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RIGHT GW-MAJORIZATION ON $\mathbf{M}_{n,m}$

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ABSTRACT. Let $\mathbf{M}_{n,m}$ be the set of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. A real or complex $n\times n$ matrix is *generalized row stochastic (g-row stochastic)* if all of its row sums equal one. For $X, Y \in \mathbf{M}_{n,m}$, we say that Y is right gw-majorized by X and write $X \succ_{rgw} Y$ if Y=XR for some g-row stochastic matrix R. Here, we characterize all strong linear preservers of $\succ_{rgw}$ on $\mathbf{M}_{n,m}$.

1. Introduction

A nonnegative real $n \times n$ matrix R is said to be *row stochastic*, if $Re = e$, where $e = (1, ..., 1)^t \in \mathbb{F}^n$. The following generalization of stochastic matrices were introduced in [5]. A complex (not necessarily nonnegative) $n \times n$ matrix R is said to be *g-row (g-doubly) stochastic*, if Re=e ($Re = e$ and $R^te = e$). The notion of *matrix majorization* was introduced by Dahl in [6]. According to that definition, a matrix A is right matrix majorized (or right weak majorized) by B if there exists an $n \times n$ row stochastic matrix R such that A=BR, and is denoted by

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$B \succ_r A$. The concept of the *left matrix majorization* is defined similarly and denoted by $\succ_l$. The definitions of *generalized majorizations (g-majorization)* are motivated by the *matrix majorization* as follows:

gs-majorization: A matrix A is said to be g-majorized by B strongly if there exists an $n \times n$ g-doubly stochastic matrix D such that $A{=}DB$, and is denoted by $B \succ_{gs} A$.

gw-majorization: A matrix A is said to be right g-majorized by B weakly if there exists an $n \times n$ g-row stochastic matrix R such that $A{=}BR$, and is denoted by $B \succ_{rgw} A$. The concept of *left g-majorization* is defined similarly and is denoted by $\succ_{lgw}$.

For a relation $\sim$ on $\mathbf{M}_{n,m}$, we say that T preserves (or strongly preserves) $\sim$ if $T(x) \sim T(y)$, whenever $x \sim y$ (or $T(x) \sim T(y)$ if and only if $x \sim y$).

In [1,2], the authors introduced the relations $\succ_{gs}$ and $\succ_{lgw}$ on $\mathbf{M}_{n,m}$. Also, they characterized all strong linear preservers of $\succ_{gs}$ and $\succ_{lgw}$ on $\mathbf{M}_{n,m}$ and $\mathbf{M}_n$ respectively. In [7] , the authors characterized all linear operator that strongly preserve the *right matrix majorization*. We prove that one of the main theorems in [7] (Theorem 4.4) may be obtained as a corollary of our Theorem 2.8. We refer the readers to [4] and [6] for more information on the type of majorization and linear preserver of majorization.

The following notations will be fixed throughout this paper:

$\mathbb{F}^n$: the set of column vectors $\mathbf{M}_{n,1}$ with standard basis $\{e_1, ..., e_n\}$.

$\mathbb{F}_m$: the set of row vectors $\mathbf{M}_{1,m}$ with standard basis $\{\epsilon_1, ..., \epsilon_m\}$.

$\mathbf{M}_n$: the set of all $n \times n$ complex matrices.

$\mathbf{GR}_n$: the set of all $n \times n$ generalized row stochastic (g-row stochastic) matrices.

$\mathbf{P}_n$: the set of all $n \times n$ permutation matrices.

tr(x): the sum of all components of a vector x.

$[x_1/x_2/.../x_n]$: an $n \times m$ matrix whose rows are $x_1, x_2, ..., x_n \in \mathbb{F}_m$.

2. Strong linear preserver

Here, we state the following statements to prove the main result of the paper.

