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AN AUTOMATON GROUP: A COMPUTATIONAL CASE STUDY

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Abstract. We introduce a two generated weakly branch contracting automaton group G which is generated by a two state automaton on a three letter alphabet. Using its branch structure and the finiteness nature of a sequence of its factor groups we compute the order of some of these factors. Furthermore some algebraic properties of G are detected.

1. Introduction

A finite automaton is a tuple $\mathfrak{A} = (S, D, \tau, \varepsilon)$, where S, the set of *states,* is a nonempty finite set, D, the set of *alphabets,* is also a finite set, $\tau : D \times S \to S$ and $\varepsilon : D \times S \to D$ are the *transition* and *exit* functions, respectively.
The automaton $\mathfrak{A}$ is said to be invertible if for each state $s \in S$ the function $\varepsilon_s = \varepsilon(., s)$ is a bijection. Therefore, if $D = \{0, 1, \ldots, d-1\}$ then $\varepsilon_s = \varepsilon(., s) : D \to D$ is an element of $Symm(D)$, the symmetric group in d letters. We extend this action to D^*, the regular $d-$ array tree, by defining

$$\varepsilon(\alpha w, s) = \varepsilon(\alpha, s)\varepsilon(w, \tau(\alpha, s)), \quad \varepsilon(\emptyset, s_1) = \emptyset,$$

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inductively for any $w \in D^*$, and $\alpha \in D$, where $\emptyset$ is the empty word.
Therefore, ε preserves the length of the word and respects the order of occurrence. For the invertible automata $\mathfrak{A}$ the group $G =< \varepsilon_s : s \in S >$ is said to be the *automaton group* generated by $\mathfrak{A}$ on the alphabet D. This group acts on D^* faithfully and so it posses an underlaying nice geometry and the automata groups are distinguished from other classes. For more on automata groups we refer to [2, 16] .
The notion of automaton group goes back to Glushkov who conjectured in early 1960's that the Burnside type problems can be solved using these groups [1, 10]. Grigorchuk solved the Burnside problem negatively by introducing a $2-$ group in [12, 8] . This group, among the new generation of automaton groups, is known as the first Grigorchuk group because of its nice properties and deep links to geometry, analysis, dynamical systems and the existence of a continuum of Grigorchuk groups. Almost at the same time Gupta and Sidki using the construction of Grigorchuk discovered their $p-$groups [17].
Finally in 1998 Grigorchuk introduced the basic notion of automaton group [13].
As mentioned above, automata groups, because of their links to many mathematical disciplines such as non-linear group theory, fractal geometry, dynamical systems, analysis, C^* algebras, topology, random walks, and so on, and, as a new field of research, have attracted the attention of many mathematicians from all over the world [8, 14, 20, 2, 4, 3, 15, 16, 7, 17, 5, 6].
Due to these characteristics there are many research problems in this field. One of the problems of interest, is a satisfactory classification of groups generated by small automata [9]. Already the 2 and 3-state automata groups on a two letter alphabet have been classified [15, 5, 6].
In this paper we study the group G generated by the automaton from Figure 1 and discover some of its properties as stated in theorems below. In this respect referring to [18], we observe that the initial automata $a = (1, 1, a)\delta$ and $b = (1, 1, b)\lambda$ of the automaton generating G are entries number 29 and 54 of table 2, respectively. So that G is one of the groups of our list of groups and may be denoted by $G_{29,54}$. Observe that Fabrykowski-Gupta and Bartholdi-Grigorchuk groups are also elements of the this list [3].
We remark that the group G, in some respect, is a ternary tree analog of the Brunner-Sidki-Veira group H [7].
The structure of the paper is as follows: Section 2 is devoted to some

notions needed throughout the sequel. In sections 3 we prove that G is a regular weakly branch group over G' and is a weakly branch group. In section 4 we obtain some algebraic properties of G mainly concerning the stabilizer sub-groups of G. In section 5 we prove that G is torsion free. We show in section 6 that G contains a free sub-semigroup and hence is of exponential growth. In section 7 we show that G has an important geometric property, namely it is contracting.

Theorem 1.1. *Let G be the automaton group generated by the automaton from Figure 1. Then*

(1) *The following relations hold in G*

$$(ab^{-1})^2 = b^{-2}a^2, \quad a^{-6}b^6 = (b^3a^{-3})^2, \quad [[a,b][b^2,a^2],[(ba)^2,(ab)^2]] = 1.$$

(2) *Let $k,l,m,n \in \mathbb{Z}$ and*

$$A_{m,n} = [a^{3m}, b^{3n}], \quad B_{m,n} = [(ba)^m, (ab)^n], \quad C_{m,n} = [(ab)^m, (b)^{3n}],$$

then

$$[A_n^k, B_n^k] = 1, \quad [A_nB_n, B_{k,l}] = 1, \quad B_{m,n}C_{n,m} = 1,$$

where $X_n = X_{n,n}$.

