# RIGHT GW-MAJORIZATION ON $\mathbf{M}_{n, m}$ 

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#### Abstract

Let $\mathbf{M}_{n, m}$ be the set of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. A real or complex $n \times n$ matrix is generalized row stochastic ( $g$-row stochastic) if all of its row sums equal one. For $X, Y \in \mathbf{M}_{n, m}$, we say that $Y$ is right gw-majorized by $X$ and write $X \succ_{r g w} Y$ if $Y=X R$ for some g-row stochastic matrix $R$. Here, we characterize all strong linear preservers of $\succ_{r g w}$ on $\mathbf{M}_{n, m}$.


## 1. Introduction

A nonnegative real $n \times n$ matrix $R$ is said to be row stochastic, if $R e=e$, where $e=(1, \ldots, 1)^{t} \in \mathbb{F}^{n}$. The following generalization of stochastic matrices were introduced in [5]. A complex (not necessarily nonnegative) $n \times n$ matrix $R$ is said to be $g$-row ( $g$-doubly) stochastic, if $R e=e\left(R e=e\right.$ and $\left.R^{t} e=e\right)$. The notion of matrix majorization was introduced by Dahl in [6]. According to that definition, a matrix $A$ is right matrix majorized (or right weak majorized) by $B$ if there exists an $n \times n$ row stochastic matrix $R$ such that $A=B R$, and is denoted by

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$B \succ_{r} A$. The concept of the left matrix majorization is defined similarly and denoted by $\succ_{l}$. The definitions of generalized majorizations ( $g$ majorization) are motivated by the matrix majorization as follows:
gs-majorization: A matrix $A$ is said to be g-majorized by $B$ strongly if there exists an $n \times n$ g-doubly stochastic matrix $D$ such that $A=D B$, and is denoted by $B \succ_{g s} A$.
$g w$-majorization: A matrix $A$ is said to be right g-majorized by $B$ weakly if there exists an $n \times n$ g-row stochastic matrix $R$ such that $A=B R$, and is denoted by $B \succ_{r g w} A$. The concept of left $g$-majorization is defined similarly and is denoted by $\succ_{l g w}$.

For a relation $\sim$ on $\mathbf{M}_{n, m}$, we say that $T$ preserves (or strongly preserves) $\sim$ if $T(x) \sim T(y)$, whenever $x \sim y$ (or $T(x) \sim T(y)$ if and only if $x \sim y$ ).

In [1,2], the authors introduced the relations $\succ_{g s}$ and $\succ_{l g w}$ on $\mathbf{M}_{n, m}$. Also, they characterized all strong linear preservers of $\succ_{g s}$ and $\succ_{l g w}$ on $\mathbf{M}_{n, m}$ and $\mathbf{M}_{n}$ respectively. In [7], the authors characterized all linear operator that strongly preserve the right matrix majorization. We prove that one of the main theorems in [7] (Theorem 4.4) may be obtained as a corollary of our Theorem 2.8. We refer the readers to [4] and [6] for more information on the type of majorization and linear preserver of majorization.

The following notations will be fixed throughout this paper:
$\mathbb{F}^{n}$ : the set of column vectors $\mathbf{M}_{n, 1}$ with standard basis $\left\{e_{1}, \ldots, e_{n}\right\}$.
$\mathbb{F}_{m}$ : the set of row vectors $\mathbf{M}_{1, m}$ with standard basis $\left\{\epsilon_{1}, \ldots, \epsilon_{m}\right\}$.
$\mathbf{M}_{n}$ : the set of all $n \times n$ complex matrices.
$\mathbf{G R}_{n}$ : the set of all $n \times n$ generalized row stochastic (g-row stochastic) matrices.
$\mathbf{P}_{n}$ : the set of all $n \times n$ permutation matrices.
$\operatorname{tr}(x)$ : the sum of all components of a vector $x$.
$\left[x_{1} / x_{2} / \ldots / x_{n}\right]:$ an $n \times m$ matrix whose rows are $x_{1}, x_{2}, \ldots, x_{n} \in \mathbb{F}_{m}$.

## 2. Strong linear preserver

Here, we state the following statements to prove the main result of the paper.

