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ORE EXTENSIONS OF SKEW π-ARMENDARIZ RINGS

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ABSTRACT. For a ring endomorphism α and an α-derivation δ, we introduce a concept, so called skew π-Armendariz ring, that is a generalization of both π-Armendariz rings and (α, δ)-compatible skew Armendariz rings. We first observe the basic properties of skew π-Armendariz rings, and extend the class of skew π-Armendariz rings through various ring extensions. We next show that all (α, δ)-compatible NI rings are skew π-Armendariz, and if a ring R is an (α, δ)-compatible 2-*primal* ring, then the polynomial ring $R[x]$ is skew π-Armendariz.

1. **Introduction**

Let R be an associative ring with unity, α an endomorphism, and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. We denote by $R[x; \alpha, \delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication is subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. Given a ring R, we use $P(R)$ and $nil(R)$ to represent the prime radical (i.e., the intersection of all prime ideals) and the set of nilpotent elements of R, respectively. A ring R is called 2-*primal* if $P(R) = nil(R)$, and a ring R is said to be an NI ring if $nil(R)$ forms an ideal. Clearly, all 2-*primal* rings are NI rings. Let I be a subset of R,

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$I[x;\alpha,\delta]$ means the set $\{a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta] \mid a_i \in I\}$. For $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha,\delta]$, we denote by $\{a_0, a_1, \cdots, a_n\}$ or C_f the set comprised of the coefficients of $f(x)$.

According to Krempa [8], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α-*rigid* if there exists a rigid endomorphism α of R. In [5], the authors introduced an (α,δ)-*compatible* ring and studied its properties. A ring R is α-*compatible* if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Clearly, this may only happen when the endomorphism α is injective. Moreover, R is said to be δ-*compatible* if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. A ring R is (α,δ)-*compatible* if it is both α-*compatible* and δ-*compatible*. Also by [5, Lemma 2.2], a ring R is α-*rigid* if and only if R is (α,δ)-*compatible* and reduced.

A ring R is said to be *Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The term Armendariz was introduced by Rege and Chhawchharia [12]. This nomenclature was used by them since it was Armendariz [4, Lemma 1] who initially showed that a *reduced* ring (i.e., a ring without nonzero nilpotent elements) always satisfies this condition. Armendariz rings are thus a generalization of reduced rings, and therefore, nilpotent elements play an important role in this class of rings. There are many examples of rings with nilpotent elements which are Armendariz. In fact, in [2], Anderson and Camillo proved that if $n > 2$, then $R[x]/(x^n)$ is an Armendariz ring if and only if R is reduced.

Huh et al. [7] have studied a generalization of Armendariz rings, which they called π-Armendariz rings. A ring R is said to be a π-Armendariz ring provided that whenever $f(x)g(x) \in nil(R[x])$ for $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$, then $a_ib_j \in nil(R)$ for all i, j. Each 2-*primal* ring is π-Armendariz by [7, Proposition 1.3] and so π-Armendariz rings are a common generalization of 2-*primal* rings and Armendariz rings. This further motivates the study of the nilpotent elements in this class of rings. What we observe is that in all the examples found in the literature of Armendariz rings, the set of nilpotent elements forms an ideal.

In this paper, we introduce the notion of skew π-Armendariz rings by considering the polynomials in the skew polynomial ring $R[x;\alpha,\delta]$ in place of the ring $R[x]$. Most of the results in the polynomial rings have been done either with the case $\delta = 0$ and α an automorphism or the case where α is the identity. We show that the notion of skew π-Armendariz rings generalizes that of skew Armendariz rings introduced by Nasr-Isfahani and Moussavi [10], in the special case where R is also assumed to be (α, δ)-*compatible*. We study skew π-Armendariz property of certain subrings of matrix rings. Consequently, some known results are obtained as special cases and new families of skew π-Armendariz rings are presented. Finally, we show that all (α, δ)-compatible NI rings are skew π-Armendariz, and if R is an (α, δ)-compatible 2-*primal* ring, then the polynomial ring $R[x]$ is skew π-Armendariz. The skew π-Armendariz conditions are strongly connected to the question of Amitsur of whether or not a polynomial ring over a nil ring is nil [1]. Since this problem was solved in the negative by Agata Smoktunowicz in [13], the affirmative results in this direction will be very useful.