(3) *For k,m,n in $\mathbb{Z}$ the following relations hold in G*

$$[[(ba)^m, b^{3n}], (ab)^k]] = 1, \quad [[(ab)^m, a^{3n}], (ba)^k]] = 1.$$

Theorem 1.2. *Let G be the automaton group generated by the automaton from Figure 1. Then*

(1) *G is weakly branch,*
(2) *G is contracting,*
(3) *G contains a free sub-semigroup of two generators,*
(4) *G is of exponential growth,*
(5) *G is torsion free.*

Theorem 1.3. *We have*

(1) $St_G(1) = < a^3, b^3, ab, ba >$.
(2) $|\frac{G}{St_G(n)}| = 3^{k_n}$, *where $n = 1,2,\ldots,7$ and $k_1 = 1, k_2 = 3, k_3 = 8, k_4 = 22, k_5 = 63, k_6 = 185$ and $k_7 = 550$.*

2. **Preliminaries**

In this section we review some basic notions needed throughout the sequel. As indicated in the introduction, the group G is generated by the automata a and b that act on the alphabet $D = \{0, 1, 2\}$ via $\delta, \lambda \in Symm(D)$, where

$$\delta = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix},$$

and

$$\lambda = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

So that $\delta\lambda = \lambda\delta$ and $\delta^3 = \lambda^3 = 1$.
Therefore, G is a sub-group of the automorphism group, $AutT$, of a rooted ternary tree T with root $\emptyset$, vertex set $V = D^*$ and edge set $E = \{(v, vi)|v \in V, i \in D\}$. In order to be able to continue our study we have to include some notions and notations that are necessary in the sequel.
The length of a vertex v is the number of letters it contains. For a non-negative integer n the nth level L_n of T is the set of vertices of T of length n. For example, $L_0 = \emptyset$ and $L_1 = D$. For a vertex v of T let T_v be the sub-tree of T hanging from v. Therefore, the set of vertices of T_v is $vw, w \in D^*$. Obviously $\chi : T_v \to T; \chi(vu) = u$ is a tree isomorphism with inverse $\chi^{-1}(u) = vu$. Therefore, T_v and T can be identified and any $g \in AutT$ is conjugate to a $g' \in AutT_v$ via this identification. In fact, let the automorphisms $g' \in Aut(T_v)$ and let $g \in AutT$ be so that the diagram

$$\begin{array}{ccc} T_v & \xrightarrow{g'} & T_v \\ \chi \downarrow & & \downarrow \chi \\ T & \xrightarrow{g} & T \end{array}$$

is commutative . Then we have $g = \chi \circ g' \circ \chi^{-1}$. This is the identification that we will use throughout the paper without mentioning.
The subgroup of G that stabilizes the level L_n point-wise, i.e., $(g(w) = w)$, is denoted by $ST_G(n)$. In the following sections we will use the notions of *self-similar*, *self-replicating* and *branch* group repeatedly without any reference and refer the reader to [2, 3] for these and other definitions used in this paper. Here only we remark that by the general properties of automaton groups, G is self-similar and self-replicating

3. G is weakly branch

In this section we study the branch structure of G. More precisely we show that it is a regular weakly branch group over it's commutator sub-group G' and it is a weakly branch group.

Lemma 3.1. *For $k, m, n \in \mathbb{Z}$ the following relations hold in G.*

(1) $[(ba)^n, (ab)^m][(ab)^m, (b)^{3n}] = 1$;
(2) $[(ba)^n, (ab)^m][(a)^{3m}, (ba)^n] = 1$;
(3) $[[(ba)^n, b^{3m}], (ab)^k]] = 1$;
(4) $[(ab)^n, a^{3m}], (ba)^k] = 1$;
(5) $[(ab)^n, a^{3m}][(ba)^n, (ab)^m][b^{3n}, (ba)^m][a^{3m}, b^{3n}] = 1$.

Proof. We prove relations (1) and (5). The other relations are proved similarly. Using $ab = (1, b, a)$ and $ba = (a, 1, b)$ we observe that

$$[(ba)^n, (ab)^m][(ab)^m, (b)^{3n}] = (1, 1, [b^n, a^m])(1, 1, [a^m, b^n]) = (1, 1, 1)$$

m, n=1,2,.... Therefore, (1) is proved.
We have

$$[(ab)^n, a^{3m}][(ba)^n, (ab)^m][b^{3n}, (ba)^m][a^{3m}, b^{3n}] =$$

$$([b^n, a^m], 1, 1)(1, 1, [b^n, a^m])(1, [b^n, a^m], 1)([a^m, b^n], [a^m, b^n], [a^m, b^n]) =$$

(1,1,1). This proves (5). □

Proposition 3.2. *The group G generated by the automaton from Figure 1 is*

(1) *regular weakly branch over G'*
(2) *a weakly branch group, i.e., $Rist_n \neq \{1\}, n = 1, 2, \ldots$.*

Proof. (1) The relations

$$ab^{-1}a^{-1}b = (b^{-1}, a^{-1}b, a), a^2b^2 = (b, ab, a), ab = (1, b, a)$$

show that

$$a^2b^2ab^{-1}a^{-2}b^{-1}a^{-1} = (1, [a, b], 1).$$

This and

$$a^2b^2ab^{-1}a^{-2}b^{-1}a^{-1} = [a, b]^a[b, a]^{(aba)},$$

where $x^y = yxy^{-1}$, show that $1 \times G' \times 1 \leq G'$. Since G is self-replicating we have $G' \times G' \times G' \leq G'$. The assertion is proved.