Lemma 2.1. *Let* $T : \mathbb{F}_n \to \mathbb{F}_n$ *be a linear operator. Then, T preserves the subspace* $\{x \in \mathbb{F}_n : tr(x) = 0\}$ *if and only if there exists a matrix* $B \in span(\mathbf{GR}_n)$ *such that* $T(x) = xB$, *for all* $x \in \mathbb{F}_n$.

Proof. Let $B = (b_{ij}) \in \mathbf{M}_n$ be the matrix representation of T with respect to the standard basis of $\mathbb{F}_n$. If $B \in span(\mathbf{GR}_n)$, then it is easy to show that T preserves the subspace $\{x \in \mathbb{F}_n : tr(x) = 0\}$. Conversely, let T preserve the subspace $\{x \in \mathbb{F}_n : tr(x) = 0\}$. Then, $tr(T(\epsilon_i - \epsilon_j)) = 0$, for all $1 \leq i, j \leq n$, so that $tr((\epsilon_i - \epsilon_j)B) = 0$, and thus $\sum_{k=1}^{n} b_{ik} = \sum_{k=1}^{n} b_{jk}$. Therefore, $B \in span(\mathbf{GR}_n)$. □

Lemma 2.2. *Let x be a nonzero vector in $\mathbb{F}_m$. Then, $x \succ_{rgw} y$, for some $y \in \mathbb{F}_m$ if and only if $tr(x) = tr(y)$.*

Proof. If $x \succ_{rgw} y$, then it is clear that $tr(x) = tr(y)$. Conversely, let $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m) \in \mathbb{F}_m$ and $tr(x) = tr(y)$. Without loss of generality, assume that $x_1 \neq 0$. Put,

$$R_y := \left(\begin{array}{c|c} r_1 & r_2 \ \cdots \ r_m \\ \hline 0 & I_{m-1} \end{array} \right),$$

where $r_1 = \frac{y_1}{x_1}$ and $r_i = \frac{y_i - x_i}{x_1}$, for every i $(2 \leq i \leq m)$. It is clear that $y = xR_y$. Since $tr(x) = tr(y)$, then it is easy to show that $R_y \in \mathbf{GR}_m$ and hence $x \succ_{rgw} y$. □

Theorem 2.3. *Let $T : \mathbb{F}_m \to \mathbb{F}_m$ be a linear operator. Then, T preserves $\succ_{rgw}$ if and only if one of the followings holds:*
(a) $Tx = \alpha x B$, for some $\alpha \in \mathbb{F}$ and some invertible $B \in \mathbf{G}R_n$.
(b) $Tx = \alpha x B$, for some $\alpha \in \mathbb{F}$ and some $B \in \mathbf{G}R_n$ such that $\{x : xB = 0\} = \{x : tr(x) = 0\}$.

Proof. Let $T : \mathbb{F}_m \to \mathbb{F}_m$ preserve $\succ_{rgw}$. If $T = 0$, then put $\alpha = 0$. So, assume that $T \neq 0$. It is easy to show that T preserves the subspace $\{x \in \mathbb{F}_m : tr(x) = 0\}$. Then, by Lemma 2.1 there exits a g-row stochastic matrix B and a scalar α such that $Tx = \alpha x B$, for all $x \in \mathbb{F}_m$. If B is invertible, then (a) holds. If B is not invertible, then (b) holds by Lemma 2.2. The converse is trivial . □

Corollary 2.4. *Let $T : \mathbb{F}_m \to \mathbb{F}_m$ be a nonzero linear preserver of $\succ_{rgw}$. Then, rank(T) is equal to m or 1.*

Proof. By Theorem 2.3, $ker(T) = \{0\}$ or $ker(T) = \{x : tr(x) = 0\}$. Then, $rank(T)=m$ or $rank(T)=1$. $\square$

Corollary 2.5. *Let $T : \mathbb{F}_m \to \mathbb{F}_m$ be a singular linear preserver of $\succ_{rgw}$. Then, there exists a vector $y \in \mathbb{F}_m$ such that $Tx = tr(x)y$ for all $x \in \mathbb{F}_m$.*

