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# PROJECTED DYNAMICAL SYSTEMS AND OPTIMIZATION PROBLEMS

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Abstract. We establish a relationship between general constrained pseudoconvex optimization problems and globally projected dynamical systems. A corresponding novel neural network model, which is globally convergent and stable in the sense of Lyapunov, is proposed. Both theoretical and numerical approaches are considered. Numerical simulations for three constrained nonlinear optimization problems are given to show that the numerical behaviors are in good agreement with the theoretical results.

## 1. Introduction

Projected dynamical system theory is a mathematical device for investigating the behaviors of dynamical systems, where solutions are restricted to a constraint set. The discipline shares connections to and applications with both the static world of optimization and equilibrium problems and the dynamical world of ordinary differential equations. A globally projected dynamical system is given by the flow to the projected

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differential equation [11],

$$\frac{dx}{dt} = P_{\Omega}(x - \alpha F(x)) - x, \qquad \forall x \in R^n.$$

where $\Omega$ is the constraint set. Globally projected dynamical systems have been introduced as models for describing network disequilibria (see Friesz et al., 1994) and have been applied to neural networks for solving a class of optimization problems (see Xia et al., 2000). In recent years, Malek and co-authors proposed various neural network models for linear [7], quadratic [8] and nonlinear [14] convex problems with different types of constraints. Here, a specific neural network model based on globally projected dynamical system, is proposed in order to solve general constrained pseudoconvex optimization problems. The motivation behind solving mathematical programming problems by neural networks is that these models may be easily implemented by a circuit [12]. Thus, proving the stability of such systems is of great importance for scientists and engineers. The proposed neural network is shown to be globally convergent and stable in the sense of Lyapunov under specific conditions.
The remainder of our work is organized as follows. In the next section, preliminary information are given. In Section 3, the problem formulation and neural network models are described. The convergence and stability of the proposed neural network are discussed in Section 4. In Section 5, illustrative examples are worked out. Finally, Section 6 gives the conclusion.

## 2. Preliminaries

This section provides the necessary mathematical background, which is used to propose the desired neural network and to study its stability and convergence.

**Definition 2.1.** *Consider the dynamical system*

$$\dot{x} = H(x(t)), \qquad x(t_0) = x_0 \in R^n. \tag{2.1}$$

*A point $\bar{x}$ is called an equilibrium point, critical point or steady state of the dynamical system, if $H(\bar{x}) = 0$. If, furthermore, there is a neighborhood $N \subset R^n$ of $\bar{x}$ such that $H(\bar{x}) = 0$ and $H(x) \neq 0$, for $x \in N \backslash \{\bar{x}\}$, then $\bar{x}$ is called an isolated equilibrium point.*

**Definition 2.2.** *Let $\Omega \subseteq R^n$ be an open neighborhood of $\bar{x}$. A continuously differentiable function $\omega : R^n \to R$ is said to be a Lyapunov function at the state $\bar{x}$ over the set $\Omega$ for (2.1), if*

$$\begin{cases} \omega(\bar{x}) = 0 \\ \frac{d\omega(x(t))}{dt} = [\nabla\omega(x(t))]^T H(x(t)) \leq 0, \quad \forall x \in \Omega. \end{cases}$$

**Definition 2.3.** *An isolated equilibrium point $\bar{x}$ is Lyapunov stable, if for any $x_0 = x(t_0)$ and any $\varepsilon > 0$, there exists a $\delta > 0$ so that if $\|x_0 - \bar{x}\| < \delta$, then $\|x(t) - \bar{x}\| < \varepsilon$, for $t \geq t_0$.*

**Definition 2.4.** *Variational inequality problem is to find $x^*$ such that*

$$VI(F, \Omega) : \qquad \langle F(x^*), (x - x^*)\rangle \geq 0, \qquad \forall x \in \Omega, \tag{2.2}$$

*where $\Omega$ is a closed convex subset of $R^n$ and $F : R^n \to R^n$ is a continuous map.*

