A SYSTEM OF GENERALIZED VARIATIONAL INCLUSIONS INVOLVING G-η-MONOTONE MAPPINGS

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ABSTRACT. We introduce a new concept of general G-η-monotone operator generalizing the general (H, η)-monotone operator [2, 3], general $H-$ monotone operator [26] in Banach spaces, and also generalizing G-η-monotone operator [29], (A, η)-monotone operator [24], A-monotone operator [23], (H, η)-monotone operator [11], H-monotone operator [10, 12], maximal η-monotone operator [8] and classical maximal monotone operators [28] in Hilbert spaces. We provide some examples and study some properties of general G-η-monotone operators. Moreover, the generalized proximal mapping associated with this type of monotone operator is defined and its Lipschitz continuity is established. Finally, using Lipschitz continuity of generalized proximal mapping under some conditions a new system of variational inclusions is solved.

1. Introduction

The variational inequality was introduced by Hartmann and Stampacchia [13] in 1966, and was later expanded by Stampacchia in several important papers; see, for example [22] and references therein. It is interesting to remark that in 1966 Karamardian also obtained existence results for variational inequalities in his Ph.D. dissertation [19], where

he used the fixed-point theory which is totally different from that used by Hartmann and Stampacchia [13].

In the early sixties we have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances and algorithmic developments. Variational inequality theory provides us with a simple, natural, general and unified framework for studying a wide class of unrelated problems arising in mechanics, physics, optimization, control, nonlinear programming, economics, transportation equilibrium, and engineering sciences; for more details, see [16, 17] and references therein. In recent years, variational inequalities have been extended and generalized in different directions using new, novel and innovative techniques, both for their own sake and for the applications. A useful and significant generalization of variational inequalities is set-valued variational inclusion. Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in many fields of pure and applied sciences.

Maximal monotone operators were first introduced in [20] and [27], and can be seen as a two-way generalization: a nonlinear generalization of linear endomorphisms with positive semi-definite matrices, and a multidimensional generalization of nondecreasing functions of a real variable; that is, of derivatives of convex and differentiable functions. Thus, not surprisingly, the main example of this kind of operator in a Banach space is the Frechet derivative of a smooth convex function, or, in the set-valued realm, the subdifferential of an arbitrary lower semi-continuous convex function. Monotone operators are the key ingredients of monotone variational inequalities and monotone variational inclusions, which extend to the realm of set-valued mappings the constrained convex minimization problem. More details can be found in [4, 18, 28] and references therein.

Proximal mapping and resolvent operator techniques play crucial roles in computing approximate solutions of generalized variational and quasi-variational inequalities, and generalized variational and quasi-variational inclusions. Rockafellar [21] used the resolvent operator associated with maximal monotone operators for solving the variational inclusion $0 \in T(x)$, where T is a maximal monotone operator on a Hilbert space. Fang and Huang [10] introduced the concept of H-monotone operators and resolvent operators associated with an H-monotone operator as a generalization of maximal monotone operators and resolvent operators

associated with maximal monotone operators. Moreover, as an extension of H-monotone operators, Fang and Huang introduced and studied a new class of monotone operators, the so-called (H, η)-monotone operators and then they studied a new system of variational inclusions involving (H, η)-monotone operators in Hilbert spaces. Further more, in [12], by the resolvent operator method associated with (H, η)-monotone operators due to Fang and Huang, the existence and uniqueness of solutions for a new system of variational inclusions are proved and also a new algorithm for approximating the solution of this system of variational inclusions is constructed and the convergence of the iterative sequence generated by this algorithm is discussed. Verma announced the notion of the A-monotone operators to the solvability of nonlinear variational inclusions and systems of nonlinear variational inclusions [23]. Very recently, Zhang in [29] and Verma in [24] independently introduced the new classes of (A, η)-monotone and G-η-monotone mappings.