2. **Skew π-Armendariz rings**

Our focus in this section is to introduce the concept of a skew π-Armendariz ring and study its properties. We prove that the notion of skew π-Armendariz rings generalizes that of skew Armendariz rings defined by Nasr-Isfahani and Moussavi [10] in case R is (α, δ)-*compatible*.

Definition 2.1. *Let R be a ring with an endomorphism α and an α-derivation δ. We say that R is a skew π-Armendariz ring, if for polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x; \alpha, \delta]$, $f(x)g(x) \in nil(R[x; \alpha, \delta])$ implies $a_i b_j \in nil(R)$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$*

It is clear that every π-Armendariz ring is skew π-Armendariz, where $\alpha = id_R$ and δ is the zero mapping, every α-Armendariz ring introduced by Hong et al, [6] is skew π-Armendariz with the case $\delta = 0$, and every subring of a skew π-Armendariz ring is skew π-Armendariz.

In the Ore extension $R[x; \alpha, \delta]$, we have

$$x^n a = \sum_{i=0}^{n} f_i^n(a) x^i, (n \geq 0)$$

where $f_i^n \in End(R, +)$ denotes the map which is the sum of all possible words in α, δ built with i letters α and $n - i$ letters δ. (In particular, $f_0^0 = 1, f_j^j = \alpha^j, f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$).

The following two Lemmas appear in [5] and [11], respectively.

Lemma 2.2. *Let R be an (α, δ)-compatible ring. Then we have the following:*

(1) If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$ for all positive integers n.

(2) If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.

(3) If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0 = \delta^m(a)\alpha^n(b)$ for all positive integers m, n.

Lemma 2.3. *Let R be an (α, δ)-compatible ring. Then we have the following:*

(1) If $abc = 0$, then $abf_i^j(c) = 0$ and $af_i^j(b)c = 0$ for all $0 \leq i \leq j$ and a, b, $c \in R$.

(2) If $ab \in nil(R)$, then $af_i^j(b) \in nil(R)$ for all $j \geq i \geq 0$ and $a, b \in R$.

(3) If $a\alpha^m(b) \in nil(R)$ for $a, b \in R$, and m is a positive integer, then $ab \in nil(R)$.

For an endomorphism α and an α-derivation δ of R, according to Nasr-Isfahani, et al. [10], a ring R is said to be skew Armendariz if for polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x; \alpha, \delta]$, $f(x)g(x) = 0$ implies $a_0 b_j = 0$ for each $0 \leq j \leq n$.

The following result shows that our definition of a skew π-Armendariz ring represents a generalization of a skew Armendariz ring when the ring R is (α, δ)-*compatible*.

Theorem 2.4. *Let R be a ring with an endomorphism α and an α-derivation δ. If R is both (α, δ)-compatible and skew Armendariz , then R is skew π-Armendariz.*

Proof. First we claim that if $f_1(x)$, $f_2(x), \ldots, f_n(x) \in R[x; \alpha, \delta]$ are such that

$$f_1(x)f_2(x)\cdots f_n(x) = 0,$$

then if $a_k \in C_{f_k}$ for $k = 1, 2, \ldots, n$, we have $a_1 a_2 \cdots a_n = 0$.

We proceed by induction on n.

Let $n = 2$. Suppose that $f_1(x) = \sum_{i=0}^{m} a_i x^i$, $f_2(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha, \delta]$ are such that

$$f_1(x)f_2(x) = (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) = 0.$$

We show that $a_i b_j = 0$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Since R is skew Armendariz, $a_0 b_j = 0$ for each $0 \leq j \leq n$, and so $a_0 f_s^t(b_j) = 0$ for every $0 \le j \le n$ and $0 \le s \le t$. Hence we obtain

$$\begin{aligned} 0 &= (a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_1x + a_2x^2 + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_1 + a_2x + \cdots + a_mx^{m-1})x(b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_1 + a_2x + \cdots + a_mx^{m-1})(\delta(b_0) + (\alpha(b_0) + \delta(b_1))x \\ &\quad + \cdots + (\alpha(b_{n-1}) + \delta(b_n))x^n + \alpha(b_n)x^{n+1}). \end{aligned}$$

By using the condition that R is skew Armendariz again, we obtain $a_1\alpha(b_n) = 0$, $a_1(\alpha(b_{k-1})+\delta(b_k)) = 0$ for each $1 \le k \le n$, and $a_1\delta(b_0) = 0$. From $a_1\alpha(b_n) = 0$, we obtain $a_1b_n = 0$, and so $a_1\delta(b_n) = 0$. Then from $a_1(\alpha(b_{n-1}) + \delta(b_n)) = 0$ and $a_1\delta(b_n) = 0$, we obtain $a_1b_{n-1} = 0$. Continuing this process yields that $a_1b_j = 0$ for every $0 \le j \le n$.