Lemma 2.1. Let $T: \mathbb{F}_{n} \rightarrow \mathbb{F}_{n}$ be a linear operator. Then, $T$ preserves the subspace $\left\{x \in \mathbb{F}_{n}: \operatorname{tr}(x)=0\right\}$ if and only if there exists a matrix $B \in \operatorname{span}\left(\mathbf{G R}_{n}\right)$ such that $T(x)=x B$, for all $x \in \mathbb{F}_{n}$.

Proof. Let $B=\left(b_{i j}\right) \in \mathbf{M}_{n}$ be the matrix representation of $T$ with respect to the standard basis of $\mathbb{F}_{n}$. If $B \in \operatorname{span}\left(\mathbf{G R}_{n}\right)$, then it is easy to show that $T$ preserves the subspace $\left\{x \in \mathbb{F}_{n}: \operatorname{tr}(x)=0\right\}$. Conversely, let $T$ preserve the subspace $\left\{x \in \mathbb{F}_{n}: \operatorname{tr}(x)=0\right\}$. Then, $\operatorname{tr}\left(T\left(\epsilon_{i}-\epsilon_{j}\right)\right)=0$, for all $1 \leq i, j \leq n$, so that $\operatorname{tr}\left(\left(\epsilon_{i}-\epsilon_{j}\right) B\right)=0$, and thus $\sum_{k=1}^{n} b_{i k}=\sum_{k=1}^{n} b_{j k}$. Therefore, $B \in \operatorname{span}\left(\mathbf{G R}_{n}\right)$.

Lemma 2.2. Let $x$ be a nonzero vector in $\mathbb{F}_{m}$. Then, $x \succ_{\text {rgw }} y$, for some $y \in \mathbb{F}_{m}$ if and only if $\operatorname{tr}(x)=\operatorname{tr}(y)$.

Proof. If $x \succ_{r g w} y$, then it is clear that $\operatorname{tr}(x)=\operatorname{tr}(y)$. Conversely, let $x=\left(x_{1}, \ldots, x_{m}\right), y=\left(y_{1}, \ldots, y_{m}\right) \in \mathbb{F}_{m}$ and $\operatorname{tr}(x)=\operatorname{tr}(y)$. Without loss of generality, assume that $x_{1} \neq 0$. Put,

$$
R_{y}:=\left(\begin{array}{c|c}
r_{1} & r_{2} \cdots r_{m} \\
--- & ---- \\
0 & I_{m-1}
\end{array}\right)
$$

where $r_{1}=\frac{y_{1}}{x_{1}}$ and $r_{i}=\frac{y_{i}-x_{i}}{x_{1}}$, for every $i(2 \leq i \leq m)$. It is clear that $y=x R_{y}$. Since $\operatorname{tr}(x)=\operatorname{tr}(y)$, then it is easy to show that $R_{y} \in \mathbf{G R}_{m}$ and hence $x \succ_{\text {rgw }} y$.

Theorem 2.3. Let $T: \mathbb{F}_{m} \rightarrow \mathbb{F}_{m}$ be a linear operator. Then, T preserves $\succ_{\text {rgw }}$ if and only if one of the followings holds:
(a) $T x=\alpha x B$, for some $\alpha \in \mathbb{F}$ and some invertible $B \in \mathbf{G} R_{n}$.
(b) $T x=\alpha x B$, for some $\alpha \in \mathbb{F}$ and some $B \in \mathbf{G} R_{n}$ such that $\{x: x B=0\}=\{x: \operatorname{tr}(x)=0\}$.

Proof. Let $T: \mathbb{F}_{m} \rightarrow \mathbb{F}_{m}$ preserve $\succ_{r g w}$. If $T=0$, then put $\alpha=0$. So, assume that $T \neq 0$. It is easy to show that $T$ preserves the subspace $\left\{x \in \mathbb{F}_{m}: \operatorname{tr}(x)=0\right\}$. Then, by Lemma 2.1 there exits a g-row stochastic matrix $B$ and a scalar $\alpha$ such that $T x=\alpha x B$, for all $x \in \mathbb{F}_{m}$. If $B$ is invertible, then (a) holds. If $B$ is not invertible, then (b) holds by Lemma 2.2. The converse is trivial.