**Definition 2.5.** *The dynamical system (2.1) is said to be globally convergent to the solution set $\Omega^* = \{x | x \quad \text{solves } (2.2)\}$, if every solution of the system satisfies $\lim_{t\to\infty} dis(x(t), \Omega^*) = 0$, where*
*$dis(x, \Omega^*) = \inf_{y\in\Omega^*} \|x - y\|$ and $\|.\|$ denotes the Euclidean norm.*

**Definition 2.6.** *Let $\Omega$ be a closed convex set in $R^n$. Then, for each $x \in R^n$, there exists a unique point $y \in \Omega$ such that $\|x - y\| \leq \|x - z\|$, $\forall z \in \Omega$. The projection of $x$ on the set $\Omega$ with respect to the Euclidean norm is $y = P_\Omega(x) = \arg\min_{z\in\Omega} \|x - z\|$.*

**Definition 2.7.** *A nonlinear mixed complementarity problem is to find a point $x \in R^n$ such that*

$$NMC(F) : \begin{cases} x_i F_i(x) = 0, \quad F_i(x) \geq 0, \quad x_i \geq 0, & \forall i \in I \\ F_i(x) = 0, & \forall i \in N \backslash I, \end{cases}$$

*where $F$ is a continuously differentiable mapping from*
*$X = \{x \in R^n | x_i \geq 0, i \in I\}$ into $R^n$, $N = \{1, 2, ..., n\}$ and $I \subseteq N$.*

**Definition 2.8.** *A differentiable function $f : R^n \to R$ is pseudoconvex on $\Omega$, if for every pair of distinct points $x, y \in \Omega$,*

$$\nabla f(x)^T (y - x) \geq 0 \Rightarrow f(y) \geq f(x).$$

**Theorem 2.9.** [9]. *Consider the dynamical system (2.1) and assume that* $H : R^n \to R^n$ *is a continuous mapping. Then, for arbitrary* $t_0 \geq 0$ *and* $x_0 \in R^n$*, there exists a local solution* $x(t)$, $t \in [t_0, \tau)$*, for (2.1), for some* $\tau > t_0$*. If, furthermore,* $H$ *is locally Lipschitz continuous at* $x_0$*, then the solution is unique. And if* $H$ *is Lipschitz continuous in* $R^n$*, then* $\tau$ *can be extended to infinity.*

**Lemma 2.10.** [9]. *An isolated equilibrium point* $\bar{x}$ *is Lyapunov stable if there exists a Lyapunov function over a neighborhood of* $\bar{x}$.

**Definition 2.11.** *Consider the function* $\Phi : E \subset R^n \to R$*. For any arbitrary constant* $\beta > 0$*, the set* $L_\beta = \{x \in E \mid \Phi(x) < \beta\}$ *is called a level set of* $\Phi$.

**Lemma 2.12.** [10]. *Let* $\Phi : D \subset R^n \to R$*, where* $D$ *is unbounded. Then, all level sets of* $\Phi$ *are bounded if and only if* $\lim_{k\to\infty} \Phi(x^k) = \infty$*, whenever* $\{x^k\} \subset D$ *and* $\lim_{k\to\infty} \|x^k\| = \infty$.

## 3. Problem formulation and neural network model

Consider the following general optimization problem:

$$Min\ f(x) \qquad s.t. \qquad g(x) \leq 0, \qquad h(x) = 0, \tag{3.1}$$

where $f(x)$ is continuously differentiable and a pseudoconvex function, $g : R^n \to R^m$ and $h : R^n \to R^r$ are continuously differentiable vector-valued functions. From now on, we make the assumptions that $g$ and $h$ are convex and linear functions. This pseudoconvex optimization problem is related intimately to a certain finite-dimensional variational inequality problem. The following well-known result reveals the relationship between these two types of problems.

**Lemma 3.1.** [6]. *Let* $\Omega$ *be a closed convex subset of* $R^n$ *and* $f : R^n \to R$ *be differentiable and pseudoconvex on* $\Omega$*. Then,* $x^* \in \Omega$ *satisfies* $\nabla f(x^*)^T(x - x^*) \geq 0, \quad \forall x \in \Omega$*, if and only if* $x^*$ *is a minimum of* $f(x)$ *in* $\Omega$.