Now suppose $i > 2$ and

$$\begin{aligned} 0 &= (a_ix^i + a_{i+1}x^{i+1} + \cdots a_mx^m)(b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_i + a_{i+1}x + \cdots + a_mx^{m-i})x^i(b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_i + a_{i+1}x + \cdots + a_mx^{m-i})(f_0^i(b_0) + (\sum_{s+t=1} f_s^i(b_t))x \\ &\quad + \cdots + (\sum_{s+t=n+i} f_s^i(b_t))x^{n+i}) \end{aligned}$$

where $0 \le s \le i$ and $0 \le t \le n$. By the condition that R is skew Armendariz, we obtain the following equations:

$$a_i(\sum_{s+t=k} f_s^i(b_t)) = 0, \quad k = 0, 1, \ldots, n+i.$$

If $s + t = n + i$, then $s = i$ and $t = n$. Thus

$$a_i(\sum_{s+t=n+i} f_s^i(b_t)) = a_if_i^i(b_n) = a_i\alpha^i(b_n) = 0.$$

Since R is (α, δ)-compatible, we obtain $a_ib_n = 0$ and so $a_if_s^t(b_n) = 0$ for each $0 \le s \le t$.

If $s + t = n + i - 1$, then

$$0 = a_i(\sum_{s+t=n+i-1} f_s^i(b_t)) = a_if_{i-1}^i(b_n) + a_i\alpha^i(b_{n-1}) = a_i\alpha^i(b_{n-1}).$$

Hence we get $a_ib_{n-1} = 0$.

Now suppose that p is a positive integer such that for all $j < p$, $a_ib_{n-j} = 0$, we show that $a_ib_{n-p} = 0$.

Let $s+t = n+i-p$. Without loss of generality, we may assume $p \ge i$. We have

$$\begin{aligned} 0 = &\ a_i(\sum_{s+t=n+i-p} f_s^i(b_t) \\ = &\ a_i(f_i^i(b_{n-p}) + f_{i-1}^i(b_{n-(p-1)}) + \cdots + f_0^i(b_{n-(p-i)})) \\ = &\ a_i\alpha^i(b_{n-p}) + a_i f_{i-1}^i(b_{n-(p-1)}) + \cdots + a_i f_0^i(b_{n-(p-i)}). \end{aligned}$$

By the inductive hypothesis, we have $a_i b_{n-j} = 0$ for each $0 \leq j < p$. Then $a_i f_s^t(b_{n-j}) = 0$ for each $0 \leq j < p$ and $0 \leq s \leq t$. Hence $a_i\alpha^i(b_{n-p}) = 0$, and so $a_i b_{n-p} = 0$. Therefore by induction, we have $a_i b_j = 0$ for each $0 \leq j \leq n$.

Then by using induction on i, we obtain $a_i b_j = 0$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$.

Now suppose $n > 2$. Consider $h(x) = f_2(x)f_3(x)\cdots f_n(x)$. Then $f_1(x)h(x) = 0$, and hence, since R is skew Armendariz, $a_1 a_h = 0$ where $a_1 \in C_{f_1}$ and $a_h \in C_h$. Therefore for all $a_1 \in C_{f_1}$, $a_1 f_2(x)f_3(x)\cdots f_n(x) = 0$, and by induction, since the coefficients of $a_1 f_2(x)$ are of the form $a_1 a_2$ where a_2 is a coefficient of $f_2(x)$, we obtain $a_1 a_2 \cdots a_n = 0$ for $a_k \in C_{f_k}$, $k = 1, 2, \ldots, n$.