The development of an efficient iterative algorithm to compute approximate solutions of variational inequalities and variational inclusions is interesting and important. One of the most efficient numerical techniques for solving variational inequalities and variational inclusions in Hilbert spaces is the projection method and its variant forms. Since the standard projection method strictly depends on the inner product property of Hilbert spaces, it can no longer be applied for variational inequalities and variational inclusions in Banach spaces. This fact motivates us to develop alterative methods to study iterative algorithms for approximating solutions of variational inclusions in Banach spaces. In this setting, Ding and Xia [7] introduced a new notion of J-proximal mapping for a nonconvex lower semicontinuous subdifferentiable proper function, and used it to study a class of completely generalized quasi-variational inequality in Banach spaces. Xia and Huang [26] introduced a new notion of general H-monotone operator, which generalizes the notions of J-proximal mapping [7] and H-monotone mapping [10] and defined proximal mapping associated with the general H-monotone operator, which is different from the resolvent operator associated with the H-accretive operator [9]. By using the proximal mapping, they introduced a new class of variational inclusion with the general H-monotone operator and constructed an iterative algorithm for solving this class of variational inclusion.

Motivated and inspired by the above recent research works, here the existence of a unique solution for a system of variational inclusions with G-η-monotone mappings is considered.

2. Preliminaries

Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces. Let X be a real Banach space with dual space X^* and let $\eta : X \times X \longrightarrow X$ be a single valued mapping. The mapping η is called γ-*Lipschitz continuous*, if there exists some $\gamma > 0$ such that $\|\eta(x, y)\| \leq \gamma\|x - y\|$, for all $x, y \in X$. For a set-valued map $T : X \multimap Y$, the *graph* of T is $Gph(T) = \{(x, y) : y \in T(x)\}$ and the *inverse* T^{-1} of T is $\{(y, x) : (x, y) \in Gph(T)\}$. For a real number c, let $cT = \{(x, cy) : (x, y) \in Gph(T)\}$. If S and T are any set-valued mappings, then define $S + T = \{(x, y + z) : (x, y) \in Gph(S), (x, z) \in Gph(T)\}$.

Definition 2.1. [2, 10, 24] *A single valued map $A : X \longrightarrow X^*$ is said to be*

   (a) *η-monotone, if $\langle A(x) - A(y), \eta(x, y)\rangle \geq 0$, for all $x, y \in X$.*
   (b) *r-strongly η-monotone, if there exists some constant $r > 0$ such that $\langle A(x) - A(y), \eta(x, y)\rangle \geq r\|x - y\|^2$, for all $x, y \in X$.*
   (c) *m-relaxed η-monotone, if there exists some constant $r > 0$ such that $\langle A(x) - A(y), \eta(x, y)\rangle \geq -m\|x - y\|^2$, for all $x, y \in X$.*
   (d) *δ-Lipschitz, if $\|A(x) - A(y)\| \leq \delta\|x - y\|$, for all $x, y \in X$.*

Definition 2.2. [2, 10, 24] *A set-valued map $T : X \multimap X^*$ is said to be*

   (a) *r-strongly monotone, if there exists some constant $r > 0$ such that $\langle x^* - y^*, x - y\rangle \geq r\|x - y\|^2$, for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.*
   (b) *r-strongly η-monotone, if there exists some constant $r > 0$ such that $\langle x^* - y^*, \eta(x, y)\rangle \geq r\|x - y\|^2$, for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.*
   (c) *m-relaxed monotone, if there exists some constant $m > 0$ such that $\langle x^* - y^*, x - y\rangle \geq -m\|x - y\|^2$, for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.*
   (d) *m-relaxed η-monotone, if there exists some constant $m > 0$ such that $\langle x^* - y^*, \eta(x, y)\rangle \geq -m\|x - y\|^2$, for all $x, y \in X$ and all $x^* \in T(x), y^* \in T(y)$.*

Definition 2.3. [10, 11] *Let $H : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping. A set-valued map $T : \mathcal{H} \multimap \mathcal{H}$ is said to be*

(a) *H-monotone operator, if it is monotone and $(H + \lambda T)(\mathcal{H}) = \mathcal{H}$, for all $\lambda > 0$.*
(b) *(H, η)-monotone operator, if it is η-monotone and $(H + \lambda T)(\mathcal{H}) = \mathcal{H}$, for all $\lambda > 0$.*

Definition 2.4. [30] *Let $\eta : X \times X \longrightarrow X^*$ be a single-valued mapping. The set-valued mapping $T : X \multimap X$ is said to be*

(a) *η-accretive, if $\langle u - v, \eta(x, y) \rangle \geq 0$, for all $x, y \in X$ and all $u \in T(x), v \in T(y)$,*
(b) *g-η-accretive, if T is η-accretive and $(g + \lambda T)(X) = X$, for every $\lambda > 0$.*