(2) Obvious as a corollary of (1).

□

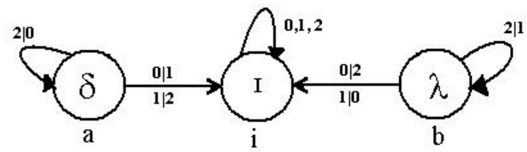

Figure 1. The automaton generating G

4. Some algebraic properties of G

Lemma 4.1. *The elements a and b are of infinite order. So that G is infinite.*

Proof. We prove the order of a in G is infinite. Let W be a power of a. Then $W = a^{3k+r} = (a^k, a^k, a^k)a^r$, for some integer k and a non-negative integer $0 \leq r \leq 2$. It is obvious that this cannot represent 1 in cases $r = 1$ and $r = 2$. In the case $r = 0$ if W represent 1 then we will have $W = a^{3k} = (a^k, a^k, a^k) = (1, 1, 1)$. Let n be the smallest positive integer with the property $a^n = 1$. Then $n = 3k$, $k < n$ and therefore from $a^{3k} = (a^k, a^k, a^k)$ we have $a^k = 1$, with $k < n$, which is a contradiction. The proof for b is similar.

□

Lemma 4.2. *We have* $|\frac{G}{St_G(1)}| = 3$.

Proof. For $g \in G$ we either have $g \in St_G(1)$ or $g \notin St_G(1)$. In the second case some $h \in St_G(1)$ exists such that $g = h\sigma$ where, $\sigma \in \{\delta, \lambda\}$. For $\sigma = \lambda$ we have $ag = ah\sigma \in St_G(1)$ and for $\sigma = \delta$ we have $a^2g = a^2h\sigma \in St_G(1)$. Therefore, $g \in St_G(1) \cup a^{-1}St_G(1) \cup a^{-2}St_G(1)$. Since $a^3 \in St_G(1)$, we have $a^{-1}St_G(1) = a^2St_G(1)$ and $a^{-2}St_G(1) = aSt_G(1)$. Also the sets $St_G(1)$, $aSt_G(1)$ and $a^2St_G(1)$ are mutually disjoint. The proof is complete. □

Lemma 4.3. *The sub-group $St_G(1)$ is generated by $\{a^3, b^3, ab, ba\}$.*

Proof. Let $H =< a^3, b^3, ab, ba >$. As $\{a^3, b^3, ab, ba\} \subset St_G(1)$, we observe that $H \leq St_G(1)$. Therefore, the elements of H fix the level L_1 of T_3 point-wise. As the elements

$$ba^3b^{-1} = ba.a^3(ba)^{-1}, b(ab)b^{-1} = ba, b(ba)b^{-1} = b^2a^2(ba)^{-1},$$
$$ab^3a^{-1} = abb^3(ab)^{-1}, a(ab)a^{-1} = a^2b^2(ab)^{-1}, a(ba)a^{-1} = ab,$$

are in H, we conclude that H is a normal sub-group of G and hence of $St_G(1)$. Now we prove $G = H \cup aH \cup a^2H$. First we observe that the sets $H \cap aH$, $aH \cap a^2H$ and $H \cap a^2H$ are all empty. As $H \cup aH \cup a^2H \subset G$ we observe that $|G/H| \geq 3$. Now we prove $|G/H| = 3$. To this end we show that every element $g = a^{\alpha_1}b^{\beta_1}\ldots a^{\alpha_m}b^{\beta_m}$ in G can be written as $g \equiv a^k(modH)$ with $k = 0, 1, 2$. This will be done by induction on the length of g with respect to $\{a, a^{-1}, b, b^{-1}\}$. We have $b = a^2(b^2a^2)^{-1}b^3, a^{-1} = a^2a^{-3}, b^{-1} = a(ba)^{-1}$. Therefore, the assertion is proved for $|g| = 1$. Let us assume that it is true for $|g| \leq n$. Consider $g = a^{\alpha_1}b^{\beta_1}\ldots a^{\alpha_m}b^{\beta_m}$ with $|g| = |\alpha_1| + |\beta_1| + \ldots |\alpha_m| + |\beta_m| = n + 1$. Four cases may occur:

(1) $\alpha_1 > 0$. Here using the assumption of induction there is $h \in H$ and $k \in \{0, 1, 2\}$ such that $g = aa^kh = a^{k+1}h$, which proves the assertion in this case.