Next we show that R is a skew π-Armendariz ring. Let $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x;\alpha,\delta]$ are such that $f(x)g(x) \in nil(R[x;\alpha,\delta])$. Then there exists some positive integer k such that $(f(x)g(x))^k = 0$. Then we obtain $a_i b_j \in nil(R)$ for every $0 \leq i \leq m$ and $0 \leq j \leq n$. Therefore R is skew π-Armendariz. $\square$

The converse of Theorem 2.4 need not hold by the following:

Example 2.5. *Let R be an Armendariz ring and let*

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

Clearly R is π-Armendariz by [7, Lemma 1.1]*, and so if we regard the polynomial ring $R[x]$ as an Ore extension $R[x; id_R, 0]$, then R is skew π-Armendariz. Now we consider the Ore extension $R_4[x; id_{R_4}, 0]$, then R_4 is skew π-Armendariz by Theorem 3.2 below. But R_4 is not skew Armendariz by* [10, Corollary 2.3] *because R_4 is not Armendariz by a simple computations.*

Let α be an endomorphism and δ an α-derivation of R. Following Moussavi and Hashemi [9], R is said to be an (α,δ)-skew Armendariz ring, if for polynomials $f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j$ in

$R[x;\alpha,\delta]$, $f(x)g(x) = 0$ implies $a_ix^ib_jx^j = 0$ for each i, j. Note that each (α,δ)-skew Armendariz ring is skew Armendariz. Thus we have the following result:

Theorem 2.6. *Let R be a ring with an endomorphism α and an α-derivation δ. If R is both (α,δ)-compatible and (α,δ)-skew Armendariz, then R is skew π-Armendariz.*

Proof. Note that every (α,δ)-skew Armendariz ring is skew Armendariz. Thus the result follows from Theorem 2.4. □

3. Extensions of Skew π-Armendariz rings

In this section we study the properties of skew π-Armendariz rings and extend the class of skew π-Armendariz rings through various ring extensions.

Let R_1, R_2, …, R_n be rings, and let α_i be an endomorphism of R_i and δ_i an α_i-derivation of R_i, $i = 1,2,\ldots,n$. Let $R = \prod_{i=1}^n R_i = \{(x_1,x_2,\cdots,x_n) \mid x_i \in R_i, i = 1,2,\ldots,n\}$. For any $(x_1,x_2,\cdots,x_n) \in R$, we define $\overline{\alpha}((x_1,x_2,\cdots,x_n)) = (\alpha_1(x_1),\alpha_2(x_2),\cdots,\alpha_n(x_n))$ and $\overline{\delta}((x_1,x_2,\cdots,x_n)) = (\delta_1(x_1),\delta_2(x_2),\cdots,\delta_n(x_n))$. Then it is easy to see that $\overline{\delta}$ is an $\overline{\alpha}$-derivation of R.

Theorem 3.1. *The ring $R = \prod_{i=1}^n R_i$ is skew π-Armendariz if and only if R_i is skew π-Armendariz for every $0 \le i \le n$.*

Proof. The proof is essentially the same as in [7, Lemma 2.1].

Since every subring of a skew π-Armendariz ring is skew π-Armendariz, it is sufficient to show that R is skew π-Armendariz when R_i is skew π-Armendariz for all $1 \le i \le n$. Consider $f(x) = \sum_{i=0}^m a_ix^i$, $g(x) = \sum_{j=0}^n b_jx^j$ in $R[x;\overline{\alpha},\overline{\delta}]$ such that $f(x)g(x) \in nil(R[x;\overline{\alpha},\overline{\delta}])$, where $a_i = (a_{i1},a_{i2},\cdots,a_{in})$, $b_j = (b_{j1},b_{j2},\cdots,b_{jn})$ in R. For every $k = 1,2,\ldots,n$, we put $f_k(x) = \sum_{i=0}^m a_{ik}x^i$, $g_k(x) = \sum_{j=0}^n b_{jk}x^j$ in $R_k[x;\alpha_k,\delta_k]$. Then $f_k(x)g_k(x) \in nil(R_k[x;\alpha_k,\delta_k])$. So by skew π-Armendariz property of R_k, $a_{ik}b_{jk} \in nil(R_k)$ for every i, j. Thus for every i, j, there exists positive integer m_{ijk} such that $(a_{ik}b_{jk})^{m_{ijk}} = 0$. Take $m_{ij} = max\{m_{ijk} \mid k = 1,2,\ldots,n\}$. Then $(a_ib_j)^{m_{ij}} = ((a_{ik}b_{jk})^{m_{ij}}) = 0$. Thus $a_ib_j \in nil(R)$ for each i, j. Therefore R is skew π-Armendariz. □

Let R be a ring and let $T_n(R)$ be the n by n upper triangular matrix ring over R. For an endomorphism α and an α-derivation δ of R, the natural extension $\overline{\alpha} : T_n(R) \longrightarrow T_n(R)$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ is an endomorphism of $T_n(R)$ and $\overline{\delta} : T_n(R) \longrightarrow T_n(R)$ defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$ is an $\overline{\alpha}$-derivation of $T_n(R)$.

