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# **RIGHT GW-MAJORIZATION ON** $M_{n,m}$

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ABSTRACT. Let  $\mathbf{M}_{n,m}$  be the set of all  $n \times m$  matrices with entries in  $\mathbb{F}$ , where  $\mathbb{F}$  is the field of real or complex numbers. A real or complex  $n \times n$  matrix is generalized row stochastic (g-row stochastic) if all of its row sums equal one. For  $X, Y \in \mathbf{M}_{n,m}$ , we say that Yis right gw-majorized by X and write  $X \succ_{rgw} Y$  if Y=XR for some g-row stochastic matrix R. Here, we characterize all strong linear preservers of  $\succ_{rgw}$  on  $\mathbf{M}_{n,m}$ .

### 1. Introduction

A nonnegative real  $n \times n$  matrix R is said to be row stochastic, if Re = e, where  $e = (1, ..., 1)^t \in \mathbb{F}^n$ . The following generalization of stochastic matrices were introduced in [5]. A complex (not necessarily nonnegative)  $n \times n$  matrix R is said to be g-row (g-doubly) stochastic, if Re=e (Re = e and  $R^te = e$ ). The notion of matrix majorization was introduced by Dahl in [6]. According to that definition, a matrix A is right matrix majorized (or right weak majorized) by B if there exists an  $n \times n$  row stochastic matrix R such that A=BR, and is denoted by

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 $B \succ_r A$ . The concept of the *left matrix majorization* is defined similarly and denoted by  $\succ_l$ . The definitions of *generalized majorizations (gmajorization)* are motivated by the *matrix majorization* as follows:

gs-majorization: A matrix A is said to be g-majorized by B strongly if there exists an  $n \times n$  g-doubly stochastic matrix D such that A=DB, and is denoted by  $B \succ_{gs} A$ .

gw-majorization: A matrix A is said to be right g-majorized by B weakly if there exists an  $n \times n$  g-row stochastic matrix R such that A=BR, and is denoted by  $B \succ_{rgw} A$ . The concept of left g-majorization is defined similarly and is denoted by  $\succ_{lgw}$ .

For a relation ~ on  $\mathbf{M}_{n,m}$ , we say that T preserves (or strongly preserves) ~ if  $T(x) \sim T(y)$ , whenever  $x \sim y$  (or  $T(x) \sim T(y)$  if and only if  $x \sim y$ ).

In [1,2], the authors introduced the relations  $\succ_{gs}$  and  $\succ_{lgw}$  on  $\mathbf{M}_{n,m}$ . Also, they characterized all strong linear preservers of  $\succ_{gs}$  and  $\succ_{lgw}$  on  $\mathbf{M}_{n,m}$  and  $\mathbf{M}_n$  respectively. In [7], the authors characterized all linear operator that strongly preserve the *right matrix majorization*. We prove that one of the main theorems in [7] (Theorem 4.4) may be obtained as a corollary of our Theorem 2.8. We refer the readers to [4] and [6] for more information on the type of majorization and linear preserver of majorization.

The following notations will be fixed throughout this paper:

 $\mathbb{F}^n$ : the set of column vectors  $\mathbf{M}_{n,1}$  with standard basis  $\{e_1, ..., e_n\}$ .  $\mathbb{F}_m$ : the set of row vectors  $\mathbf{M}_{1,m}$  with standard basis  $\{\epsilon_1, ..., \epsilon_m\}$ .  $\mathbf{M}_n$ : the set of all  $n \times n$  complex matrices.

**GR**<sub>n</sub>: the set of all  $n \times n$  generalized row stochastic (g-row stochastic) matrices.

 $\mathbf{P}_n$ : the set of all  $n \times n$  permutation matrices.

tr(x): the sum of all components of a vector x.

 $[x_1/x_2/.../x_n]$ : an  $n \times m$  matrix whose rows are  $x_1, x_2, ..., x_n \in \mathbb{F}_m$ .

### 2. Strong linear preserver

Here, we state the following statements to prove the main result of the paper.