Corollary 2.4. Let $T: \mathbb{F}_{m} \rightarrow \mathbb{F}_{m}$ be a nonzero linear preserver of $\succ_{\text {rgw }}$. Then, $\operatorname{rank}(T)$ is equal to $m$ or 1 .

Proof. By Theorem 2.3, $\operatorname{ker}(T)=\{0\}$ or $\operatorname{ker}(T)=\{x: \operatorname{tr}(x)=0\}$. Then, $\operatorname{rank}(T)=m$ or $\operatorname{rank}(T)=1$.

Corollary 2.5. Let $T: \mathbb{F}_{m} \rightarrow \mathbb{F}_{m}$ be a singular linear preserver of $\succ_{\text {rgw }}$. Then, there exists a vector $y \in \mathbb{F}_{m}$ such that $T x=\operatorname{tr}(x) y$ for all $x \in \mathbb{F}_{m}$.

Proof. Consider the basis $\left\{e^{t}, \epsilon_{1 j}: 2 \leq j \leq m\right\}$ for $\mathbb{F}_{m}$, where $\epsilon_{1 j}=$ $\epsilon_{1}-\epsilon_{j}$. Then, for every $x \in \mathbb{F}_{m}, \quad x=\frac{1}{m} \operatorname{tr}(x) e^{t}+\sum_{j=2}^{m} \alpha_{1 j} \epsilon_{1 j}$, for some $\alpha_{1 j} \in \mathbb{F}$. Since $\operatorname{ker}(T)=\{x: \operatorname{tr}(x)=0\}$ by Theorem 2.3, then $T x=\operatorname{tr}(x) y$, where $y=\frac{1}{m} T\left(e^{t}\right)$.

Lemma 2.6. Let $A \in \mathbf{M}_{m}$ and $\alpha$ be a nonzero scalar in $\mathbb{F}$. Then, $A=\gamma I$ for some $\gamma \in \mathbb{F}$ if and only if we have,

$$
\begin{equation*}
\alpha x A+y \succ_{\text {rgw }} \alpha x R A+y R, \forall x, y \in \mathbb{F}_{m}, \forall R \in \mathbf{G R}_{m} \tag{2.2}
\end{equation*}
$$

Proof. If $A=\gamma I$, for some $\gamma \in \mathbb{F}$, then it is clear that (2.2) holds. Conversely, let (2.2) hold. For every $x \in \mathbb{F}_{m}$, put $x$ and $y$ in (2.2), where $y=-\alpha x A$. Then, $\alpha x R A-\alpha x A R=0$ for all $x$ in $\mathbb{F}_{m}$. So, $R A=A R$, for all $R \in \mathbf{G R}_{m}$, and hence $A=\gamma I$, for some $\gamma \in \mathbb{F}$.

Remark 2.7. Every strong linear preserver of $\succ_{r g w}$ is invertible.

Now, we state the main result of this paper. The following theorem holds for $n=1$ obviously, and thus we assume that $n \geq 2$.

For every $i, j(1 \leq i, j \leq n)$, consider the embedding $E^{j}: \mathbb{F}_{m} \rightarrow \mathbf{M}_{n, m}$ and the projection $E_{i}: \mathbf{M}_{n, m} \rightarrow \mathbb{F}_{m}$, where $E^{j}(x)=e_{j} x$ and $E_{i}(A)=$ $\epsilon_{i} A$. It is easy to show that for every linear operator $T: \mathbf{M}_{n, m} \rightarrow \mathbf{M}_{n, m}$, $T(X)=T\left[x_{1} / \cdots / x_{n}\right]=\left[\sum_{j=1}^{n} T_{1}^{j} x_{j} / \cdots / \sum_{j=1}^{n} T_{n}^{j} x_{j}\right]$, where $x_{j}$ is the $j$ th row of X and $T_{i}^{j}=E_{i} \circ T \circ E^{j}$.

Theorem 2.8. Let $T: \mathbf{M}_{n, m} \rightarrow \mathbf{M}_{n, m}$ be a linear operator. Then, $T$ strongly preserves $\succ_{\text {rgw }}$ if and only if $T(X)=M X A$, for some invertible matrices $M \in \mathbf{M}_{n}$ and $A \in \mathbf{G R}_{m}$.