Theorem 3.2. *Let α be an endomorphism and δ an α-derivation of R. Then the following conditions are equivalent:*

(1) R is skew π-Armendariz.

(2) For any n, $T_n(R)$ is skew π-Armendariz.

Proof. (1) $\Rightarrow$ (2) Note that $T_n(R)[x;\overline{\alpha},\overline{\delta}] \cong T_n(R[x;\alpha,\delta])$. Let $f(x) = \sum_{i=0}^{p} A_i x^i$ and $g(x) = \sum_{j=0}^{q} B_j x^j$ be in $T_n(R)[x;\overline{\alpha},\overline{\delta}]$ such that $f(x)g(x) \in nil(T_n(R)[x;\overline{\alpha},\overline{\delta}])$. Let

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b_{22}^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^j \end{pmatrix}$$

for $0 \le i \le p$, and $0 \le j \le q$. Then $f_s(x) = \sum_{i=0}^{p} a_{ss}^i x^i$ and $g_s(x) = \sum_{j=0}^{q} b_{ss}^j x^j \in R[x;\alpha,\delta]$ and $f_s(x)g_s(x) \in nil(R[x;\alpha,\delta])$ for every $1 \le s \le n$. Since R is a skew π-Armendariz ring, there exists some positive integer m_{ijs} such that $(a_{ss}^i b_{ss}^j)^{m_{ijs}} = 0$ for any s and any i, j. Let $m_{ij} = max\{m_{ijs} \mid 1 \le s \le n\}$. Then $((A_i B_j)^{m_{ij}})^n = 0$. Therefore, $T_n(R)$ is skew π-Armendariz.

(2) $\Rightarrow$ (1) It is clear that every subring of a skew π-Armendariz ring is also skew π-Armendariz. □

Let R be a ring and let

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

with $n \ge 2$; and let

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\}$$

with $n \geq 2$, and let $T(R,R)$ be the trivial extension of R by R. Any endomorphism α of R can be extended to an endomorphism $\overline{\alpha}$ of $S_n(R)$ (or $T(R,n)$, or $T(R,R)$) defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$, and any α-derivation δ can be extended to an $\overline{\alpha}$-derivation $\overline{\delta}$ of $S_n(R)$ (or $T(R,n)$, or $T(R,R)$) defined by $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$.

Using the same method in the proof of Theorem 3.2, we obtain the following results.

Theorem 3.3. *Let α be an endomorphism and δ an α-derivation of R. Then the following conditions are equivalent:*

(1) R is a skew π-Armendariz ring.
(2) $S_n(R)$ is a skew π-Armendariz ring.
(3) $T(R,n)$ is a skew π-Armendariz ring.
(4) $T(R,R)$ is a skew π-Armendariz ring.

Corollary 3.4. *Let α be an endomorphism and δ an α-derivation of R. Then the following conditions are equivalent:*

(1) R is a skew π-Armendariz ring.
(2) $R[x]/(x^n)$ is a skew π-Armendariz ring with $n \geq 2$.

Proof. Observe that $R[x]/(x^n) \cong T(R,n)$ for every positive integer $n \geq 2$. Thus the result follows from Theorem 3.3. □

Theorem 3.5. *All (α,δ)-compatible NI rings are skew π-Armendariz.*

Proof. First we claim that $nil(R[x;\alpha,\delta]) \subseteq nil(R)[x;\alpha,\delta]$ when R is an (α,δ)-compatible NI ring. Suppose $p(x) = p_0 + p_1x + \cdots + p_nx^n \in nil(R[x;\alpha,\delta])$. There exists some positive integer k such that $p(x)^k = (p_0 + p_1x + \cdots + p_nx^n)^k = 0$. Then

$$0 = p(x)^k = \text{“}lower\ terms\text{”} + p_n\alpha^n(p_n)\alpha^{2n}(p_n)\cdots\alpha^{(k-1)n}(p_n)x^{nk}.$$

