RIGHT GW-MAJORIZATION ON $M_{n,m}$

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Abstract. Let $M_{n,m}$ be the set of all $n \times m$ matrices with entries in $F$, where $F$ is the field of real or complex numbers. A real or complex $n \times n$ matrix is generalized row stochastic (g-row stochastic) if all of its row sums equal one. For $X, Y \in M_{n,m}$, we say that $Y$ is right gw-majorized by $X$ and write $X \succ_{rgw} Y$ if $Y=XR$ for some g-row stochastic matrix $R$. Here, we characterize all strong linear preservers of $\succ_{rgw}$ on $M_{n,m}$.

1. Introduction

A nonnegative real $n \times n$ matrix $R$ is said to be row stochastic, if $Re = e$, where $e = (1, ..., 1)^t \in \mathbb{R}^n$. The following generalization of stochastic matrices were introduced in [5]. A complex (not necessarily nonnegative) $n \times n$ matrix $R$ is said to be g-row (g-doubly) stochastic, if $Re=e$ ($Re = e$ and $R^t e = e$). The notion of matrix majorization was introduced by Dahl in [6]. According to that definition, a matrix $A$ is right matrix majorized (or right weak majorized) by $B$ if there exists an $n \times n$ row stochastic matrix $R$ such that $A=BR$, and is denoted by

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The concept of the left matrix majorization is defined similarly and denoted by $\succ_l$. The definitions of generalized majorizations (g-majorization) are motivated by the matrix majorization as follows:

**gs-majorization:** A matrix $A$ is said to be g-majorized by $B$ strongly if there exists an $n \times n$ g-doubly stochastic matrix $D$ such that $A=DB$, and is denoted by $B \succ_{gs} A$.

**gw-majorization:** A matrix $A$ is said to be right g-majorized by $B$ weakly if there exists an $n \times n$ g-row stochastic matrix $R$ such that $A=BR$, and is denoted by $B \succ_{rgw} A$. The concept of left g-majorization is defined similarly and is denoted by $\succ_{lgw}$.

For a relation $\sim$ on $M_{n,m}$, we say that $T$ preserves (or strongly preserves) $\sim$ if $T(x) \sim T(y)$, whenever $x \sim y$ (or $T(x) \sim T(y)$ if and only if $x \sim y$).

In [1,2], the authors introduced the relations $\succ_{gs}$ and $\succ_{lgw}$ on $M_{n,m}$. Also, they characterized all strong linear preservers of $\succ_{gs}$ and $\succ_{lgw}$ on $M_{n,m}$ and $M_n$ respectively. In [7], the authors characterized all linear operator that strongly preserve the right matrix majorization. We prove that one of the main theorems in [7] (Theorem 4.4) may be obtained as a corollary of our Theorem 2.8. We refer the readers to [4] and [6] for more information on the type of majorization and linear preserver of majorization.

The following notations will be fixed throughout this paper:

- $\mathbb{F}^n$: the set of column vectors $M_{n,1}$ with standard basis $\{e_1, ..., e_n\}$.
- $\mathbb{F}_m$: the set of row vectors $M_{1,m}$ with standard basis $\{\epsilon_1, ..., \epsilon_m\}$.
- $M_n$: the set of all $n \times n$ complex matrices.
- $GR_n$: the set of all $n \times n$ generalized row stochastic (g-row stochastic) matrices.
- $P_n$: the set of all $n \times n$ permutation matrices.
- $tr(x)$: the sum of all components of a vector $x$.
- $[x_1/x_2/.../x_n]$: an $n \times m$ matrix whose rows are $x_1, x_2, ..., x_n \in \mathbb{F}_m$.

## 2. Strong linear preserver

Here, we state the following statements to prove the main result of the paper.