Proof. Consider the basis $\{e^t, \epsilon_{1j} : 2 \le j \le m\}$ for $\mathbb{F}_m$, where $\epsilon_{1j} = \epsilon_1 - \epsilon_j$. Then, for every $x \in \mathbb{F}_m$, $x = \frac{1}{m} tr(x)e^t + \sum_{j=2}^{m} \alpha_{1j}\epsilon_{1j}$, for some $\alpha_{1j} \in \mathbb{F}$. Since $ker(T) = \{x : tr(x) = 0\}$ by Theorem 2.3, then $Tx = tr(x)y$, where $y = \frac{1}{m}T(e^t)$. $\square$

Lemma 2.6. *Let $A \in \mathbf{M}_m$ and α be a nonzero scalar in $\mathbb{F}$. Then, $A = \gamma I$ for some $\gamma \in \mathbb{F}$ if and only if we have,*

$$\alpha x A + y \succ_{rgw} \alpha x RA + yR, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m. \tag{2.2}$$

Proof. If $A = \gamma I$, for some $\gamma \in \mathbb{F}$, then it is clear that (2.2) holds. Conversely, let (2.2) hold. For every $x \in \mathbb{F}_m$, put x and y in (2.2), where $y = -\alpha x A$. Then, $\alpha x RA - \alpha x AR = 0$ for all x in $\mathbb{F}_m$. So, $RA = AR$, for all $R \in \mathbf{GR}_m$, and hence $A = \gamma I$, for some $\gamma \in \mathbb{F}$. $\square$

Remark 2.7. Every strong linear preserver of $\succ_{rgw}$ is invertible.

Now, we state the main result of this paper. The following theorem holds for n=1 obviously, and thus we assume that $n \ge 2$.

For every i, j $(1 \le i, j \le n)$, consider the embedding $E^j : \mathbb{F}_m \to \mathbf{M}_{n,m}$ and the projection $E_i : \mathbf{M}_{n,m} \to \mathbb{F}_m$, where $E^j(x) = e_j x$ and $E_i(A) = \epsilon_i A$. It is easy to show that for every linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$, $T(X) = T[x_1/\cdots/x_n] = [\sum_{j=1}^{n} T_1^j x_j/\cdots/\sum_{j=1}^{n} T_n^j x_j]$, where x_j is the j *th* row of X and $T_i^j = E_i \circ T \circ E^j$.

Theorem 2.8. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then, T strongly preserves $\succ_{rgw}$ if and only if $T(X) = MXA$, for some invertible matrices $M \in \mathbf{M}_n$ and $A \in \mathbf{GR}_m$.*

Proof. First assume that T strongly preserves $\succ_{rgw}$. It is clear that for every i, j $(1 \le i, j \le n)$, $T_i^j : \mathbb{F}_m \to \mathbb{F}_m$ preserves $\succ_{rgw}$. Then, by Theorem 2.3, there exist $A_i^j \in \mathbf{GR}_m$ and $\alpha_i^j \in \mathbb{F}$ such that

$$T[x_1/\cdots/x_n] = [\sum_{j=1}^{n} \alpha_1^j x_j A_1^j/\cdots/\sum_{j=1}^{n} \alpha_n^j x_j A_n^j].$$

Now, we consider some steps.

Step 1: Let there exist p and q $(1 \le p, q \le n)$ such that A_p^q is invertible and $\alpha_p^q \neq 0$. We want to replace A_p^j by A_p^q with suitable coefficient β_p^j. For every $x, y \in \mathbb{F}_m$, let $X = [x_1/\cdots/x_n]$ where $x_p = x$, $x_j = y$ and $x_i = 0$, for $i \neq p, j$. Since $X \succ_{rgw} XR$, $\forall R \in \mathbf{GR}_n$, then $T(X) \succ_{rgw} T(XR)$ and hence,

$$\alpha_p^q x A_p^q + \alpha_p^j y A_p^j \succ_{rgw} \alpha_p^q x R A_p^q + \alpha_p^j y R A_p^j \ . \ \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m \ .$$

Therefore, by Lemma 2.6, $A_p^j = \beta_p^j A_p^q$, for some $\beta_p^j \in \mathbb{F}$.

