) = \begin{cases} \frac{1}{2}, & \omega \in B^{-1}C \setminus B^{-2}C \\ 1, & \omega \in B^{-2}C \\ 0, & \omega \in \bar{B}^{-1}C \end{cases}$$

and

$$m_0(\omega) = \begin{cases} 0, & \omega \in B^{-1}C \setminus B^{-2}C \\ \frac{1}{2}, & \omega \in B^{-2}C \setminus B^{-3}C \\ 1, & \omega \in B^{-3}C \end{cases},$$

$$m_1(\omega) = \begin{cases} 0, & \omega \in B^{-1}C \setminus B^{-2}C \\ \frac{1}{2}, & \omega \in B^{-2}C \setminus B^{-3}C \\ 0, & \omega \in B^{-3}C \end{cases},$$

$$m_2(\omega) = \begin{cases} 0, & \omega \in B^{-1}C \setminus B^{-2}C \\ 0, & \omega \in B^{-2}C \setminus B^{-3}C \\ 1, & \omega \in B^{-3}C \end{cases}.$$

Now we extend $m_0(\omega), m_1(\omega), m_2(\omega)$ to C such that the equalities

$$|m_0(\omega)|^2 + |m_1(\omega)|^2 + |m_2(\omega)|^2 = 1$$

and

$$m_0(\omega)\overline{m_0(\omega + \beta)} - m_1(\omega)\overline{m_1(\omega + \beta)} - m_2(\omega)\overline{m_2(\omega + \beta)} = 0$$

are satisfied for all $\omega \in C$, where we take $\beta = (\frac{1}{2}, \frac{1}{2})$, and extend $m_0(\omega), m_1(\omega), m_2(\omega)$ to \mathbb{R}^2 by Z^2 -periodicity. From the definitions of the functions φ and m_0, m_1, m_2 , we easily deduce that φ is a pseudo-scaling function and $M := \{m_0, m_1, m_2\}$ is a generalized low pass matrix filter. Finally, we define

$$\hat{\psi}_1(B\omega) = e^{2\pi i \omega} \overline{m_1(\omega)} \hat{\varphi}(\omega),$$

$$\hat{\psi}_2(B\omega) = e^{2\pi i \omega} \overline{m_2(\omega)} \hat{\varphi}(\omega).$$

Therefore, by Theorem 3.4, we know that we get an MRA Parseval frame multiwavelet. However, from [17], we know that this MRA Parseval

frame multiwavelet is not a Parseval frame wavelet associated to MRA, which does not permit fast algorithm.

5. Conclusion

Multiwavelets can be constructed with more flexibility than traditional scalar wavelets, giving rise to wavelets with desirable properties that are not possible for scalar wavelets. In this paper, we study a class of generalized low pass matrix filters that allows us to define (and construct) the subclass of MRA tight frame multiwavelets. We also investigate several properties of these classes of generalized multiwavelets, scaling functions and matrix filters. Our results are generalizations of the construction of FMWs introduced in [19] to the PFMWs from generalized low-pass filters. However, our ways and results are different from the original ones.

Acknowledgments

The authors would like to express their gratitude to the referee for his (or her) valuable comments and suggestions that lead to a significant improvement of our manuscript. This work was supported by the Natural Science Foundation for the Education Department of Henan Province of China (No. 102300410205).

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