Definition 2.5. [5, 29] *Suppose $A, B : \mathcal{H} \longrightarrow \mathcal{H}$ are two single valued mappings. B is said to be s-strongly monotone with respect to A, if $\langle B(u) - B(v), A(u) - A(v) \rangle \geq s\|x - y\|^2$, for all $x, y \in \mathcal{H}$.*

Definition 2.6. [5, 29] *Suppose X is a nonempty set.*

(a) *The map $f : \mathcal{H} \times X \longrightarrow \mathcal{H}$ is said to be r-strongly η-monotone with respect to A in first argument, if $f(., x)$ is r-strongly monotone with respect to A, for all $x \in X$.*
(b) *The map $g : X \times \mathcal{H} \longrightarrow \mathcal{H}$ is said to be r-strongly η-monotone with respect to A in second argument, if $g(x, .)$ is r-strongly monotone with respect to A, for all $x \in X$.*

Definition 2.7. [5, 29] *Suppose X_1 and X_2 are two real Banach spaces.*

(a) *The map $f : X_1 \times X_2 \longrightarrow X_1$ is δ-Lipschitz continuous in first argument, if $f(., y)$ is δ-Lipschitz continuous, for all $y \in X_2$.*
(b) *The map $g : X_1 \times X_2 \longrightarrow X_2$ is δ-Lipschitz continuous in second argument, if $g(x, .)$ is δ-Lipschitz continuous, for all $x \in X_1$.*

3. Main results

Here, we introduce a new concept of the general A-monotone and general G-η-monotone operators, give the definition of the proximal mapping, and prove the Lipschitz continuity of the proximal mapping in Banach spaces. In terms of these results, we deal with existence of a unique solution for a system of variational inclusions with the G-η-monotone mappings.

Definition 3.1. [3] *Let $H : X \longrightarrow X^*$ be a single-valued mapping. The set-valued map $T : X \multimap X^*$ is said to be a general (H, η)-monotone operator, if T is η-monotone and $(H + \lambda T)(X) = X^*$ holds, for every $\lambda > 0$.*

Example 3.2. [3] *Let $X = \mathbb{R}$ and $\eta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\eta(x, y) = y^3 - x^3$. Consider the set-valued mapping $T : \mathbb{R} \multimap \mathbb{R}$, defined by*

$$T(x) = \begin{cases} \{-x - 1\} & x > 0 \\ \{-1, 1\} & x = 0 \\ \{-x + 1\} & x < 0. \end{cases}$$

Then, T is η-monotone and $(I + \lambda T)(\mathbb{R}) \neq \mathbb{R}$, for $\lambda \geq 1$. Therefore, T is not maximal η-monotone. On the other hand, for single valued mapping $H : \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$H(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0, \end{cases}$$

T is a general (H, η)-monotone operator.

Example 3.3. [3] *Let $X = \mathbb{R}$ and $\eta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $\eta(x, y) = x^3 - y^3$. Then, $T : \mathbb{R} \multimap \mathbb{R}$, defined by $T(x) = \{x\}$, is maximal η-monotone. Also, for single valued mapping $H : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $H(x) = x^2$, we have T is not a general (H, η)-monotone operator.*

Definition 3.4. *A set-valued map $T : X \multimap X^*$ satisfying $(A + \lambda T)(X) = X^*$ is said to be*

    (a) general A-monotone mapping, if it is m-relaxed monotone,

    (b) general G-η-monotone mapping, if it is m-relaxed η-monotone.

Example 3.5. [29] *Let $X = \mathbb{R}$, $T(x) = 2x$, $A(x) = x^3$, $\eta(x, y) = y - x$, for all $x, y \in X$. Then, T is G-η-monotone and also clearly it is not g-η-accretive.*

Example 3.6. [29] *Let $X = \mathbb{R}$,*

$$T(x) = \begin{cases} [-\frac{\sqrt[4]{\lambda^3}}{2}, \frac{\sqrt[4]{\lambda^3}}{2}] & x = 0 \\ \{\frac{1}{x}\} & x \neq 0, \end{cases}$$

$A(x) = x^3$, $\eta(x, y) = xy(x - y)$, for all $x, y \in X$. Then, T is G-η-monotone, but it is not A-monotone and H-monotone.