**Lemma 2.1.** Let $T: \mathbb{F}_n \to \mathbb{F}_n$ be a linear operator. Then, $T$ preserves the subspace $\{x \in \mathbb{F}_n : tr(x) = 0\}$ if and only if there exists a matrix $B \in \text{span}(GR_n)$ such that $T(x) = xB$, for all $x \in \mathbb{F}_n$. 
Proof. Let $B = (b_{ij}) \in M_n$ be the matrix representation of $T$ with respect to the standard basis of $\mathbb{F}_n$. If $B \in \text{span}(\text{GR}_n)$, then it is easy to show that $T$ preserves the subspace $\{x \in \mathbb{F}_n : \text{tr}(x) = 0\}$. Conversely, let $T$ preserve the subspace $\{x \in \mathbb{F}_n : \text{tr}(x) = 0\}$. Then, $\text{tr}(T(\varepsilon_i - \varepsilon_j)) = 0$, for all $1 \leq i, j \leq n$, so that $\sum_{k=1}^{n} b_{ik} = \sum_{k=1}^{n} b_{jk}$. Therefore, $B \in \text{span}(\text{GR}_n)$. □

Lemma 2.2. Let $x$ be a nonzero vector in $\mathbb{F}_m$. Then, $x \succ_{rgw} y$, for some $y \in \mathbb{F}_m$ if and only if $\text{tr}(x) = \text{tr}(y)$.

Proof. If $x \succ_{rgw} y$, then it is clear that $\text{tr}(x) = \text{tr}(y)$. Conversely, let $x = (x_1, ..., x_m), \; y = (y_1, ..., y_m) \in \mathbb{F}_m$ and $\text{tr}(x) = \text{tr}(y)$. Without loss of generality, assume that $x_1 \neq 0$. Put,

$$R_y := \begin{pmatrix} r_1 & r_2 & \cdots & r_m \\ - & - & \cdots & - \\ 0 & - & \cdots & I_{m-1} \end{pmatrix},$$

where $r_1 = \frac{y_1}{x_1}$ and $r_i = \frac{y_i - x_i}{x_1}$, for every $2 \leq i \leq m$. It is clear that $y = xR_y$. Since $\text{tr}(x) = \text{tr}(y)$, then it is easy to show that $R_y \in \text{GR}_m$ and hence $x \succ_{rgw} y$. □

Theorem 2.3. Let $T : \mathbb{F}_m \rightarrow \mathbb{F}_m$ be a linear operator. Then, $T$ preserves $\succ_{rgw}$ if and only if one of the followings holds:

(a) $Tx = \alpha xB$, for some $\alpha \in \mathbb{F}$ and some invertible $B \in \text{GR}_n$.

(b) $Tx = \alpha xB$, for some $\alpha \in \mathbb{F}$ and some $B \in \text{GR}_n$ such that $\{x : xB = 0\} = \{x : \text{tr}(x) = 0\}$.

Proof. Let $T : \mathbb{F}_m \rightarrow \mathbb{F}_m$ preserve $\succ_{rgw}$. If $T = 0$, then put $\alpha = 0$. So, assume that $T \neq 0$. It is easy to show that $T$ preserves the subspace $\{x \in \mathbb{F}_m : \text{tr}(x) = 0\}$. Then, by Lemma 2.1 there exits a g-row stochastic matrix $B$ and a scalar $\alpha$ such that $Tx = \alpha xB$, for all $x \in \mathbb{F}_m$. If $B$ is invertible, then (a) holds. If $B$ is not invertible, then (b) holds by Lemma 2.2. The converse is trivial. □

Corollary 2.4. Let $T : \mathbb{F}_m \rightarrow \mathbb{F}_m$ be a nonzero linear preserver of $\succ_{rgw}$. Then, rank$(T)$ is equal to $m$ or 1.
**Proof.** By Theorem 2.3, \( \ker(T) = \{0\} \) or \( \ker(T) = \{x : \text{tr}(x) = 0\} \).

Then, \( \text{rank}(T) = m \) or \( \text{rank}(T) = 1 \). \( \square \)

**Corollary 2.5.** Let \( T : \mathbb{F}_m \to \mathbb{F}_m \) be a singular linear preserver of \( \succ_{rgw} \). Then, there exists a vector \( y \in \mathbb{F}_m \) such that \( Tx = \text{tr}(x)y \) for all \( x \in \mathbb{F}_m \).