**Lemma 2.1.** Let  $T : \mathbb{F}_n \to \mathbb{F}_n$  be a linear operator. Then, T preserves the subspace  $\{x \in \mathbb{F}_n : tr(x) = 0\}$  if and only if there exists a matrix  $B \in span(\mathbf{GR}_n)$  such that T(x) = xB, for all  $x \in \mathbb{F}_n$ . Right gw-majorization on  $\mathbf{M}_{n,m}$ 

**Proof.** Let  $B = (b_{ij}) \in \mathbf{M}_n$  be the matrix representation of T with respect to the standard basis of  $\mathbb{F}_n$ . If  $B \in span(\mathbf{GR}_n)$ , then it is easy to show that T preserves the subspace  $\{x \in \mathbb{F}_n : tr(x) = 0\}$ . Conversely, let T preserve the subspace  $\{x \in \mathbb{F}_n : tr(x) = 0\}$ . Then,  $tr(T(\epsilon_i - \epsilon_j)) = 0$ , for all  $1 \leq i, j \leq n$ , so that  $tr((\epsilon_i - \epsilon_j)B) = 0$ , and thus  $\sum_{k=1}^n b_{ik} = \sum_{k=1}^n b_{jk}$ . Therefore,  $B \in span(\mathbf{GR}_n)$ .

**Lemma 2.2.** Let x be a nonzero vector in  $\mathbb{F}_m$ . Then,  $x \succ_{rgw} y$ , for some  $y \in \mathbb{F}_m$  if and only if tr(x) = tr(y).

**Proof.** If  $x \succ_{rgw} y$ , then it is clear that tr(x) = tr(y). Conversely, let  $x = (x_1, ..., x_m), y = (y_1, ..., y_m) \in \mathbb{F}_m$  and tr(x) = tr(y). Without loss of generality, assume that  $x_1 \neq 0$ . Put,

$$R_y := \begin{pmatrix} r_1 & | & r_2 \cdots r_m \\ --- & | & ---- \\ 0 & | & I_{m-1} \end{pmatrix},$$

where  $r_1 = \frac{y_1}{x_1}$  and  $r_i = \frac{y_i - x_i}{x_1}$ , for every  $i \ (2 \le i \le m)$ . It is clear that  $y = xR_y$ . Since tr(x) = tr(y), then it is easy to show that  $R_y \in \mathbf{GR}_m$  and hence  $x \succ_{rgw} y$ .

**Theorem 2.3.** Let  $T : \mathbb{F}_m \to \mathbb{F}_m$  be a linear operator. Then, T preserves  $\succ_{rgw}$  if and only if one of the followings holds:

- (a)  $Tx = \alpha x B$ , for some  $\alpha \in \mathbb{F}$  and some invertible  $B \in \mathbf{G}R_n$ .
- (b)  $Tx = \alpha x B$ , for some  $\alpha \in \mathbb{F}$  and some  $B \in \mathbf{G}R_n$  such that
  - $\{x : xB = 0\} = \{x : tr(x) = 0\}.$

**Proof.** Let  $T : \mathbb{F}_m \to \mathbb{F}_m$  preserve  $\succ_{rgw}$ . If T = 0, then put  $\alpha = 0$ . So, assume that  $T \neq 0$ . It is easy to show that T preserves the subspace  $\{x \in \mathbb{F}_m : tr(x) = 0\}$ . Then, by Lemma 2.1 there exits a g-row stochastic matrix B and a scalar  $\alpha$  such that  $Tx = \alpha xB$ , for all  $x \in \mathbb{F}_m$ . If B is invertible, then (a) holds. If B is not invertible, then (b) holds by Lemma 2.2. The converse is trivial.

**Corollary 2.4.** Let  $T : \mathbb{F}_m \to \mathbb{F}_m$  be a nonzero linear preserver of  $\succ_{rgw}$ . Then, rank(T) is equal to m or 1.

**Proof.** By Theorem 2.3,  $ker(T) = \{0\}$  or  $ker(T) = \{x : tr(x) = 0\}$ . Then, rank(T) = m or rank(T) = 1.

**Corollary 2.5.** Let  $T : \mathbb{F}_m \to \mathbb{F}_m$  be a singular linear preserver of  $\succ_{rgw}$ . Then, there exists a vector  $y \in \mathbb{F}_m$  such that Tx = tr(x)y for all  $x \in \mathbb{F}_m$ .