(2) $\alpha_1 = 0, \beta_1 < 0$. We write

$$g = b^{\beta_1}\ldots a^{\alpha_m}b^{\beta_m} = b^{-1}b^{\beta_1+1}\ldots a^{\alpha_m}b^{\beta_m}.$$

Now the length of $g' = b^{\beta_1+1}\ldots a^{\alpha_m}b^{\beta_m}$ is at most n and the induction assumption can be invoked for it. Therefore, $g' = a^kh'$ for some $k = 0, 1, 2$ and $h' \in H$. We have $g = b^{-1}g' = b^{-1}a^kh'$. From this relation for $k = 0$ we obtain $g = b^{-1}g' = a(a^{-1}b^{-1})h' \in aH$. Also for $k = 1$ we have $g = b^{-1}g' = b^{-1}a^1h' = b^{-2}bah' = bb^{-3}bah' = a^{-1}abb^{-3}bah' = a^2a^{-3}abb^{-3}bah' \in a^2H$. In the case $k = 2$ we have $g = b^{-1}g' = b^{-1}a^2h' = b^{-1}a^{-1}a^3h' \in H$.
The proofs of the remaining two case

(3) $\alpha_1 = 0, \beta_1 > 0$

(4) $\alpha_1 < 0$

are similar. Therefore $\frac{G}{H} \leq 3$ and finally we have, $|\frac{G}{H}| = 3$.
Since $|\frac{G}{H}| = 3 = |\frac{G}{St_G(1)}|$ and $H \leq St_G(1)$ we have, $St_G(1) = H$.
The proof is complete.

□

Now since H is a sub-group of $St_G(1)$ the above Lemma shows that $St_G(1) = H$.

Remark For any positive integer n, each element $g \in G$ induces a permutation $\pi_n(g)$ on L_n, the nth level of T. As an example for $n = 1$ and $g = a$, we have $L_1 = \{0, 1, 2\}$ and labeling these vertices by $1, 2, 3$, respectively, the induced permutation on L_1 is the cycle

$$\pi_1(a) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1, 2, 3).$$

In general, let $n \geq 1$ be a positive integer. To find the permutations induced by a and b on L_n we label the 3^n words of length n in alphabet $0, 1, 2$ belonging to L_n as follows: using the element $O = 0 \dots 0$ of L_n we have $L_n = \{a^0(O) := O, a(O), a^2(O), \dots, a^{3^n-1}(O)\}$. Now we consider $l = k+1 : k = 0, \dots, 3^n - 1$ as the label of the element $a^k(O), k = 0, 1, \dots, 3^n - 1$, respectively. It is clear that the permutation $\pi_n(a) \in Symm(3^n)$ induced by a on L_n is the cycle $(1, \dots, 3^n)$. To find $\pi_n(b)$ we have to determine the numbers

$$b(l), l = 1, \dots, 3^n.$$

For example for $n = 2$, denoting each element with its label we have
$L_2 = \{1 = a^0(00) = 00, 2 = a^1(00) = 10, 3 = a^2(00) = 20, 4 = a^3(00) = 01, 5 = a^4(00) = 11, 6 = a^5(00) = 21, 7 = a^6(00) = 02, 8 = a^7(00) = 12, 9 = a^8(00) = 22\}$ and $\pi_2(a)$ is the cycle $(1, 2, 3, 4, 5, 6, 7, 8, 9)$. For b we have $b(1) = 3, b(2) = 1, b(3) = 8, b(4) = 6, b(5) = 4, b(6) = 2, b(7) = 9, b(8) = 7, b(9) = 5$. Therefore, the permutation induced by b on L_2 is the cycle:

$$\pi_2(b) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 8 & 6 & 4 & 2 & 9 & 7 & 5 \end{pmatrix} = (1, 3, 8, 7, 9, 5, 4, 6, 2).$$

We extend π_n to a homomorphism from G to $Symm(3^n)$ and use it several times in the sequel concerning the first and second isomorphism theorems .
We also remark that since $G =< a, b >$, for any positive integer n we have

$$Im(\pi_n) =< \pi_n(a), \pi_n(b) > .$$

Lemma 4.4. *We have* $|\frac{St_G(1)}{St_G(2)}| = 3^2$.

Proof. Let $Symm(9)$ be the symmetric group on 9 elements. Using the above remark, consider the homomorphism $\pi' := \pi_2|_{St_G(1)} : St_G(1) \to Symm(9)$. We have

$$\pi'(a^3) = (1, 4, 7)(2, 5, 8)(3, 6, 9), \quad \pi'(b^3) = (1, 7, 4)(2, 8, 5)(3, 9, 6),$$

and

$$\pi'(ab) = (1, 4, 7)(3, 9, 6), \quad \pi'(ba) = (2, 8, 5)(3, 6, 9).$$

We observe that $St_G(2) = \ker \pi'$. In fact if $\pi'(g) = 1$ then g induces the identity permutation on L_2 and hence $g \in St_G(2)$. Now using the above Lemma and the GAP program we observe that $|Im(\pi')| = 9$. Therefore, by the first isomorphism theorem the proof is complete. $\square$

Corollary 4.5. *We have* $|\frac{G}{St_G(2)}| = 3^3$.

Proof. The result follows from the above two Lemmas and the second isomorphism theorem for groups. □

Corollary 4.6. *The set* $\{1, ab, ba, a^3, b^3, (ab)^3, (ba)^3, a^4b, b^4a\}$ *is a coset representative* $mod(St_G)(2)$ *of* $St_G(1)$.

Proof. Computing the values of the homomorphism π_2 for the elements of the given set we observe that the set of these values is $Im(\pi_2)$. The proof is complete. □

Lemma 4.7. *We have*

(1) $|\frac{G}{St_G(3)}| = 3^8$,
(2) $|\frac{G}{St_G(4)}| = 3^{22}$,
(3) $|\frac{G}{St_G(5)}| = 3^{63}$,
(4) $|\frac{G}{St_G(6)}| = 3^{185}$,
(5) $|\frac{G}{St_G(7)}| = 3^{550}$.