Step 2: We show that there is no p, q $(1 \le p, q \le n)$ such that A_p^q is singular and $\alpha_p^q \neq 0$. Assume, if possible, that there exist p and q such that A_p^q is singular and $\alpha_p^q \neq 0$. So, by Lemma 2.6, A_p^j is singular, for every j $(1 \le j \le n)$, for which $\alpha_p^j \neq 0$. Without loss of generality, assume that *p=1*. So, by Theorem 2.3, $ker(A_1^j) = \{x : tr(x) = 0\}$, for every j $(1 \le j \le n)$, Therefore, by this and Step 1, for every i $(2 \le i \le n)$, either we can assume $A_i^1 = \cdots = A_i^n$ or we have $ker(A_i^1) = \cdots = ker(A_i^n) = \{x : tr(x) = 0\}$. Since the vectors $(\alpha_2^1, \cdots, \alpha_n^1), \cdots, (\alpha_2^n, \cdots, \alpha_n^n)$ are linearly dependent in $\mathbb{F}_{n-1}$, then there exist scalars (not all zero) $t_1, \cdots, t_n \in \mathbb{F}$ such that $\sum_{j=1}^{n} t_j(\alpha_2^j, \cdots, \alpha_n^j) = 0$. For every j $(1 \le j \le n)$. put $x_j = (t_j, -t_j, 0, \cdots, 0) \in \mathbb{F}_m$, and then $X = [x_1/\cdots/x_n] \neq 0$. It is easy to check that *TX=0*, which contradicts the injectivity of T. Therefore, by Step 1,

$$T[x_1/\cdots/x_n] = [\sum_{j=1}^{n} \beta_1^j x_j A_1/\cdots/\sum_{j=1}^{n} \beta_n^j x_j A_n],$$

for some invertible matrices $A_i \in \mathbf{GR}_m$ and some scalars $\beta_i^j \in \mathbb{F}$ $(1 \leq i, j \leq n)$.

Step 3: Here, we show that $A_i = A_1$, for every i $(1 \leq i \leq n)$. By Step 2, $T[x_1/\cdots/x_n] = [b_1XA_1/\cdots/b_nXA_n]$, where $b_i = (\beta_i^1, \cdots, \beta_i^n)$, for every i $(1 \leq i \leq n)$. If $\{b_1, \cdots, b_n\} \subseteq span\{b\}$, for some $b \in \mathbb{F}_n$, then there exists a nonzero vector $a \in \mathbb{F}_n$ such that $b_1a^t = 0$, because $n \geq 2$. Let $X \in \mathbf{M}_{n,m}$ be such that the first column of X is a^t and the other columns of X are zero. Then, $X \neq 0$ and $TX = 0$, which is a contradiction. Therefore, at least two elements of $\{b_1, \cdots, b_m\}$ are linearly independent. Without loss of generality, assume that $\{b_1, b_2\}$ is linearly independent. For every $X \in \mathbf{M}_{n,m}$ and every $R \in \mathbf{GR}_m$, $TX \succ_{rgw} T(RX)$. Then,

$$b_1XA_1 + b_2XA_2 \succ_{rgw} b_1XRA_1 + b_2XRA_2 \quad \forall X \in \mathbf{M}_{n,m} \quad , \quad \forall R \in \mathbf{GR}_m. \tag{2.3}$$

Since $\{b_1, b_2\}$ is a linearly independent set in $\mathbb{F}_n$, then for every $x, y \in \mathbb{F}_m$ there exists $X_{x,y} \in \mathbf{M}_{n,m}$ such that $b_1X_{x,y} = x$ and $b_2X_{x,y} = y$. Therefore, by (2.3) we have,

$$xA_1 + yA_2 \succ_{rgw} xRA_1 + yRA_2, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m.$$