**Proof.** Consider the basis \( \{e^t, \epsilon_1j : 2 \leq j \leq m\} \) for \( \mathbb{F}_m \), where \( \epsilon_{1j} = \epsilon_1 - \epsilon_j \). Then, for every \( x \in \mathbb{F}_m \), \( x = \frac{1}{m} \text{tr}(x)e^t + \sum_{j=2}^{m} \alpha_{1j}\epsilon_{1j} \), for some \( \alpha_{1j} \in \mathbb{F} \). Since \( \ker(T) = \{x : \text{tr}(x) = 0\} \) by Theorem 2.3, then \( Tx = \text{tr}(x)y \), where \( y = \frac{1}{m}T(e^t) \). \( \square \)

**Lemma 2.6.** Let \( A \in \mathbb{M}_m \) and \( \alpha \) be a nonzero scalar in \( \mathbb{F} \). Then, \( A = \gamma I \) for some \( \gamma \in \mathbb{F} \) if and only if we have,

\[
(2.2) \quad \alpha xA + y \succ_{rgw} \alpha xRA + yR, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbb{GR}_m.
\]

**Proof.** If \( A = \gamma I \), for some \( \gamma \in \mathbb{F} \), then it is clear that (2.2) holds. Conversely, let (2.2) hold. For every \( x \in \mathbb{F}_m \), put \( x \) and \( y \) in (2.2), where \( y = -\alpha xA \). Then, \( \alpha xRA - \alpha xAR = 0 \) for all \( x \) in \( \mathbb{F}_m \). So, \( RA = AR \), for all \( R \in \mathbb{GR}_m \), and hence \( A = \gamma I \), for some \( \gamma \in \mathbb{F} \).

**Remark 2.7.** Every strong linear preserver of \( \succ_{rgw} \) is invertible.

Now, we state the main result of this paper. The following theorem holds for \( n=1 \) obviously, and thus we assume that \( n \geq 2 \).

For every \( i, j \) \( (1 \leq i, j \leq n) \), consider the embedding \( E^j : \mathbb{F}_m \to \mathbb{M}_{n,m} \) and the projection \( E_i : \mathbb{M}_{n,m} \to \mathbb{F}_m \), where \( E^j(x) = e_jx \) and \( E_i(A) = \epsilon_iA \). It is easy to show that for every linear operator \( T : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m} \),

\[
T(X) = T[x_1/ \cdots / x_n] = [\sum_{j=1}^{n} T^j \epsilon_jx_j/ \cdots / \sum_{j=1}^{n} T^j \epsilon_jx_j], \text{ where } x_j \text{ is the } j \text{ th row of } X \text{ and } T^j_i = E_i \circ T \circ E^j.
\]
Theorem 2.8. Let $T : M_{p,m} \rightarrow M_{n,m}$ be a linear operator. Then, $T$ strongly preserves $\succ_{rgw}$ if and only if $T(X) = MXA$, for some invertible matrices $M \in M_n$ and $A \in GR_m$.

Proof. First assume that $T$ strongly preserves $\succ_{rgw}$. It is clear that for every $i,j \ (1 \leq i,j \leq n)$, $T^i_j : F_m \rightarrow F_m$ preserves $\succ_{rgw}$. Then, by Theorem 2.3, there exist $A^i_j \in GR_m$ and $\alpha^i_j \in F$ such that

$$T[x_1/\cdots/x_n] = \left[ \sum_{j=1}^n \alpha^i_j x_j A^i_j / \cdots / \sum_{j=1}^n \alpha^n_j x_j A^n_j \right].$$

Now, we consider some steps.

Step 1: Let there exist $p$ and $q$ (1 $\leq p,q \leq n$) such that $A^i_p \neq 0$. We want to replace $A^i_p$ by $A^q_p$ with suitable coefficient $\beta^i_p$. For every $x,y \in F_m$, let $X = [x_1/\cdots/x_n]$ where $x_p = x$, $x_j = y$ and $x_i = 0$, for $i \neq p,j$. Since $X \succ_{rgw} XR$, for every $R \in GR_n$, we have $T(X) \succ_{rgw} T(XR)$ and hence,

$$\alpha^i_p x A^i_p + \alpha^q_p y A^q_p \succ_{rgw} \alpha^i_p x A^q_p + \alpha^q_p y A^i_p \forall x,y \in F_m, \forall R \in GR_m.$$

Therefore, by Lemma 2.6, $A^i_p = \beta^i_p A^q_p$, for some $\beta^i_p \in F$.