Proof. 1) Define the homomorphism $\pi_3 : G \to Symm(27)$ by induced permutations

$\pi_3(a) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27)$

and

$\pi_3(b) = (1, 3, 8, 7, 9, 23, 22, 24, 20, 19, 21, 26, 25, 27, 14, 13, 15, 11, 10, 12, 17, 16, 18, 5, 4, 6, 2)$ on G.

Using the GAP software we obtain $|Im(\pi_3)| = 6561 = 3^8$.

2) Let the permutation induced on L_4 by a be $\pi_4(a) = (1, 2, \ldots, 81)$. Then the induced permutation by b will be

$\pi_4(b) = (1, 3, 8, 7, 9, 23, 22, 24, 20, 19, 21, 26, 25, 27, 68, 67, 69, 65, 64, 66, 71, 70, 72, 59, 58, 60, 56, 55, 57, 62, 61, 63, 77, 76, 78, 74, 73, 75, 80, 79, 81, 41, 40, 42, 38, 37, 39, 44, 43, 45, 32, 31, 33, 29, 28, 30, 35, 34, 36, 50, 49, 51, 47, 46, 48, 53, 52, 54, 14, 13, 15, 11, 10, 12, 17, 16, 18, 5, 4, 6, 2)$.

The subgroup of the symmetric group $Symm(81)$ generated by these permutations is of order $31381059609 = 3^{22}$.

3) As above, let the permutation induced on L_5 by a be $\pi_5(a) = (1, \ldots, 243)$. Then the induced permutation by b will be

$\pi_5(b) = (1, 3, 8, 7, 9, 23, 22, 24, 20, 19, 21, 26, 25, 27, 68, 67, 69, 65, 64, 66, 71, 70, 72, 59, 58, 60, 56, 55, 57, 62, 61, 63, 77, 76, 78, 74, 73, 75, 80, 79, 81, 203, 202, 204, 200, 199, 201, 206, 205, 207, 194, 193, 195, 191, 190, 192, 197, 196,$

198, 212, 211, 213, 209, 208, 210, 215, 214, 216, 176, 175, 177, 173, 172, 174, 179, 178, 180, 167, 166, 168, 164, 163, 165, 170, 169, 171, 185, 184, 186, 182, 181, 183, 188, 187, 189, 230, 229, 231, 227, 226, 228, 233, 232, 234, 221, 220, 222, 218, 217, 219, 224, 223, 225, 239, 238, 240, 236, 235, 237, 242, 241, 243, 122, 121, 123, 119, 118, 120, 125, 124, 126, 113, 112, 114, 110, 109, 111, 116, 115, 117, 131, 130, 132, 128, 127, 129, 134, 133, 135, 95, 94, 96, 92, 91, 93, 98, 97, 99, 86, 85, 87, 83, 82, 84, 89, 88, 90, 104, 103, 105, 101, 100, 102, 107, 106, 108, 149, 148, 150, 146, 145, 147, 152, 151, 153, 140, 139, 141, 137, 136, 138, 143, 142, 144, 158, 157, 159, 155, 154, 156, 161, 160, 162, 41, 40, 42, 38, 37, 39, 44, 43, 45, 32, 31, 33, 29, 28, 30, 35, 34, 36, 50, 49, 51, 47, 46, 48, 53, 52, 54, 14, 13, 15, 11, 10, 12, 17, 16, 18, 5, 4, 6, 2).

Now using GAP we obtain

$$|\frac{G}{St_G(5)}| = 1144561273430837494885949696427 = 3^{63}$$

as desired.

4) Again let the permutation induced on L_6 by a be $\pi_6(a) = (1, 2, \ldots, 729)$. Then calculation by hand shows that the induced permutation by b is the cycle shown in Figure 2. Using GAP we observe that the permutation group $< \pi_6(a), \pi_6(b) >$ is a subgroup of order

$$185110955751455333384474038361095815053532267407820328039323804$$
$$76024584374357028238763043 = 3^{185}$$

of S_{729}.

5) Here hand calculation of $\pi_7(a)$and $\pi_7(b)$ are not useful. Instead one can use the part of GAP relevant to automata groups which has been added recently to GAP [19]. One only needs to factor the 263 digit number to see that the assertion $|\frac{G}{St_G(7)}| = 3^{550}$ is really true.