According to Lemma 2, the optimization problem converts to the following variational inequalities:

$$\langle \nabla f(x^*), (x - x^*)\rangle \geq 0, \qquad \forall x \in S, \tag{3.2}$$

where $S = \{x \in R^n | g(x) \leq 0, h(x) = 0\}$.

**Lemma 3.2.** [2]. *$x^* \in S$ is a solution of problem (3.2) if and only if there exists $y^* \geq 0$, $y^* \in R^m$ and $z^* \in R^r$ such that $u^* = ((x^*)^T, (y^*)^T, (z^*)^T)^T$ is a solution of $NMC(G)$, where $G : s_0 \to R^{n+m+r}$ is defined by*

$$G(u) = \begin{pmatrix} \nabla_x f(x) + \nabla_x g(x)^T y + \nabla_x h(x)^T z \\ -g(x) \\ -h(x) \end{pmatrix},$$

*with $s_0 = \{u = (x^T, y^T, z^T)^T | y \geq 0\}$.*

**Theorem 3.3.** *$u^* \in s_0$ is a solution of $NMC(G)$ if and only if it is a solution of*

$$\langle G(u^*), u - u^*\rangle \geq 0, \qquad \forall u \in s_0. \tag{3.3}$$

*proof.* First, note that if $u^*$ is a solution of $NMC(G)$, then

$$\langle G(u^*), u\rangle \geq 0, \quad \forall u \in s_0.$$

Thus,

$$\langle G(u^*), u - u^*\rangle = \langle G(u^*), u\rangle - \langle G(u^*), u^*\rangle = \langle G(u^*), u\rangle \geq 0.$$

Second, suppose that $u^*$ is a solution to the variational inequality (3.3). Then, $v = u^* + e_i$, $e_i = (0, ..., 0, 1, 0, ..., 0)^T$ (1 in the $i$th place) is an element of $s_0$. Thus,

$$\begin{aligned} \langle G(u^*), u^* + e_i - u^*\rangle &= \langle G(u^*), e_i\rangle \\ &= G_i(u^*) \geq 0, \ \forall i \in N = \{1, 2, ..., n + m + r\}, \end{aligned}$$

Therefore,

$$G(u^*) \geq 0. \tag{3.4}$$

Now, let $v' = x^* - e'_i, e'_i = (0, ..., 0, 1, 0, ..., 0)^T$ (1 in the $i$th place), $i \in N \backslash I$, and $I = \{n + 1, ..., n + m\}$. It is obvious that $v' \in s_0$. Thus,

$$\langle G(u^*), u^* - e_i - u^*\rangle = \langle G(u^*), -e_i\rangle = -G_i(u^*) \geq 0$$

or

$$G_i(u^*) \leq 0, \qquad \forall i \in N \backslash I.$$

Using relation (3.4) implies that $G_i(u^*) = 0$, for $i \in N \backslash I$.
Let $0 \in s_0$. Then, $0 \leq \langle G(u^*), 0 - u^* \rangle = -\langle G(u^*), u^* \rangle$. Thus,

$$\langle G(u^*), u^* \rangle \leq 0$$

which means that

$$\langle G(u^*), u^* \rangle = \sum_{i \in I} G_i(u^*) u_i^* + \sum_{i \in N \backslash I} G_i(u^*) u_i^* = \sum_{i \in I} G_i(u^*) u_i^* \leq 0. \tag{3.5}$$

According to (3.4), $G(u^*) \in R_+^n$ and $u^* \in s_0$. Hence,

$$G_i(u^*) u_i^* \geq 0, \qquad \forall i \in I. \tag{3.6}$$

Finally, from (3.5) and (3.6), we obtain that $u_i^* G_i(u^*) = 0$, for all $i \in I$.