Proof. First assume that $T$ strongly preserves $\succ_{\text {rgw }}$. It is clear that for every $i, j(1 \leq i, j \leq n), T_{i}^{j}: \mathbb{F}_{m} \rightarrow \mathbb{F}_{m}$ preserves $\succ_{r g w}$. Then, by Theorem 2.3, there exist $A_{i}^{j} \in \mathbf{G R}_{m}$ and $\alpha_{i}^{j} \in \mathbb{F}$ such that

$$
T\left[x_{1} / \cdots / x_{n}\right]=\left[\sum_{j=1}^{n} \alpha_{1}^{j} x_{j} A_{1}^{j} / \cdots / \sum_{j=1}^{n} \alpha_{n}^{j} x_{j} A_{n}^{j}\right]
$$

Now, we consider some steps.
Step 1: Let there exist $p$ and $q(1 \leq p, q \leq n)$ such that $A_{p}^{q}$ is invertible and $\alpha_{p}^{q} \neq 0$. We want to replace $A_{p}^{j}$ by $A_{p}^{q}$ with suitable coefficient $\beta_{p}^{j}$. For every $x, y \in \mathbb{F}_{m}$, let $X=\left[x_{1} / \cdots / x_{n}\right]$ where $x_{p}=x$, $x_{j}=y$ and $x_{i}=0$, for $i \neq p, j$. Since $X \succ_{r g w} X R, \forall R \in \mathbf{G R}_{n}$, then $T(X) \succ_{\text {rgw }} T(X R)$ and hence,
$\alpha_{p}^{q} x A_{p}^{q}+\alpha_{p}^{j} y A_{p}^{j} \succ_{r g w} \alpha_{p}^{q} x R A_{p}^{q}+\alpha_{p}^{j} y R A_{p}^{j} . \forall x, y \in \mathbb{F}_{m}, \forall R \in \mathbf{G R}_{m}$.
Therefore, by Lemma $2.6, A_{p}^{j}=\beta_{p}^{j} A_{p}^{q}$, for some $\beta_{p}^{j} \in \mathbb{F}$.
Step 2: We show that there is no $p, q(1 \leq p, q \leq n)$ such that $A_{p}^{q}$ is singular and $\alpha_{p}^{q} \neq 0$. Assume, if possible, that there exist $p$ and $q$ such that $A_{p}^{q}$ is singular and $\alpha_{p}^{q} \neq 0$. So, by Lemma 2.6, $A_{p}^{j}$ is singular, for every $j$ $(1 \leq j \leq n)$, for which $\alpha_{p}^{j} \neq 0$. Without loss of generality, assume that $p=1$. So, by Theorem 2.3, $\operatorname{ker}\left(A_{1}^{j}\right)=\{x: \operatorname{tr}(x)=0\}$, for every $j(1 \leq$ $j \leq n)$, Therefore, by this and Step 1, for every $i(2 \leq i \leq n)$, either we can assume $A_{i}^{1}=\cdots=A_{i}^{n}$ or we have $\operatorname{ker}\left(A_{i}^{1}\right)=\cdots=\operatorname{ker}\left(A_{i}^{n}\right)=\{x:$ $\operatorname{tr}(x)=0\}$. Since the vectors $\left(\alpha_{2}^{1}, \cdots, \alpha_{n}^{1}\right), \cdots,\left(\alpha_{2}^{n}, \cdots, \alpha_{n}^{n}\right)$ are linearly dependent in $\mathbb{F}_{n-1}$, then there exist scalars (not all zero) $t_{1}, \cdots, t_{n} \in \mathbb{F}$ such that $\sum_{j=1}^{n} t_{j}\left(\alpha_{2}^{j}, \cdots, \alpha_{n}^{j}\right)=0$. For every $j(1 \leq j \leq n)$. put $x_{j}=\left(t_{j},-t_{j}, 0, \cdots, 0\right) \in \mathbb{F}_{m}$, and then $X=\left[x_{1} / \cdots / x_{n}\right] \neq 0$. It is easy to check that $T X=0$, which contradicts the injectivity of $T$. Therefore, by Step 1,

$$
T\left[x_{1} / \cdots / x_{n}\right]=\left[\sum_{j=1}^{n} \beta_{1}^{j} x_{j} A_{1} / \cdots / \sum_{j=1}^{n} \beta_{n}^{j} x_{j} A_{n}\right]
$$

for some invertible matrices $A_{i} \in \mathbf{G R}_{m}$ and some scalars $\beta_{i}^{j} \in \mathbb{F}(1 \leq$ $i, j \leq n)$.