$\pi_6(b) = (1, 3, 8, 7, 9, 23, 22, 24, 20, 19, 21, 26, 25, 27, 68, 67, 69, 65, 64, 66, 71, 70, 72, 59, 58, 60, 56, 55, 57, 62, 61, 63, 77, 76, 78, 74, 73, 75, 80, 79, 81, 203, 202, 204, 200, 199, 201, 206, 205, 207, 194, 193, 195, 191, 190, 192, 197, 196, 198, 212, 211, 213, 209, 208, 210, 215, 214, 216, 176, 175, 177, 173, 172, 174, 179, 178, 180, 167, 166, 168, 164, 163, 165, 170, 169, 171, 185, 184, 186, 182, 181, 183, 188, 187, 189, 230, 229, 231, 227, 226, 228, 233, 232, 234, 221, 220, 222, 218, 217, 219, 224, 223, 225, 239, 238, 240, 236, 235, 237, 242, 241, 243, 608, 607, 609, 605, 604, 606, 611, 610612, 599, 598, 600, 596, 595, 597, 602, 601, 603, 617, 616, 618, 614, 613, 615, 620, 619, 621, 581, 580, 582, 578, 577, 579, 584, 583, 585, 572, 571, 573, 569, 568, 570, 575, 574, 576, 590, 589, 591, 587, 586, 588, 593, 592, 594, 635, 634, 636, 632, 631, 633, 638, 637, 639, 626, 625, 627, 623, 622, 624, 629, 628, 630, 644, 643, 645, 641, 640, 642, 647,$

646, 648, 527, 526, 528, 524, 523, 525, 530, 529, 531, 518, 517, 519, 515, 514,
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36, 50, 49, 51, 47, 46, 48, 53, 52, 54, 14, 13, 15, 11, 10, 12, 17, 16, 18, 5, 4, 6, 2)

Figure 2. The cycle $\pi_6(b)$

□

5. G is torsion free

In this section we prove that for any integer $k \geq 2$ the equation $X^k = 1$ has no nontrivial solution in G. In other words, we prove the following proposition.

Proposition 5.1. *The group G is torsion free.*

We prove this assertion through the following Lemmas. But first we need a definition.

Definition 5.2. *For any reduced word $W = a^{\alpha_1}b^{\beta_1}\dots a^{\alpha_n}b^{\beta_n}$ in alphabet $\{a, b, a^{-1}, b^{-1}\}$, where $\alpha_1, \beta_n \in \mathbb{Z}$ and $\beta_1 \dots \alpha_{n-1} \in \mathbb{Z}-\{0\}$, the numbers $\sum_1^n \alpha_k$ and $\sum_1^n \beta_k$ are called the exponent sums of a and b in W and are denoted by $|a|_W$ and $|b|_W$, respectively.*

Lemma 5.3. *For any $g \in G$ with $|g| = n > 2$ and $g = (g_0, g_1, g_2)\sigma$, for some $\sigma \in sym(D)$, we have $|g_i| < |g|, i = 0, 1, 2$. Specifically, for any $g \in St_G(1)$ we have $|g_i| < |g|, i = 0, 1, 2$.*

Proof. We use induction on $n = |g|$. As direct computation shows that the assertion is true for $n = 3$. Now let it be true for all g with $|g| = n \geq 3$ and consider $h \in G$ with $|h| = n + 1$. Then $h = gu$ for some $g \in G$ with $|g| = n$ and $u \in \{a, a^{-1}, b, b^{-1}\}$. Writing $h = (h_0, h_1, h_2)\sigma$ and $g = (g_0, g_1, g_2)\mu$ we observe that the relation $|g_i| = |h_i|$ holds for two components of g and h, and, a third component of them differ by 1 in length. Let $|h_3| = |g_3| \pm 1$. We observe that

$$|h_3| = |g_3| \pm 1 < |g| \pm 1 \leq |h|.$$

Now by induction hypothesis the proof of the first part is complete.
To prove the second assertion we remark that except for the elements $a^{-1}b$ and $b^{-1}a$ the assertion is also true for all elements of length 2 in G and hence for elements $g \in St_G(1)$.

□

Lemma 5.4. *If a reduced word w in alphabet $\{a, b, a^{-1}, b^{-1}\}$ represent the unit element of G then $|a|_W = |b|_W = 0$.*

Proof. We prove the Lemma by induction on the length $|W|$ of W. For $|W| = 1$ or $|W| = 2$ there is no reduced word that represent 1 in G.
Let the assertion be true for all words W with $|W| \leq n$ and consider the reduced word W with $|w| = n + 1$. As W represents 1 in G, it can be written $W = (W_0, W_1, W_2)$. Therefore, $W_i = 1, i = 0, 1, 2$ represent the

trivial element of G as well. Since $|W_i| < |W|, i = 0, 1, 2$, the induction hypothesis implies that

$$|a|_{W_i} = |b|_{W_i} = 0, i = 0, 1, 2.$$

From $a = (1, 1, a)\delta$ and $b = (1, 1, b)\lambda$ we conclude that

$$|a|_W = |a|_{W_0 W_1 W_2} = |a|_{W_0} + |a|_{W_1} + |a|_{W_2} = 0,$$

and

$$|b|_W = |b|_{W_0 W_1 W_2} = |b|_{W_0} + |b|_{W_1} + |b|_{W_2} = 0.$$

The proof is complete. □

Lemma 5.5. *The equation $x^3 = 1$ has no solution in G.*

Proof. In contrary let W be a solution of the given equation with least length in alphabet $\{a, a^{-1}, b, b^{-1}\}$. If $W = (W_0, W_1, W_2)$ is a solution of the given equation then the relations $|W_i| < |W|, i = 0, 1, 2$ will contradict the assumption. So let $W = W'a$ for some $W' \in St_G(1) - \{1\}$. Write $W' = (W_0, W_1, W_2)$. Using emma (4.3) W' can be represented as a word in alphabet $\{a^{\pm 3}, b^{\pm 3}, (ab)^{\pm 1}, (ba)^{\pm 1}\}$. Using this representation and the relations