Theorem 3.7. *Suppose $A : X \longrightarrow X^*$ is an r-strongly η-monotone mapping and $T : X \multimap X^*$ is a general G-η-monotone mapping. Then, $(A + \lambda T)^{-1} : X^* \longrightarrow X$ is single-valued, for $0 < \lambda < \frac{r}{m}$.*

Proof. Let $x^* \in X^*$, and x and y be two distinct elements in $(A + \lambda T)^{-1}(x^*)$. Then, $x^* \in (A + \lambda T)(x)$ and $x^* \in (A + \lambda T)(y)$. Hence, there exist $t^* \in T(x)$ and $s^* \in T(y)$, for which $x^* = A(x) + \lambda t^*$ and $x^* = A(y) + \lambda s^*$. Since A is r-strongly η-monotone and since $-\frac{r}{\lambda} < -m$, so

$$
\begin{aligned}
\langle t^* - s^*, \eta(x, y) \rangle &= \langle \frac{x^* - A(x)}{\lambda} - \frac{x^* - A(y)}{\lambda}, \eta(x, y) \rangle \\
&= -\frac{1}{\lambda} \langle A(x) - A(y), \eta(x, y) \rangle \\
&\leq -\frac{r}{\lambda} \| x - y \|^2 \\
&< -m \| x - y \|^2.
\end{aligned}
$$

This contradicts the fact that T is m-relaxed η-monotone. $\square$

Definition 3.8. *Suppose $A : X \longrightarrow X^*$ is an r-strongly η-monotone mapping and $T : X \multimap X^*$ is a general G-η-monotone mapping. For any $\lambda > 0$, the generalized proximal mapping $R_{T,\lambda}^{A,\eta} : X^* \longrightarrow X$ is defined by $R_{T,\lambda}^{A,\eta}(x^*) = (A + \lambda T)^{-1}(x^*)$.*

Remark 3.9. For appropriate and suitable choices of X, A, T and η one can obtain many known resolvent operators and proximal mappings considered in the recent literature; for example, see [1, 2, 3, 10, 11, 12, 14, 23, 24, 25, 26, 28, 29] and references therein.

Theorem 3.10. *Let $A : X \longrightarrow X^*$ be an r-strongly η-monotone single-valued mapping, $\eta : X \times X \longrightarrow X$ be γ-Lipschitz continuous and $T : X \multimap X^*$ be a general G-η-monotone mapping. Then, the generalized proximal mapping $R_{T,\lambda}^{A,\eta}$ is $\frac{\gamma}{r - \lambda m}$-Lipschitz continuous.*

Proof. For any two points $x^*, y^* \in X^*$, with $\| R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*) \| \neq 0$, since

$$
R_{T,\lambda}^{A,\eta}(x^*) = (A + \lambda T)^{-1}(x^*) \text{ and } R_{T,\lambda}^{A,\eta}(y^*) = (A + \lambda T)^{-1}(y^*),
$$

we have

(3.1)
$$
\frac{x^* - A(R_{T,\lambda}^{A,\eta}(x^*))}{\lambda} \in T(R_{T,\lambda}^{A,\eta}(x^*))
$$

and

(3.2) $$\frac{y^* - A(R_{T,\lambda}^{A,\eta}(y^*))}{\lambda} \in T(R_{T,\lambda}^{A,\eta}(y^*)).$$

T is a general G-η-monotone mapping, and thus (3.1) and (3.2) imply that

$$\langle \frac{x^* - A(R_{T,\lambda}^{A,\eta}(x^*))}{\lambda} - \frac{y^* - A(R_{T,\lambda}^{A,\eta}(y^*))}{\lambda} \, , \, \eta(R_{T,\lambda}^{A,\eta}(x^*), R_{T,\lambda}^{A,\eta}(y^*))\rangle$$
$$\geq -m\|R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*)\|^2.$$

Therefore,

$$\langle x^* - y^*, \quad \eta(R_{T,\lambda}^{A,\eta}(x^*), R_{T,\lambda}^{A,\eta}(y^*))\rangle \geq \langle A(R_{T,\lambda}^{A,\eta}(x^*)) - A(R_{T,\lambda}^{A,\eta}(y^*)),$$
$$\eta(R_{T,\lambda}^{A,\eta}(x^*), R_{T,\lambda}^{A,\eta}(y^*))\rangle - \lambda m\|R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*)\|^2.$$