Step 3: Here, we show that $A_{i}=A_{1}$, for every $i(1 \leq i \leq n)$. By Step 2, $T\left[x_{1} / \cdots / x_{n}\right]=\left[b_{1} X A_{1} / \cdots / b_{n} X A_{n}\right]$, where $b_{i}=\left(\beta_{i}^{1}, \cdots, \beta_{i}^{n}\right)$, for every $i(1 \leq i \leq n)$. If $\left\{b_{1}, \cdots, b_{n}\right\} \subseteq \operatorname{span}\{b\}$, for some $b \in \mathbb{F}_{n}$, then there exists a nonzero vector $a \in \mathbb{F}_{n}$ such that $b_{1} a^{t}=0$, because $n \geq 2$. Let $X \in \mathbf{M}_{n, m}$ be such that the first column of $X$ is $a^{t}$ and the other columns of $X$ are zero. Then, $X \neq 0$ and $T X=0$, which is a contradiction. Therefore, at least two elements of $\left\{b_{1}, \cdots, b_{m}\right\}$ are linearly independent. Without loss of generality, assume that $\left\{b_{1}, b_{2}\right\}$ is linearly independent. For every $X \in \mathbf{M}_{n, m}$ and every $R \in \mathbf{G R}_{m}$, $T X \succ_{\text {rgw }} T(R X)$. Then,

$$
\begin{array}{rll}
b_{1} X A_{1}+b_{2} X A_{2} & \succ_{r g w} & b_{1} X R A_{1}+b_{2} X R A_{2}  \tag{2.3}\\
\forall X \in \mathbf{M}_{n, m} & , & \forall R \in \mathbf{G R}_{m} .
\end{array}
$$

Since $\left\{b_{1}, b_{2}\right\}$ is a linearly independent set in $\mathbb{F}_{n}$, then for every $x, y \in$ $\mathbb{F}_{m}$ there exists $X_{x, y} \in \mathbf{M}_{n, m}$ such that $b_{1} X_{x, y}=x$ and $b_{2} X_{x, y}=y$. Therefore, by (2.3) we have,

$$
x A_{1}+y A_{2} \succ_{r g w} x R A_{1}+y R A_{2}, \forall x, y \in \mathbb{F}_{m}, \forall R \in \mathbf{G R}_{m}
$$

So, by Lemma 2.6, $A_{2}=\alpha A_{1}$, for some $\alpha \in \mathbb{F}$. For every $j(2 \leq j \leq n)$, if $b_{j}=0$, then we can replace $A_{j}$ by $A_{1}$. If $b_{j} \neq 0$, then $\left\{b_{1}, b_{j}\right\}$ or $\left\{b_{2}, b_{j}\right\}$ is a linearly independent set. Similar to the above argument, $A_{j}=\beta A_{1}$ or $A_{j}=\beta A_{2}$, for some $\beta \in \mathbb{F}$. Put $A=A_{1}$. Then, $T(X)=$ $\left[b_{1} X A \mid b_{2} X\left(\alpha_{2} A\right) / \cdots / b_{n} X\left(\alpha_{n} A\right)\right]$, for some $\alpha_{2}, \cdots, \alpha_{n} \in \mathbb{F}$, and hence $T(X)=M X A$, for all $X \in \mathbf{M}_{n, m}$, where $M=\left[b_{1} / \alpha_{2} b_{2} / \cdots / \alpha_{n} b_{n}\right]$. The other side of the theorem is easy to establish.

The following lemma was proved in [3].
Lemma 2.9. Let $R$ be a nonsingular row stochastic matrix. If $R^{-1}$ is nonnegative, then $R$ is a permutation matrix.