$$a^3 = (a, a, a), b^3 = (b, b, b), ab = (b, 1, a), ba = (1, a, b)$$

we deduce that

$$|a|_{W_0 W_2 W_1} \equiv |b|_{W_0 W_2 W_1} (mod 3). \tag{5.1}$$

In order that $W = W'a$ satisfy the given equation we must have

$$W^3 = ((W_0, W_1, W_2)a)^3 = ((W_0, W_1, W_2 a)\delta)^3 =$$

$$(W_0 W_1 W_2 a, W_1 W_2 a W_0, W_2 a W_0 W_1) = 1.$$

Therefore, $W_0 W_1 W_2 a$ must represents the trivial element of the group. Now using Lemma (5.4) we have on one hand

$$|a|_{W_0 W_1 W_2 a} = |b|_{W_0 W_1 W_2 a} = 0,$$

and on the other hand

$$|b|_{W_0 W_1 W_2 a} = |b|_{W_0 W_1 W_2}, |a|_{W_0 W_1 W_2 a} = |a|_{W_0 W_1 W_2} + 1.$$

Therefore,

$$|b|_{W_0 W_1 W_2} = 0, \qquad |a|_{W_0 W_1 W_2} + 1 = 0,$$

which contradicts (5.1) The proof is complete. □

Lemma 5.6. *The equation $x^2 = 1$ has no solution in G.*

Proof. Let $W \in G$ be a solution of the equation with a least length. Then we must have $W^2 = (W_0^2, W_1^2, W_2^2) = 1$. Therefore, $W_i, i = 0, 1, 2$ are also solutions of the equation. This along with $|W_i| < |W|, i = 0, 1, 2$ contradicts the assumption. □

To complete the proof of the proposition (5.1) using Lemmas (5.4) and (5.5) the following observation will be sufficient.

Lemma 5.7. *For any integer $k \geq 2$ the equation $x^k = 1$ has no non-trivial solution in G.*

6. **Free sub-semigroup**

In this section we show that G is of exponential growth.

Proposition 6.1. *The semigroup M generated by the three state automaton from Figure 1 is of exponential growth.*

This proposition is a corollary of the following Lemmas.

Lemma 6.2. *If as an automorphism of the rooted ternary tree the word W in alphabet $\{a, b\}$, say $W = a^{k_1} b^{l_1} \dots a^{k_n} b^{l_n}$, is an element of $St_1(G)$ then $|a|_W \equiv |b|_W (mod 3)$.*

Proof. Using $a = (1, 1, a)\delta$ and $b = (1, 1, b)\lambda$ we observe that the exponents of δ and λ in W are $k_1 + \dots + k_n$ and $l_1 + \dots + l_n$, respectively. This along with $\delta = \lambda^{-1}$ implies the assertion. □

Lemma 6.3. *If a word W in alphabet $\{a, b\}$ as an automorphism fixes the vertices $0, 1, 2$ of T_3, then writing $W = (W_0, W_1, W_2)$ we have $|W_i| < |W|, i = 0, 1, 2$.*

Proof. This is a partial case of the Lemma (5.3) □

Lemma 6.4. *The semigroup M is free.*

Proof. As calculations show there are no two words in alphabet $\{a, b\}$ of the same length l with $l \leq 4$ representing the same element of G. Now consider two different words U and V in alphabet $\{a, b\}$ that represent the same element g of G and such that $l = max(|U|, |V|)$ is minimal. Write $U = (U_0, U_1, U_2)\sigma$ and $V = (V_0, V_1, V_2)\sigma$, with $\sigma \in \{1, \delta, \lambda\}$. For $\sigma = 1$ we observe that each of the three pairs U_0, V_0; U_1, V_1 and U_2, V_2 represent the same element say g_0, g_1 and g_2, respectively, of G. Since U and V are different words in $\{a, b\}$, U_i is different from V_i for some $i \in \{0, 1, 2\}$. By the Lemma (6.3) for this i we have $max(|U_i|, |V_i|) < l$. This contradicts the minimality of l. For $\sigma = \delta$ consider $Ub = (U_0b, U_1, U_2)$ and $Vb = (V_0b, V_1, V_2)$. We observe that each of the three pairs U_0b, V_0b; U_1, V_1 and U_2, V_2 represent the same element, say h_0, h_1 and h_2 of G, respectively, and either U_0b is different from V_0b or U_i is different from V_i for some $i \in \{1, 2\}$. In particular, the pair U_0, V_0 represent the same element of G. In any case, using Lemma (5.3) we reach to a contradiction with the choice of l. For $\sigma = \lambda$ we consider the words $Ua = (U_0, U_1a, U_2)$ and $Va = (V_0, V_1a, V_2)$ and proceed as above. □

7. G is a contracting group

Proposition 7.1. *Let Γ be an automaton group generated by a finite automaton over the alphabet $\{0, 1, \ldots, d-1\}$. If Γ contains a finite index contracting subgroup Φ. Then Γ is contracting .*