**Proof.** Consider the basis  $\{e^t, \epsilon_{1j} : 2 \leq j \leq m\}$  for  $\mathbb{F}_m$ , where  $\epsilon_{1j} = \epsilon_1 - \epsilon_j$ . Then, for every  $x \in \mathbb{F}_m$ ,  $x = \frac{1}{m} tr(x)e^t + \sum_{j=2}^m \alpha_{1j}\epsilon_{1j}$ , for some  $\alpha_{1j} \in \mathbb{F}$ . Since  $ker(T) = \{x : tr(x) = 0\}$  by Theorem 2.3, then Tx = tr(x)y, where  $y = \frac{1}{m}T(e^t)$ .

**Lemma 2.6.** Let  $A \in \mathbf{M}_m$  and  $\alpha$  be a nonzero scalar in  $\mathbb{F}$ . Then,  $A = \gamma I$  for some  $\gamma \in \mathbb{F}$  if and only if we have,

(2.2)  $\alpha xA + y \succ_{rgw} \alpha xRA + yR, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m.$ 

**Proof.** If  $A = \gamma I$ , for some  $\gamma \in \mathbb{F}$ , then it is clear that (2.2) holds. Conversely, let (2.2) hold. For every  $x \in \mathbb{F}_m$ , put x and y in (2.2), where  $y = -\alpha x A$ . Then,  $\alpha x R A - \alpha x A R = 0$  for all x in  $\mathbb{F}_m$ . So, RA = A R, for all  $R \in \mathbf{GR}_m$ , and hence  $A = \gamma I$ , for some  $\gamma \in \mathbb{F}$ .

**Remark 2.7.** Every strong linear preserver of  $\succ_{rgw}$  is invertible.

Now, we state the main result of this paper. The following theorem holds for n=1 obviously, and thus we assume that  $n \ge 2$ .

For every i, j  $(1 \le i, j \le n)$ , consider the embedding  $E^j : \mathbb{F}_m \to \mathbf{M}_{n,m}$ and the projection  $E_i : \mathbf{M}_{n,m} \to \mathbb{F}_m$ , where  $E^j(x) = e_j x$  and  $E_i(A) = \epsilon_i A$ . It is easy to show that for every linear operator  $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ ,  $T(X) = T[x_1/\cdots/x_n] = [\sum_{j=1}^n T_1^j x_j/\cdots/\sum_{j=1}^n T_n^j x_j]$ , where  $x_j$  is the j th row of X and  $T_i^j = E_i \circ T \circ E^j$ .

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**Theorem 2.8.** Let  $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$  be a linear operator. Then, T strongly preserves  $\succ_{rgw}$  if and only if T(X) = MXA, for some invertible matrices  $M \in \mathbf{M}_n$  and  $A \in \mathbf{GR}_m$ .

**Proof.** First assume that T strongly preserves  $\succ_{rgw}$ . It is clear that for every i, j  $(1 \leq i, j \leq n), T_i^j : \mathbb{F}_m \to \mathbb{F}_m$  preserves  $\succ_{rgw}$ . Then, by Theorem 2.3, there exist  $A_i^j \in \mathbf{GR}_m$  and  $\alpha_i^j \in \mathbb{F}$  such that

$$T[x_1/\dots/x_n] = [\sum_{j=1}^n \alpha_1^j x_j A_1^j/\dots/\sum_{j=1}^n \alpha_n^j x_j A_n^j].$$

Now, we consider some steps.

fore, by Step 1,

Step 1: Let there exist p and q  $(1 \leq p, q \leq n)$  such that  $A_p^q$  is invertible and  $\alpha_p^q \neq 0$ . We want to replace  $A_p^j$  by  $A_p^q$  with suitable coefficient  $\beta_p^j$ . For every  $x, y \in \mathbb{F}_m$ , let  $X = [x_1/\cdots/x_n]$  where  $x_p = x$ ,  $x_j = y$  and  $x_i = 0$ , for  $i \neq p, j$ . Since  $X \succ_{rgw} XR$ ,  $\forall R \in \mathbf{GR}_n$ , then  $T(X) \succ_{rgw} T(XR)$  and hence,

$$\alpha_p^q x A_p^q + \alpha_p^j y A_p^j \succ_{rgw} \alpha_p^q x R A_p^q + \alpha_p^j y R A_p^j \, . \, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m \, .$$

Therefore, by Lemma 2.6,  $A_p^j = \beta_p^j A_p^q$ , for some  $\beta_p^j \in \mathbb{F}$ .