Since $\eta : X \times X \longrightarrow X$ is γ-Lipschitz continuous and A is r-strongly η-monotone,

$$\begin{aligned}
\gamma\|x^* - y^* \quad \| \quad \|R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*)\| \\
\geq \quad \|x^* - y^*\|\|\eta(R_{T,\lambda}^{A,\eta}(x^*), R_{T,\lambda}^{A,\eta}(y^*))\| \\
\geq \quad \langle x^* - y^*, \eta(R_{T,\lambda}^{A,\eta}(x^*), R_{T,\lambda}^{A,\eta}(y^*))\rangle \\
\geq \quad \langle A(R_{T,\lambda}^{A,\eta}(x^*)) - A(R_{T,\lambda}^{A,\eta}(y^*)), \eta(R_{T,\lambda}^{A,\eta}(x^*), R_{T,\lambda}^{A,\eta}(y^*))\rangle \\
- \lambda m\|R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*)\|^2 \\
\geq \quad (r - \lambda m)\|R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*)\|^2.
\end{aligned}$$

That $R_{T,\lambda}^{A,\eta}$ is $\frac{\gamma}{r+\lambda\beta}$-Lipschitz continuous follows from the fact that $\|R_{T,\lambda}^{A,\eta}(x^*) - R_{T,\lambda}^{A,\eta}(y^*)\| \neq 0$. $\square$

Problem 3.11. *Consider single valued mappings $\eta_i : X_1 \times X_2 \longrightarrow X_i$, $A_i : X_i \longrightarrow X_i^*$ and $S_i : X_1 \times X_2 \longrightarrow X_i^*$, for $i = 1, 2$. Also, suppose that the set-valued mappings $T_i : X_i \multimap X_i^*$ are general G-η_i-monotone operator. Our problem is finding $(x_1, x_2) \in X_1 \times X_2$, for which*

$$\begin{cases} 0 \in S_1(x_1, x_2) + T_1(x_1) \\ 0 \in S_2(x_1, x_2) + T_2(x_2). \end{cases}$$

Remark 3.12. By appropriate and suitable choices of X_1, X_2, A_1, A_2, $S_1, S_2, \eta_1, \eta_2, T_1$ and T_2 one can obtain many known and new classes of variational inequalities and variational inclusions as special cases of the Problem 3.11. For example, see [1, 6, 12, 15, 25] and references therein.

Theorem 3.13. *Suppose $A_i : X_i \longrightarrow X_i$ is an r_i-strongly η_i-monotone operator and $T_i : X_i \multimap X_i$ is a general G-η_i-monotone operator, for $i = 1, 2$. Define $F : X_1 \times X_2 \longrightarrow X_1$, $G : X_1 \times X_2 \longrightarrow X_2$ and $Q : X_1 \times X_2 \longrightarrow X_1 \times X_2$, respectively, by $F(x_1, x_2) = R_{T_1,\lambda}^{A_1,\eta_1}[A_1(x_1) - \lambda S_1(x_1, x_2)]$, $G(x_1, v) = R_{T_2,\mu}^{A_2,\eta_2}[A_2(x_2) - \mu S_2(x_1, x_2)]$ and $Q(x_1, x_2) = (F(x_1, x_2), G(x_1, x_2))$, for all $(x_1, x_2) \in X_1 \times X_2$. Then, the following statements are equivalent.*

(a) *$(x_1, x_2) \in X_1 \times X_2$ is a solution of* Problem 3.11.
(b) *$x_1 = R_{T_1,\lambda}^{A_1,\eta_1}[A_1(x_1) - \lambda S_1(x_1, x_2)]$ and $x_2 = R_{T_2,\mu}^{A_2,\eta_2}[A_2(x_2) - \mu S_2(x_1, x_2)]$.*
(c) *(x_1, x_2) is a fixed point of Q.*

Proof. These are immediate consequences of Definition 3.8.