Proof. The proof is based on the generating set $\{a_1, a_2, \ldots, a_n\}$ coming from the finite automaton. Since Φ is contracting, there are constants $n_0 \in \mathbb{N}$,$0 < \lambda < 1$ and C such that

$$|s_i| < \lambda|s| + C \tag{7.1}$$

for $s \in \Phi$ with ith component s_i and $|s| > n_0$.
Choose $R = \{u_0 = 1, u_1, \ldots, u_{m-1}\}$ as a system of representatives for Γ mod (Φ). Let $l_i = |u_i|, i = 1, 2, \ldots, m-1$ be the length of u_i with respect to the generating set $\{a_1, a_2, \ldots, a_n\}$ and $l = \max\{l_1, \ldots, l_{m-1}\}$. Let u_{ij} be the jth component of u_i on decomposing u_i with respect to the structure of the automaton generating Γ. Suppose $M = \max\{|u_{ij}| : i = 1, \ldots, m-1; j = 0, 1, \ldots, d-1\}$.
For $g \in \Gamma$, $g = u_is$ for some $u_i \in R$ and some $s \in \Phi$, we have $s = u_i^{-1}g$ and so

$$|s| < |u_i| + |g|. \tag{7.2}$$

Also $g_i = u_{ij}s_k$, where g_i ,u_{ij} and s_k are the ith, ijth and kth component of g, u_i and s, respectively. This implies

$$|g_i| < |u_{ij}| + |s_k|. \tag{7.3}$$

Using (7.1) and (7.3) we obtain

$$|g_i| < |u_{ij}| + \lambda|s| + C. \tag{7.4}$$

This and (7.2) imply that

$$|g_i| < |u_{ij}| + \lambda(|u_i|) + |g|) + C = \lambda|g| + |u_{ij}|\lambda|u_i| + C. \tag{7.5}$$

Now (7.4) implies

$$|g_i| < M + \lambda l + \lambda|g| + C = \lambda|g| + (M + \lambda l + C).$$

The proof is complete. □

Corollary 7.2. *The group G generated by the automaton from Figure 1 is contracting .*

Proof. By Lemma (4.2), $St_G(1)$ is of finite index in G and we have $G = St_G(1) \cup St_G(1)a \cup St_G(1)b$. First we prove $St_G(1)$ is contracting. Consider $g \in St_G(1)$ with a decomposition $g = (g_0, g_1, g_2)$. For $g \in \{a^{\pm 3}, b^{\pm 3}, (ab)^{\pm 1}, (ba)^{\pm 1}\}$ a direct computation shows that

$$|g_i| < \frac{1}{2}|g| + 1, \qquad i = 0, 1, 2.$$

Suppose the assertion is true for g with $|g| < n$ and consider $g \in St_G(1)$ with $|g| = n$. Then by Lemma (4.3) g can be written in a unique way as

(1) $g = ug'$ with $u \in \{a^{\pm 3}, b^{\pm 3}\}$, $|g| = 3 + |g'|$ and $g' \in St_G(1)$; or
(2) $g = vg'$ with $v \in \{(ab)^{\pm 3}, (ba)^{\pm 3}\}$, $|g| = 2 + |g'|$ and $g' \in St_G(1)$.

Writing $g' = (g'_0, g'_1, g'_2)$ from items (1) and (2) we obtain
(1') $g_i = xg'_i, x \in \{a^{\pm 1}, b^{\pm 1}\}, |g_i| = |g'_i| + 1, i = 0, 1, 2$, and
(2')$g_i = xg'_i, x \in \{1, a^{\pm 1}, b^{\pm 1}\}, |g_i| \leq |g'_i| + 1, i = 0, 1, 2$, respectively.
Using $|g'| < |g| = n$ and the induction in the case (1') we get

$$|g_i| = |g'_i| < \frac{1}{2}(|g'| + 1) = 1 + \frac{1}{2}(|g| - 2) = \frac{1}{2}|g| < \frac{1}{2}(|g| + 1), i = 0, 1, 2$$

in case (2') we either obtain

$$|g_i| = 1 + |g'_i| < 1 + \frac{1}{2}(|g'| + 1) = \frac{1}{2}(|g| - 1) < \frac{1}{2}(|g| + 1), i = 0, 1, 2$$

or

$$|g_i| = 1 + |g'_i| < 1 + \frac{1}{2}(|g'| + 1) = 1 + \frac{1}{2}(|g| - 1) = \frac{1}{2}(|g| + 1), i = 0, 1, 2.$$

Now consider $h = ga \in St_G(1)a$ with $g = (g'_0, g'_1, g'_2) \in St_G(1)$, and let $|h| = |g| + 1$. Then $h = (g'_0, g'_1, g'_2a)\delta$ and

$$|g'_2a| < \frac{1}{2}(|g| + 1) + 1 < \frac{1}{2}(|h|) + 1.$$

In case $|h| = |g| - 1$ we have

$$|g'_2a| < \frac{1}{2}(|g| + 1) + 1 < \frac{1}{2}(|h| + 2) + 1 = \frac{1}{2}|h| + 2.$$

Other cases are treated similarly. Therefore choosing $\lambda = \frac{1}{2}$ and $C = 2$ we observe that G is contracting. □

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