Step 2: We show that there is no p, q  $(1 \le p, q \le n)$  such that  $A_p^q$  is singular and  $\alpha_p^q \ne 0$ . Assume, if possible, that there exist p and q such that  $A_p^q$  is singular and  $\alpha_p^q \ne 0$ . So, by Lemma 2.6,  $A_p^j$  is singular, for every j  $(1 \le j \le n)$ , for which  $\alpha_p^j \ne 0$ . Without loss of generality, assume that p=1. So, by Theorem 2.3,  $ker(A_1^j) = \{x : tr(x) = 0\}$ , for every j  $(1 \le j \le n)$ , Therefore, by this and Step 1, for every i  $(2 \le i \le n)$ , either we can assume  $A_i^1 = \cdots = A_i^n$  or we have  $ker(A_i^1) = \cdots = ker(A_i^n) = \{x : tr(x) = 0\}$ . Since the vectors  $(\alpha_2^1, \cdots, \alpha_n^1), \cdots, (\alpha_2^n, \cdots, \alpha_n^n)$  are linearly dependent in  $\mathbb{F}_{n-1}$ , then there exist scalars (not all zero)  $t_1, \cdots, t_n \in \mathbb{F}$  such that  $\sum_{j=1}^n t_j(\alpha_2^j, \cdots, \alpha_n^j) = 0$ . For every j  $(1 \le j \le n)$ . put  $x_j = (t_j, -t_j, 0, \cdots, 0) \in \mathbb{F}_m$ , and then  $X = [x_1/\cdots/x_n] \ne 0$ . It is easy to check that TX=0, which contradicts the injectivity of T. There-

$$T[x_1/\cdots/x_n] = [\sum_{j=1}^n \beta_1^j x_j A_1/\cdots/\sum_{j=1}^n \beta_n^j x_j A_n],$$

for some invertible matrices  $A_i \in \mathbf{GR}_m$  and some scalars  $\beta_i^j \in \mathbb{F}$   $(1 \le i, j \le n)$ .

Step 3: Here, we show that  $A_i = A_1$ , for every i  $(1 \le i \le n)$ . By Step 2,  $T[x_1/\cdots/x_n] = [b_1XA_1/\cdots/b_nXA_n]$ , where  $b_i = (\beta_i^1, \cdots, \beta_i^n)$ , for every i  $(1 \le i \le n)$ . If  $\{b_1, \cdots, b_n\} \subseteq span\{b\}$ , for some  $b \in \mathbb{F}_n$ , then there exists a nonzero vector  $a \in \mathbb{F}_n$  such that  $b_1a^t = 0$ , because  $n \ge 2$ . Let  $X \in \mathbf{M}_{n,m}$  be such that the first column of X is  $a^t$  and the other columns of X are zero. Then,  $X \ne 0$  and TX = 0, which is a contradiction. Therefore, at least two elements of  $\{b_1, \cdots, b_m\}$  are linearly independent. Without loss of generality, assume that  $\{b_1, b_2\}$ is linearly independent. For every  $X \in \mathbf{M}_{n,m}$  and every  $R \in \mathbf{GR}_m$ ,  $TX \succ_{rgw} T(RX)$ . Then,

(2.3) 
$$b_1 X A_1 + b_2 X A_2 \succ_{rgw} b_1 X R A_1 + b_2 X R A_2$$
$$\forall X \in \mathbf{M}_{n,m} \quad , \quad \forall R \in \mathbf{GR}_m.$$

Since  $\{b_1, b_2\}$  is a linearly independent set in  $\mathbb{F}_n$ , then for every  $x, y \in \mathbb{F}_m$  there exists  $X_{x,y} \in \mathbf{M}_{n,m}$  such that  $b_1 X_{x,y} = x$  and  $b_2 X_{x,y} = y$ . Therefore, by (2.3) we have,