Hence $p_n\alpha^n(p_n)\alpha^{2n}(p_n)\cdots\alpha^{(k-1)n}(p_n) = 0$, and α-compatibility of R gives $p_n \in nil(R)$. So by Lemma 2.3, $p_n = 1 \cdot p_n \in nil(R)$ implies $1 \cdot f_i^j(p_n) = f_i^j(p_n) \in nil(R)$ for each $0 \leq i \leq j$. Let $Q = p_0 + a_1x + \cdots + p_{n-1}x^{n-1}$. Then we have

$$\begin{aligned} 0 &= (Q + p_nx^n)^k \\ &= (Q + p_nx^n)(Q + p_nx^n)\cdots(Q + p_nx^n) \\ &= (Q^2 + Q \cdot p_nx^n + p_nx^n \cdot Q + p_nx^n \cdot p_nx^n)(Q + p_nx^n)\cdots(Q + p_nx^n) \\ &= \cdots = Q^k + \Delta, \end{aligned}$$

where $\Delta \in R[x;\alpha,\delta]$. Now we consider Δ. Note that the coefficients of Δ can be written as sums of monomials in p_i and $f_u^v(p_j)$ where p_i,

$p_j \in \{p_0, p_1, \cdots, p_n\}$ and $v \geq u \geq 0$ are positive integers, and every monomial has p_n or $f_s^t(p_n)$. Since $nil(R)$ of an NI ring R is an ideal, we obtain that each monomial is in $nil(R)$, and so $\Delta \in nil(R)[x;\alpha,\delta]$. Thus we obtain

$$\begin{aligned} & (p_0 + p_1 x + \cdots + p_{n-1}x^{n-1})^k \\ = \quad & \text{“}lower\ terms\text{”} + p_{n-1}\alpha^{n-1}(p_{n-1})\cdots\alpha^{(n-1)(k-1)}(p_{n-1})x^{(n-1)k} \end{aligned}$$

$\in nil(R)[x;\alpha,\delta]$. Hence

$$p_{n-1}\alpha^{n-1}(p_{n-1})\cdots\alpha^{(n-1)(k-1)}(p_{n-1}) \in nil(R),$$

and so $p_{n-1} \in nil(R)$ by Lemma 2.3. Using induction on n we obtain $p_i \in nil(R)$ for each $0 \leq i \leq n$.

Next we show that R is skew π-Armendariz. Assume that $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in R[x;\alpha,\delta]$ are such that $f(x)g(x) \in nil(R[x;\alpha,\delta])$. Then

$$\begin{aligned} f(x)g(x) &= \sum_{k=0}^{m+n}\left(\sum_{s+t=k}(\sum_{i=s}^m a_i f_s^i(b_t))\right)x^k \\ &= \sum_{k=0}^{m+n}\Delta_k x^k \in nil(R[x;\alpha,\delta]) \subseteq nil(R)[x;\alpha,\delta], \end{aligned}$$

where $0 \leq s \leq m$ and $0 \leq t \leq n$. Then we have the following equations:

$$\Delta_k = \sum_{s+t=k}(\sum_{i=s}^m a_i f_s^i(b_t)), \quad k = 1, 2, \ldots, m+n,$$

and $\Delta_k \in nil(R)$ for every $0 \leq k \leq m+n$. Using the same method as in the proof of [11, Proposition 3.8], we can show that $a_i b_j \in nil(R)$ for every $0 \leq i \leq m$ and $0 \leq j \leq n$. Therefore R is skew π-Armendariz. $\square$

Corollary 3.6. *Let α be an endomorphism and δ an α-derivation of R. Then we have the following:*

(1) All (α,δ)-compatible 2-primal rings are skew π-Armendariz.
(2) All α-rigid rings are skew π-Armendariz.

Let α an endomorphism and δ an α-derivation of R. Then the map $\overline{\alpha} : R[x] \longrightarrow R[x]$ defined by $\sum_{i=0}^m a_i x^i \longrightarrow \sum_{i=0}^m \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α, and the α-derivation δ of R is also extended to $\overline{\delta} : R[x] \longrightarrow R[x]$ defined by $\overline{\delta}(\sum_{i=0}^m a_i x^i) = \sum_{i=0}^m \delta(a_i)x^i$. We can easily see that $\overline{\delta}$ is an $\overline{\alpha}$-derivation of $R[x]$.

Anderson and Camillo [2, Theorem 2] have shown that a ring R is Armendariz if and only if $R[x]$ is Armendariz. For skew π-Armendariz ring, we have the following results.