**Remark 3.4.** *Let $s_0$ be the set as defined in Lemma 3.2. Then, by Definition 2.6,*

$$P_{s_0}(u) = (x_1, ..., x_n, (y_1)^+, ..., (y_m)^+, z_1, ..., z_r),$$

*where $(y_i)^+ = max\{0, y_i\}$.*

**Lemma 3.5.** [3]. *The vector $u^* \in s_0$ is a solution of the variational inequality (3.3) if and only if for any arbitrary constant $\alpha \geq 0$, it satisfies the relation*

$$u^* = P_{s_0}(u^* - \alpha G(u^*)).$$

According to the above procedure, we propose the following projection neural network model for solving the problem (3.1):

$$\frac{du}{dt} = \frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} -(\nabla_x f(x) + \nabla_x g(x)^T y + \nabla_x h(x)^T z) \\ (y + g(x))^+ - y \\ h(x) \end{pmatrix}, \tag{3.7}$$

where $\lambda$ is a positive scaling factor. The proposed neural network has one layer structure with the low model complexity.

## 4. **Stability and convergence analysis**

In this section, we study some basic properties of the dynamic system (3.7) and prove its global convergence and Lyapunov stability. For complete analysis, first we prove the existence of the solution for the ordinary differential equation (3.7).

**Theorem 4.1.** *For any initial point $u(t_0) = u_0 \in R^{n+m+r}$, there exists a unique solution $u(t)$ for the proposed neural network model. Moreover, if $u_0 \in s_0$ then, $u(t) \in s_0$.*

*proof.* Let $k(t) = P_{s_0}(u - G(u)) - u$. Then, $\nabla k(u) = -I_{n+m+r}$, where $I_{n+m+r}$ is $(n+m+r)\times(n+m+r)$ identity matrix. So, $\|k(u) - k(u')\| = \|\nabla_u k(\hat{u})(u - u')\| \leq \|u - u'\|, \quad \forall u, u' \in s_0$, where $\hat{u} = \lambda' u + (1 - \lambda')u', \quad \lambda' \in [0, 1]$. Hence, $k(t)$ is a locally Lipschitz continuous function. According to Theorem 2.9, there exists a unique solution $u(t)$, $t \in [t_0, \tau)$, for (3.7), for some $\tau > t_0$. Next, it is sufficient to show that if $u_0 \in s_0$, then $y(t) \geq 0$. Since $\frac{dy}{dt} + y = \lambda(y + g(x))^+$, we have

$$\int_{t_0}^{t} (\frac{dy}{dt} + y)e^s ds = \lambda \int_{t_0}^{t} (y + g(x))^+ e^s ds,$$

$$e^s y(s)|_{t_0}^{t} = \lambda \int_{t_0}^{t} (y + g(x))^+ e^s ds,$$

$$y(t) = e^{-(t-t_0)} y(t_0) + \lambda e^{-t} \int_{t_0}^{t} (y + g(x))^+ e^s ds \geq 0.$$

**Lemma 4.2.** [6]. *Assume that the set $K$ is closed convex in $R^n$. We have the followings:*
*(i) For any $w \in R^n$ and any $v \in K$,*

$$(P_K(w) - w)^T (P_K(w) - v) \geq 0.$$

*(ii) For any $w, v \in R^n$,*

$$\|P_K(w) - P_K(v)\| \leq \|w - v\|.$$

**Theorem 4.3.** *If $\nabla_x^2 f(x) + \nabla_x^2 g(x)^T y + \nabla_x^2 h(x)^T z$ is positive definite on $s_0$, then the proposed neural network model is stable in the sense of Lyapunov and globally convergent to a stationary point $u^* = ((x^*)^T, (y^*)^T, (z^*)^T)^T$ of (3.7), where $x^*$ is a solution of the problem (3.1).*

*Proof.* Let $u(t)$ be a solution of (3.7) with the initial point $u_0 \in s_0$. We have $u(t) \in s_0$, using Theorem 4.1. suppose that $G(u)$ be the function as defined in Lemma 3.1. Consider the following function on $s_0$:

$$V(u(t)) = \frac{1}{2}\{P_{s_0}(u - \alpha G(u)) - u\}^T\{P_{s_0}(u - \alpha G(u)) - (u - G(u)) + G(u)\} - \frac{1}{2}\|u(t) - u^*\|^2. \tag{4.1}$$