 $xA_1 + yA_2 \succ_{rqw} xRA_1 + yRA_2, \forall x, y \in \mathbb{F}_m, \forall R \in \mathbf{GR}_m.$ 

So, by Lemma 2.6,  $A_2 = \alpha A_1$ , for some  $\alpha \in \mathbb{F}$ . For every j  $(2 \le j \le n)$ , if  $b_j = 0$ , then we can replace  $A_j$  by  $A_1$ . If  $b_j \ne 0$ , then  $\{b_1, b_j\}$  or  $\{b_2, b_j\}$  is a linearly independent set. Similar to the above argument,  $A_j = \beta A_1$  or  $A_j = \beta A_2$ , for some  $\beta \in \mathbb{F}$ . Put  $A = A_1$ . Then,  $T(X) = [b_1 XA | b_2 X(\alpha_2 A) / \cdots / b_n X(\alpha_n A)]$ , for some  $\alpha_2, \cdots, \alpha_n \in \mathbb{F}$ , and hence T(X) = MXA, for all  $X \in \mathbf{M}_{n,m}$ , where  $M = [b_1/\alpha_2 b_2/\cdots / \alpha_n b_n]$ . The other side of the theorem is easy to establish.  $\Box$ 

The following lemma was proved in [3].

**Lemma 2.9.** Let R be a nonsingular row stochastic matrix. If  $R^{-1}$  is nonnegative, then R is a permutation matrix.

**Proof.** Let  $R = (r_{ij})$ . Suppose that R has two nonzero entries in some row. Without loss of generality, we may assume that the first row of R has at least two nonzero entries. Since  $R^{-1}$  is invertible, then the first column of  $R^{-1}$  has a nonzero entry, say in row *i*, and then the *ith* row

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of  $R^{-1}R$  must have at least two nonzero entries. This is a contradiction, since  $R^{-1}R=I$ . Thus, every row of R has exactly one nonzero entry. Since R is invertible, then therefore R is a permutation matrix .

**Corollary 2.10.** ([7], Theorem 4.4) A linear operator  $T : \mathbf{M}_n \to \mathbf{M}_n$ strongly preserves right matrix majorization  $\succ_r$  if and only if T(X) = MXP, where P is a permutation and  $M \in \mathbf{M}_n$  is invertible.

**Proof.** Assume that T strongly preserves right matrix majorization. We show that T is a strong linear preserver of the right *qw*-majorization. Let  $A \succ_{raw} B$ . Then, there exists a g-row stochastic matrix R such that B=AR. For the g-row stochastic matrix R, there exist scalars  $r_1, ..., r_k$ and row stochastic matrices  $R_1, ..., R_k$  such that  $\sum_{i=1}^k r_i = 1$  and  $R = \sum_{i=1}^k r_i R_i$ . For every i  $(1 \le i \le k), A \succ_r R_i A$ , and hence  $T(A) \succ_r T(R_i A)$ . Thus, there exist row stochastic matrices  $S_i$   $(1 \leq i \leq k)$ , such that  $T(R_iA) = T(A)S_i$ . Put  $S = \sum_{i=1}^{k} r_iS_i$ . It is clear that S is a g-row sto-chastic matrix and T(B) = T(A)S. Therefore,  $T(A) \succ_{rgw} T(B)$ . For the other side, replace T by  $T^{-1}$  and similarly conclude that  $A \succ_{raw} B$ whenever  $T(A) \succ_{rqw} T(B)$ . Then, T strongly preserves the right gwmajorization. Therefore, by Theorem 2.8, there exist invertible matrices  $A \in \mathbf{GR}_n$  and  $M \in \mathbf{M}_n$  such that T(X) = MXA for all  $X \in \mathbf{M}_n$ . For every row stochastic matrix R, it is clear that  $I \succ_r R$ . So,  $T(I) \succ_r T(R)$ , for every row stochastic matrix R. Then,  $MIA \succ_r MRA$ , and hence  $A^{-1}RA$  is a row stochastic matrix, for every row stochastic matrix R. So, it is easy to show that  $A^{-1}$  is a row stochastic matrix . Similarly, A is a row stochastic matrix too, and hence A is a permutation matrix by Lemma 2.9. To complete the proof, put P=A. 

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