Theorem 3.7. *If R is an (α, δ)-compatible 2-primal ring, then $R[x]$ is a skew π-Armendariz ring.*

The key in proving Theorem 3.7 is to show the following three crucial Lemmas.

Lemma 3.8. *If R is a 2-primal ring and $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then $f(x) \in nil(R[x])$ if and only if $a_i \in nil(R)$ for every $0 \leq i \leq n$, that is, we have $nil(R[x]) = nil(R)[x]$ when R is a 2-primal ring.*

Proof. Suppose $f(x) = \sum_{i=0}^{n} a_i x^i \in nil(R[x])$. Then by [7, Proposition 1.3], we obtain $a_i \in nil(R)$ for every $0 \leq i \leq n$. Now assume that $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ with $a_i \in nil(R)$ for every $0 \leq i \leq n$. Consider the finite subset $\{a_0, a_1, \cdots, a_n\} \subseteq nil(R)$. Since R is a 2-*primal* ring, there exists a positive integer k such that any product of k elements $a_{i1} a_{i2} \cdots a_{ik}$ from $\{a_0, a_1, \cdots, a_n\}$ is zero. Hence we obtain $(f(x))^{k+1} = 0$, and so $f(x) \in nil(R[x])$. $\square$

R. Antoine [3, Corollary 5.3] proved that if R is an Armendariz ring then $nil(R[x]) = nil(R)[x]$. Now we can deduce from Lemma 3.8 that if R is a 2-*primal* ring, then $R[x]$ is an NI ring and $nil(R[x]) = nil(R)[x]$.

We do not know whether $R[x]$ is $(\overline{\alpha}, \overline{\delta})$- compatible when R is (α, δ)-compatible. Hence in order to prove Theorem 3.7, we need the following two Lemmas.

Lemma 3.9. *Let R be an (α, δ)-compatible 2-primal ring, and $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Then we have the following:*

(1) If $f(x)g(x) \in nil(R[x])$, then $f(x)\overline{\alpha}^n(g(x)) \in nil(R[x])$, and $\overline{\alpha}^m(f(x))g(x) \in nil(R[x])$ for positive integers m and n.

(2) $f(x)g(x) \in nil(R[x]) \Rightarrow f(x)\overline{\delta}^s(g(x)) \in nil(R[x])$ for every positive integer s.

(3) If $\overline{\alpha}^k(f(x))g(x) \in nil(R[x])$ for some positive integers k, then $f(x)g(x) \in nil(R[x])$.

(4) If $f(x)\overline{\alpha}^k(g(x)) \in nil(R[x])$ for some positive integers k, then $f(x)g(x) \in nil(R[x])$.

(5) If $f(x)g(x) \in nil(R[x])$, then $f(x)f_s^t(g(x)) \in nil(R[x])$ for every $0 \leq s \leq t$, where $f_s^t \in End(R[x], +)$ denotes the map which is the sum of all possible words in $\overline{\alpha}$, $\overline{\delta}$ built with s letters $\overline{\alpha}$ and $t-s$ letters $\overline{\delta}$.

Proof. (1) Note that $f(x)g(x) \in nil(R[x]) \Leftrightarrow a_ib_j \in nil(R)$ for all i, j $\Leftrightarrow a_i\alpha^n(b_j) \in nil(R)$, $\alpha^m(a_i)b_j \in nil(R)$ for all i, $j \Leftrightarrow f(x)\overline{\alpha}^n(g(x)) \in nil(R[x])$, $\overline{\alpha}^m(f(x))g(x) \in nil(R[x])$.

Similarly, we can prove (2), (3), (4) and (5). □

Lemma 3.10. *Let R be an (α, δ)-compatible 2-primal ring, and $p(y) = a_0 + a_1y + \cdots + a_ny^n \in nil(R[x][y; \overline{\alpha}, \overline{\delta}])$ where $a_i \in R[x]$ for every $0 \leq i \leq n$. Then $a_i \in nil(R[x])$ for each $0 \leq i \leq n$.*

Proof. Suppose $p(y) = a_0 + a_1y + \cdots + a_ny^n \in nil(R[x][y; \overline{\alpha}, \overline{\delta}])$ where $a_i \in R[x]$ for every $0 \leq i \leq n$. There exists some positive integer k such that $p(y)^k = (a_0 + a_1y + \cdots + a_ny^n)^k = 0$. Then

$$0 = p(y)^k = \text{“}\textit{lower terms}\text{”} + a_n\overline{\alpha}^n(a_n)\overline{\alpha}^{2n}(a_n)\cdots\overline{\alpha}^{(k-1)n}(a_n)y^{nk}.$$