Step 2: We show that there is no $p,q$ (1 $\leq p,q \leq n$) such that $A^i_p$ is singular and $\alpha^i_p \neq 0$. Assume, if possible, that there exist $p$ and $q$ such that $A^i_p$ is singular and $\alpha^i_p \neq 0$. So, by Lemma 2.6, $A^i_p$ is singular, for every $j \ (1 \leq j \leq n)$, for which $\alpha^i_j \neq 0$. Without loss of generality, assume that $p=1$. So, by Theorem 2.3, $\ker(A^i_1) = \{ x : tr(x) = 0 \}$, for every $j \ (1 \leq j \leq n)$, Therefore, by this and Step 1, for every $i \ (2 \leq i \leq n)$, either we can assume $A^i_1 = \cdots = A^n_1$ or we have $\ker(A^i_1) = \cdots = \ker(A^n_1) = \{ x : tr(x) = 0 \}$. Since the vectors $(\alpha^2_1, \cdots, \alpha^n_1)$ are linearly dependent in $F_{n-1}$, then there exist scalars (not all zero) $t_1, \cdots, t_n \in F$ such that

$$\sum_{j=1}^n t_j (\alpha^2_j, \cdots, \alpha^n_j) = 0.$$

For every $j \ (1 \leq j \leq n)$, put $x_j = (t_j, -t_j, 0, \cdots, 0) \in F_m$, and then $X = [x_1/\cdots/x_n] \neq 0$. It is easy to check that $TX = 0$, which contradicts the injectivity of $T$. Therefore, by Step 1,

$$T[x_1/\cdots/x_n] = \left[ \sum_{j=1}^n \beta^i_j x_j A_1 / \cdots / \sum_{j=1}^n \beta^n_j x_j A_n \right],$$
for some invertible matrices \( A_i \in \mathbf{GR}_m \) and some scalars \( \beta^i_j \in \mathbb{F} \) \((1 \leq i, j \leq n)\).

Step 3: Here, we show that \( A_i = A_1 \), for every \( i \) \((1 \leq i \leq n)\). By Step 2, \( T[x_1/\cdots/x_n] = [b_1X_1/\cdots/b_nX_n] \), where \( b_i = (\beta^1_i, \cdots, \beta^n_i) \), for every \( i \) \((1 \leq i \leq n)\). If \( \{b_1, \cdots, b_n\} \subseteq \text{span}\{b\} \), for some \( b \in \mathbb{F}_n \), then there exists a nonzero vector \( a \in \mathbb{F}_n \) such that \( b_1a^i = 0 \), because \( n \geq 2 \). Let \( X \in \mathbf{M}_{n,m} \) be such that the first column of \( X \) is \( a^i \) and the other columns of \( X \) are zero. Then, \( X \neq 0 \) and \( TX = 0 \), which is a contradiction. Therefore, at least two elements of \( \{b_1, \cdots, b_n\} \) are linearly independent. Without loss of generality, assume that \( \{b_1, b_2\} \) is linearly independent. For every \( X \in \mathbf{M}_{n,m} \) and every \( R \in \mathbf{GR}_m \), \( TX \succ_{rgw} T(RX) \). Then,

\[
(2.3) \quad b_1XA_1 + b_2XA_2 \succ_{rgw} b_1XRA_1 + b_2XRA_2 \quad \forall X \in \mathbf{M}_{n,m}, \forall R \in \mathbf{GR}_m.
\]

Since \( \{b_1, b_2\} \) is a linearly independent set in \( \mathbb{F}_n \), then for every \( x, y \in \mathbb{F}_m \) there exists \( X_{x,y} \in \mathbf{M}_{n,m} \) such that \( b_1X_{x,y} = x \) and \( b_2X_{x,y} = y \). Therefore, by (2.3) we have,

\[
xA_1 + yA_2 \succ_{rgw} xRA_1 + yRA_2, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m.
\]