So, by Lemma 2.6, $A_2 = \alpha A_1$, for some $\alpha \in \mathbb{F}$. For every j $(2 \leq j \leq n)$, if $b_j = 0$, then we can replace A_j by A_1. If $b_j \neq 0$, then $\{b_1, b_j\}$ or $\{b_2, b_j\}$ is a linearly independent set. Similar to the above argument, $A_j = \beta A_1$ or $A_j = \beta A_2$, for some $\beta \in \mathbb{F}$. Put $A = A_1$. Then, $T(X) = [b_1XA|b_2X(\alpha_2A)/\cdots/b_nX(\alpha_nA)]$, for some $\alpha_2, \cdots, \alpha_n \in \mathbb{F}$, and hence $T(X) = MXA$, for all $X \in \mathbf{M}_{n,m}$, where $M = [b_1/\alpha_2b_2/\cdots/\alpha_nb_n]$. The other side of the theorem is easy to establish. □

The following lemma was proved in [3].

Lemma 2.9. *Let R be a nonsingular row stochastic matrix. If R^{-1} is nonnegative, then R is a permutation matrix.*

Proof. Let $R = (r_{ij})$. Suppose that R has two nonzero entries in some row. Without loss of generality, we may assume that the first row of R has at least two nonzero entries. Since R^{-1} is invertible, then the first column of R^{-1} has a nonzero entry, say in row i, and then the *ith* row of $R^{-1}R$ must have at least two nonzero entries. This is a contradiction, since $R^{-1}R$=I. Thus, every row of R has exactly one nonzero entry. Since R is invertible, then therefore R is a permutation matrix . □

Corollary 2.10. ([7], Theorem 4.4) *A linear operator* $T : \mathbf{M}_n \to \mathbf{M}_n$ *strongly preserves right matrix majorization* $\succ_r$ *if and only if* $T(X) = MXP$*, where* P *is a permutation and* $M \in \boldsymbol{M}_n$ *is invertible.*

Proof. Assume that T strongly preserves right matrix majorization. We show that T is a strong linear preserver of the *right gw-majorization.* Let $A \succ_{rgw} B$. Then, there exists a g-row stochastic matrix R such that B=AR. For the g-row stochastic matrix R, there exist scalars $r_1, ..., r_k$ and row stochastic matrices $R_1, ..., R_k$ such that $\sum_{i=1}^{k} r_i = 1$ and R=$\sum_{i=1}^{k} r_i R_i$. For every i $(1 \le i \le k)$, $A \succ_r R_i A$, and hence $T(A) \succ_r T(R_i A)$. Thus, there exist row stochastic matrices S_i $(1 \le i \le k)$, such that $T(R_i A) = T(A)S_i$. Put $S = \sum_{i=1}^{k} r_i S_i$. It is clear that S is a g-row stochastic matrix and $T(B) = T(A)S$. Therefore, $T(A) \succ_{rgw} T(B)$. For the other side, replace T by T^{-1} and similarly conclude that $A \succ_{rgw} B$ whenever $T(A) \succ_{rgw} T(B)$. Then, T strongly preserves the *right gw-majorization.* Therefore, by Theorem 2.8, there exist invertible matrices $A \in \mathbf{GR}_n$ and $M \in \mathbf{M}_n$ such that $T(X) = MXA$ for all $X \in \mathbf{M}_n$. For every row stochastic matrix R, it is clear that $I \succ_r R$. So, $T(I) \succ_r T(R)$, for every row stochastic matrix R. Then, $MIA \succ_r MRA$, and hence $A^{-1}RA$ is a row stochastic matrix, for every row stochastic matrix R. So, it is easy to show that A^{-1} is a row stochastic matrix . Similarly, A is a row stochastic matrix too, and hence A is a permutation matrix by Lemma 2.9. To complete the proof, put P=A . □

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