Theorem 3.14. *Suppose X_1 and X_2 are two real Banach spaces. Also, suppose, for $i = 1, 2$,*

(a) *$\eta_i : X_i \times X_i \longrightarrow X_i$ is γ_i-Lipschitz continuous,*
(b) *$A_i : X_i \longrightarrow X_i$ is r_i-strongly η_i-monotone and θ_i-Lipschitz continuous,*
(c) *$T_i : X_i \multimap X_i$ is general G-η_i-monotone operator,*
(d) *$S_i : X_1 \times X_2 \longrightarrow X_i$ is δ_i-Lipschitz continuous in first argument and ξ_i-Lipschitz continuous in second argument. If*

$$
\begin{cases}
\frac{\gamma_1}{r_1 - \lambda m}(\theta_1 + \lambda \delta_1) + \frac{\gamma_2 \mu \xi_2}{r_2 - \rho m} < 1, & 0 < \lambda < \frac{r_1}{m} \\
\frac{\gamma_1 \lambda \xi_1}{r_1 - \lambda m} + \frac{\gamma_2}{r_2 - \mu m}(\theta_2 + \mu \delta_2) < 1, & 0 < \mu < \frac{r_2}{m},
\end{cases}
$$

then Problem 3.11 *has a unique solution.*

Proof. Let F, G and Q be defined as in Theorem 3.13. For any elements (x_1, x_2) and (x_1', x_2') in $X_1 \times X_2$, it follows from Theorem 3.10 that

$$
\begin{aligned}
\| \ &F(x_1, x_2) - F(x_1', x_2')\| = \\
&\|R_{T_1,\lambda}^{A_1,\eta_1}[A_1(x_1) - \lambda S_1(x_1, x_2)] - R_{T_1,\lambda}^{A_1,\eta_1}[A_1(x_1') - \lambda S_1(x_1', x_2')]\| \\
&\leq \frac{\gamma_1}{r_1 - \lambda m}(\|A_1(x_1) - A_1(x_1')\| + \lambda\|S_1(x_1, x_2) - S_1(x_1', x_2')\|) \\
&\leq \frac{\gamma_1}{r_1 - \lambda m}(\theta_1\|x_1 - x_1'\| + \lambda\|S_1(x_1, x_2) - S_1(x_1', x_2)\| \\
&\qquad\qquad\qquad\qquad +\lambda\|S_1(x_1', x_2) - S_1(x_1', x_2')\|) \\
&\leq \frac{\gamma_1}{r_1 - \lambda m}(\theta_1\|x_1 - x_1'\| + \lambda\delta_1\|x_1 - x_1'\| + \lambda\xi_1\|x_2 - x_2'\|)
\end{aligned}
$$

$$= \frac{\gamma_1}{r_1 - \lambda m}(\theta_1 + \lambda \delta_1)\|x_1 - x_1'\| + \frac{\gamma_1 \lambda \xi_1}{r_1 - \lambda m}\|x_2 - x_2'\|,$$

that is,

$\|F(x_1, x_2) - F(x_1', x_2')\|$

(3.3) $\leq \quad \frac{\gamma_1}{r_1 - \lambda m}(\theta_1 + \lambda \delta_1)\|x_1 - x_1'\| + \frac{\gamma_1 \lambda \xi_1}{r_1 - \lambda m}\|x_2 - x_2'\|.$

Similarly, one can deduce that

$\|G(x_1, x_2) - G(x_1', x_2')\| \leq$

(3.4) $\frac{\gamma_2 \mu \xi_2}{r_2 - \mu m}\|x_1 - x_1'\| + \frac{\gamma_2}{r_2 - \mu m}(\theta_2 + \mu \delta_2)\|x_2 - x_2'\|.$

Set

(3.5) $k := \max\{\frac{\gamma_1}{r_1 - \lambda m}(\theta_1 + \lambda \delta_1) + \frac{\gamma_2 \mu \xi_2}{r_2 - \mu m},$

$$\frac{\gamma_1 \lambda \xi_1}{r_1 - \lambda m} + \frac{\gamma_2}{r_2 - \mu m}(\theta_2 + \mu \delta_2)\}.$$

Equip $X_1 \times X_2$ with $\|(x_1, x_2)\|_\times = \|x_1\| + \|x_2\|$, for all $(x_1, x_2) \in X_1 \times X_2$. It is well known that $(X_1 \times X_2, \|(.,.)\|_\times)$ is a Banach space. On the other hand, $\|Q(x_1, x_2) - Q(x_1', x_2')\|_\times = \|F(x_1, x_2) - F(x_1', x_2')\| + \|G(x_1, x_2) - G(x_1', x_2')\|$. It follows from assumption, (3.3), (3.4) and (3.5) that Q is a contraction map. Now, the Banach contraction theorem implies that Q has a unique fixed point. That Problem 3.11 has a unique solution, follows from Theorem 3.13.