Hence $a_n\overline{\alpha}^n(a_n)\overline{\alpha}^{2n}(a_n)\cdots\overline{\alpha}^{(k-1)n}(a_n) = 0 \in nil(R[x])$. By Lemma 3.9, we obtain

$$\begin{array}{rl} & a_n\overline{\alpha}^n(a_n)\overline{\alpha}^{2n}(a_n)\cdots\overline{\alpha}^{(k-1)n}(a_n) \in nil(R[x]) \\ \Rightarrow & a_n\overline{\alpha}^n(a_n)\overline{\alpha}^{2n}(a_n)\cdots\overline{\alpha}^{(k-2)n}(a_n)a_n \in nil(R[x]) \\ \Rightarrow & a_n\overline{\alpha}^n(a_n)\overline{\alpha}^{2n}(a_n)\cdots\overline{\alpha}^{(k-2)n}(a_na_n) \in nil(R[x]) \\ \Rightarrow & a_n\overline{\alpha}^n(a_n)\overline{\alpha}^{2n}(a_n)\cdots\overline{\alpha}^{(k-3)n}(a_n)a_na_n \in nil(R[x]) \\ \Rightarrow & \cdots \Rightarrow a_n \in nil(R[x]). \end{array}.$$

So by Lemma 3.9, $a_n = 1 \cdot a_n \in nil(R[x])$ implies $1 \cdot f_i^j(a_n) = f_i^j(a_n) \in nil(R[x])$ for each $0 \leq i \leq j$. Let $Q = a_0 + a_1y + \cdots + a_{n-1}y^{n-1}$. Then we have

$$\begin{array}{rcl} 0 & = & (Q + a_ny^n)^k \\ & = & (Q + a_ny^n)(Q + a_ny^n)\cdots(Q + a_ny^n) \\ & = & Q^k + \Delta, \end{array}$$

where $\Delta \in nil(R[x])[y; \overline{\alpha}, \overline{\delta}]$. Then by analogy with the proof of Theorem 3.5 and using Lemma 3.9, we obtain $a_i \in nil(R[x])$ for every $0 \leq i \leq n$. □

With these results in hand, the proof of Theorem 3.7 becomes greatly simplified.

Proof. Suppose that $p(y) = a_0 + a_1y + \cdots + a_my^m$ and $q(y) = b_0 + b_1y + \cdots + b_ny^n$ in $R[x][y; \overline{\alpha}, \overline{\delta}])$ are such that $p(y)q(y) \in nil(R[x][y; \overline{\alpha}, \overline{\delta}])$, where $a_i \in R[x]$ and $b_j \in R[x]$ for all $0 \leq i \leq m$, $0 \leq j \leq n$. Then

$$p(y)q(y) = \sum_{k=0}^{m+n}\left(\sum_{s+t=k}(\sum_{i=s}^{m} a_if_s^i(b_t))\right)y^k = \sum_{k=0}^{m+n}\Delta_ky^k \in nil(R[x][y; \overline{\alpha}, \overline{\delta}]).$$

Then we have the following equations:

$$\begin{aligned}
&\Delta_{m+n} = a_m\alpha^m(b_n),\\
&\Delta_{m+n-1} = a_m\alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_m f^m_{m-1}(b_n),\\
&\Delta_{m+n-2} = a_m\alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} a_i f^i_{m-1}(b_{n-1}) + \sum_{i=m-2}^{m} a_i f^i_{m-2}(b_n),\\
&\qquad \vdots\\
&\Delta_k = \sum_{s+t=k}\Big(\sum_{i=s}^{m} a_i f^i_s(b_t)\Big),
\end{aligned}$$

where $\Delta_{m+n} \in nil(R[x])$, $\Delta_{m+n-1} \in nil(R[x])$, $\Delta_{m+n-2} \in nil(R[x])$, $\ldots$, $\Delta_k \in nil(R[x])$ by Lemma 3.10. Then by analogy with the proof of [11, Proposition 3.6], and using Lemma 3.9, we obtain $a_ib_j \in nil(R[x])$ for every $0 \le i \le n$ and $0 \le j \le n$. Therefore $R[x]$ is a skew π-Armendariz ring. □

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