Proof. Let $R=\left(r_{i j}\right)$. Suppose that $R$ has two nonzero entries in some row. Without loss of generality, we may assume that the first row of $R$ has at least two nonzero entries. Since $R^{-1}$ is invertible, then the first column of $R^{-1}$ has a nonzero entry, say in row $i$, and then the $i$ th row
of $R^{-1} R$ must have at least two nonzero entries. This is a contradiction, since $R^{-1} R=\mathrm{I}$. Thus, every row of $R$ has exactly one nonzero entry. Since $R$ is invertible, then therefore $R$ is a permutation matrix .

Corollary 2.10. ([7], Theorem 4.4) A linear operator $T: \mathbf{M}_{n} \rightarrow \mathbf{M}_{n}$ strongly preserves right matrix majorization $\succ_{r}$ if and only if $T(X)=$ $M X P$, where $P$ is a permutation and $M \in M_{n}$ is invertible.

Proof. Assume that $T$ strongly preserves right matrix majorization. We show that $T$ is a strong linear preserver of the right gw-majorization. Let $A \succ_{\text {rgw }} B$. Then, there exists a g-row stochastic matrix $R$ such that $B=A R$. For the g-row stochastic matrix $R$, there exist scalars $r_{1}, \ldots, r_{k}$ and row stochastic matrices $R_{1}, \ldots, R_{k}$ such that $\sum_{i=1}^{k} r_{i}=1$ and $R=\sum_{i=1}^{k} r_{i} R_{i}$. For every $i(1 \leq i \leq k), A \succ_{r} R_{i} A$, and hence $T(A) \succ_{r} T\left(R_{i} A\right)$. Thus, there exist row stochastic matrices $S_{i}(1 \leq i \leq k)$, such that $T\left(R_{i} A\right)=T(A) S_{i}$. Put $S=\sum_{i=1}^{k} r_{i} S_{i}$. It is clear that $S$ is a g-row stochastic matrix and $T(B)=T(A) S$. Therefore, $T(A) \succ_{\text {rgw }} T(B)$. For the other side, replace T by $T^{-1}$ and similarly conclude that $A \succ_{\text {rgw }} B$ whenever $T(A) \succ_{r g w} T(B)$. Then, $T$ strongly preserves the right $g w$ majorization. Therefore, by Theorem 2.8, there exist invertible matrices $A \in \mathbf{G R}_{n}$ and $M \in \mathbf{M}_{n}$ such that $T(X)=M X A$ for all $X \in \mathbf{M}_{n}$. For every row stochastic matrix $R$, it is clear that $I \succ_{r} R$. So, $T(I) \succ_{r} T(R)$, for every row stochastic matrix $R$. Then, MIA $\succ_{r} M R A$, and hence $A^{-1} R A$ is a row stochastic matrix, for every row stochastic matrix $R$. So, it is easy to show that $A^{-1}$ is a row stochastic matrix . Similarly, $A$ is a row stochastic matrix too, and hence $A$ is a permutation matrix by Lemma 2.9. To complete the proof, put $P=A$.

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## References

[1] A. Armandnejad and A. Salemi, Strong linear preservers of gw-majorization on $M_{n}$, Journal of Dynamical Systems and Geometric Theories 5(2) (2007) 168-165.
[2] A. Armandnejad and A. Salemi, The structure of linear preservers of gsmajorization, Bull. Iranian Math. Soc. 32(2) (2006) 31-42.
[3] B. Beasley, S.G. Lee and Y.H. Lee, A characterization of strong preservers of matrix majorization, Linear Algebra Appl. 367 (2003) 341-346.
[4] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.
[5] H. Chiang and C.K. Li, Generalized doubly stochastic matrices and linear preservers, Linear and Multilinear Algebra 53, Issue 1 (2005) 1-11.
[6] G. Dahl, Matrix majorization, Linear Algebra Appl. 288 (1999) 53-73.
[7] A.M. Hasani and M. Radjabalipour, The structure of linear operators strongly preserving majorizations of matrices, Electronic Journal of Linear Algebra 15 (2006) 260-268.

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