In the rest of our work, the $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ are single valued mappings, for $i = 1, 2$. As a corollary of Theorem 3.14, we can resolve the following problem which was considered in [1]. $\square$

Problem 3.15. *Consider single valued mappings $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$, $A_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ and $S_i : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_i$, for $i = 1, 2$. Also, consider the set-valued mappings $T_i : \mathcal{H}_i \multimap \mathcal{H}_i$. Suppose T_i is an (A_i, η_i)-monotone operator, for $i = 1, 2$. Our problem is finding $(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$, for which*

$$\begin{cases} 0 \in S_1(u, v) + T_1(u) \\ 0 \in S_2(u, v) + T_2(v). \end{cases}$$

Corollary 3.16. [1] *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces. Also, suppose for $i = 1, 2$,*

(a) *$\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is a γ_i-Lipschitz continuous mapping,*

(b) *$A_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is r_i-strongly η_i-monotone and θ_i-Lipschitz continuous operator,*

    (c) $T_i : \mathcal{H}_i \multimap \mathcal{H}_i$ is an (A_i, η_i)-monotone operator,

    (d) $S_i : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_i$ is δ_i-Lipschitz continuous in the ith argument and s_i-strongly monotone with respect to A_i in the ith argument,

    (e) S_1 is ξ_1-Lipschitz continuous in second argument and S_2 is ξ_2-Lipschitz continuous in first argument. If

$$
\begin{cases}
\frac{\gamma_1}{r_1 - \lambda m} \sqrt{\theta_1^2 - 2\lambda s_1 + \lambda^2 \delta_1^2} + \frac{\mu \gamma_2 \xi_2}{r_2 - \mu m} < 1, & 0 < \lambda < \frac{r_1}{m} \\
\frac{\lambda \gamma_1 \xi_1}{r_1 - \lambda m} + \frac{\gamma_2}{r_2 - \mu m} \sqrt{\theta_2^2 - 2\mu s_2 + \mu^2 \delta_2^2} < 1, & 0 < \mu < \frac{r_2}{m},
\end{cases}
$$

then Problem 3.15 *has a unique solution.*

Corollary 3.17. [12] *Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous operator with constant σ (see Remark 3.18). Let $H_1 : \mathcal{H}_1 \to \mathcal{H}_1$ be a strongly η-monotone, Lipschitz continuous operator with constants γ_1 and τ_1, and $H_2 : \mathcal{H}_2 \to \mathcal{H}_2$ be a strongly η-monotone, Lipschitz continuous operator with constants γ_2 and τ_2. Let $M : \mathcal{H}_1 \multimap \mathcal{H}_1$ be (H_1, η)-monotone and $N : \mathcal{H}_2 \multimap \mathcal{H}_2$ be (H_2, η)-monotone. Let $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ be an operator, such that, for any given $(a, b) \in \mathcal{H}_1 \times \mathcal{H}_2$, $F(., b)$ is strongly monotone with respect to H_1 and Lipschitz continuous with constants r_l and s_l, respectively, and $F(a, .)$ is Lipschitz continuous with the constant θ. Let $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be an operator, such that, for any given $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $G(x, .)$ is strongly monotone with respect to H_2 and Lipschitz continuous with constant r_2 and s_2, respectively, and $G(., y)$ is Lipschitz continuous with the constant ξ. Let there exist constants $\rho > 0$ and $\lambda > 0$ such that*

$$
\gamma_2 \sigma \sqrt{\tau_1^2 - 2\rho r_1 + \rho^2 s_1^2} + \gamma_1 \sigma \lambda \xi < \gamma_1 \gamma_2
$$

$$
\gamma_1 \sigma \sqrt{\tau_2^2 - 2\lambda r_2 + \lambda^2 s_2^2} + \gamma_2 \sigma \rho \theta < \gamma_1 \gamma_2.
$$

Then, the following system admits a unique solution:

$$
\begin{cases}
0 \in F(a, b) + M(a) \\
0 \in G(a, b) + N(b).
\end{cases}
$$

Remark 3.18. In the context of Theorem 4.1 in [12], there is a minor mistake. In fact, they should use $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$, instead of $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$.

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