So, by Lemma 2.6, \( A_2 = \alpha A_1 \), for some \( \alpha \in \mathbb{F} \). For every \( j \) \((2 \leq j \leq n)\), if \( b_j = 0 \), then we can replace \( A_j \) by \( A_1 \). If \( b_j \neq 0 \), then \( \{b_1, b_j\} \) or \( \{b_2, b_j\} \) is a linearly independent set. Similar to the above argument, \( A_j = \beta A_1 \) or \( A_j = \beta A_2 \), for some \( \beta \in \mathbb{F} \). Put \( A = A_1 \). Then, \( T(X) = [b_1XA_1b_2X(\alpha_2A)\cdots/b_nX(\alpha_nA)] \), for some \( \alpha_2, \cdots, \alpha_n \in \mathbb{F} \), and hence \( T(X) = MA_1 \) for all \( X \in \mathbf{M}_{n,m} \), where \( M = [b_1/\alpha_2b_2/\cdots/\alpha_nb_n] \). The other side of the theorem is easy to establish.

The following lemma was proved in [3].

**Lemma 2.9.** Let \( R \) be a nonsingular row stochastic matrix. If \( R^{-1} \) is nonnegative, then \( R \) is a permutation matrix.

**Proof.** Let \( R = (r_{ij}) \). Suppose that \( R \) has two nonzero entries in some row. Without loss of generality, we may assume that the first row of \( R \) has at least two nonzero entries. Since \( R^{-1} \) is invertible, then the first column of \( R^{-1} \) has a nonzero entry, say in row \( i \), and then the \( ith \) row
of $R^{-1}R$ must have at least two nonzero entries. This is a contradiction, since $R^{-1}R = I$. Thus, every row of $R$ has exactly one nonzero entry. Since $R$ is invertible, then therefore $R$ is a permutation matrix. □

**Corollary 2.10.** ([7], Theorem 4.4) A linear operator $T : \mathbf{M}_n \rightarrow \mathbf{M}_n$ strongly preserves right matrix majorization $\succ_r$ if and only if $T(X) = MXP$, where $P$ is a permutation and $M \in \mathbf{M}_n$ is invertible.

**Proof.** Assume that $T$ strongly preserves right matrix majorization. We show that $T$ is a strong linear preserver of the right gw-majorization. Let $A \succ_{rgw} B$. Then, there exists a g-row stochastic matrix $R$ such that $B = AR$. For the g-row stochastic matrix $R$, there exist scalars $r_1, \ldots, r_k$ and row stochastic matrices $R_1, \ldots, R_k$ such that $\sum_{i=1}^{k} r_i = 1$ and $R = \sum_{i=1}^{k} r_i R_i$.

For every $i$ $(1 \leq i \leq k)$, $A \succ_r R_i A$, and hence $T(A) \succ_r T(R_i A)$. Thus, there exist row stochastic matrices $S_i$ $(1 \leq i \leq k)$, such that $T(R_i A) = T(A)S_i$. Put $S = \sum_{i=1}^{k} r_i S_i$. It is clear that $S$ is a g-row stochastic matrix and $T(B) = T(A)S$. Therefore, $T(A) \succ_{rgw} T(B)$. For the other side, replace $T$ by $T^{-1}$ and similarly conclude that $A \succ_{rgw} B$ whenever $T(A) \succ_{rgw} T(B)$. Then, $T$ strongly preserves the right gw-majorization. Therefore, by Theorem 2.8, there exist invertible matrices $A \in \mathbf{GR}_n$ and $M \in \mathbf{M}_n$ such that $T(X) = M X A$ for all $X \in \mathbf{M}_n$. For every row stochastic matrix $R$, it is clear that $I \succ_r R$. So, $T(I) \succ_r T(R)$, for every row stochastic matrix $R$. Then, $MIA \succ_r MRA$, and hence $A^{-1}RA$ is a row stochastic matrix, for every row stochastic matrix $R$. So, it is easy to show that $A^{-1}$ is a row stochastic matrix. Similarly, $A$ is a row stochastic matrix too, and hence $A$ is a permutation matrix by Lemma 2.9. To complete the proof, put $P = A$. □

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