In order to prove the Lyapunov stability of (3.7), according to Lemma 2.10, it is sufficient to show that $V(u(t))$ is a Lyapunov function at state $u^*$ over the set $s_0$. First, it is obvious that $V(u^*) = 0$. Second, let $b(u) = P_{s_0}(u - G(u))$, for any $\lambda \geq 0$. Then,

$$\frac{dV(u)}{dt} = \left(\frac{dV(u)}{du}\right)^T \frac{du}{dt} = \lambda\{G(u) - (\nabla G(u) - I)b(u) + (u - u^*)\}^T b(u),$$

or

$$\frac{dV(u)}{dt} = \lambda\{G(u) + (u - u^*)\}^T b(u) + \lambda\|b(u)\|^2 - \lambda b(u)^T \nabla G(u) b(u), \tag{4.2}$$

where,

$$\nabla G(u) = \begin{pmatrix} \nabla_x^2 f(x) + \nabla_x^2 g(x)^T y + \nabla_x^2 h(x)^T z & \nabla_x g(x) & \nabla_x h(x) \\ -\nabla_x g(x) & 0_{m\times m} & 0_{r\times r} \\ -\nabla_x h(x) & 0_{m\times m} & 0_{r\times r} \end{pmatrix},$$

and 0 is used for zero matrices of appropriate dimensions. It is obvious that $\nabla G(u)$ is asymmetric matrix and since $\nabla_x^2 f(x) + \nabla_x^2 g(x)^T y + \nabla_x^2 h(x)^T z$ is positive definite, we can see that $\nabla G(u)$ is positive definite.

In the first inequality of Lemma 4.2, let $w = u - G(u)$, $v = u^*$ and $K = s_0$. We get

$$(b(u) + (u - u^*))^T(-b(u) - G(u)) \geq 0,$$

or

$$(G(u) + (u - u^*))^T b(u) \leq -G(u)^T(u - u^*) - \|b(u)\|^2. \tag{4.3}$$

Thus, from (4.2) and (4.3), we have

$$\frac{dV(u(t))}{dt} \leq -\lambda G(u)^T(u - u^*) - \lambda\|b(u)\|^2 + \lambda\|b(u)\|^2 - \lambda b(u)^T \nabla G(u) b(u)$$

$$= -\lambda G(u)^T(u - u^*) - \lambda b(u)^T \nabla G(u) b(u), \tag{4.4}$$

that is,

$$\frac{dV(u(t))}{dt} \leq -\lambda b(u)^T \nabla G(u) b(u) \leq 0. \tag{4.5}$$

This shows the Lyapunov stability of the proposed neural network model (3.7).
Next, by using the first inequality of Lemma 3.5, when $w = u - G(u)$, $v = u$ and $K = s_0$, we have

$$G(u)^T b(u) \leq - \|b(u)\|^2 . \tag{4.6}$$

Now, from (4.1) and (4.6), we may write:

$$V(u(t)) \geq \frac{1}{2} \|u - u^*\|^2 .$$

Hence, $V(u(t_k)) \to \infty$, as $\|u(t_k)\| \to \infty$. So, by Lemma 2.12, all the level sets of $V$ are bounded.
According to inequality (4.5),

$$\{u(t)\} \subset X_0 = \{u \in s_0 \mid V(u) \leq V(u_0)\},$$

and thus for any initial point $u_0 \in s_0$, the solution trajectory $u(t)$ is bounded. By the invariant set Theorem [9], we see that all the solution trajectories of (3.7) converge to largest invariant set $\prod$, where $dV(u(t))/dt = 0$. We now prove that $dV/dt = 0 \Leftrightarrow du/dt = 0$. Clearly, if $du/dt = 0$, then $dV(u(t))/dt = (dV/du)^T (du/dt) = 0$. To prove the converse, let $\hat{u} = (\hat{x}^T, \hat{y}^T, \hat{z}^T)^T \in \Pi$. It is enough to show that $d\hat{x}/dt = 0$, $d\hat{y}/dt = 0$ and $d\hat{z}/dt = 0$. Form (4.4) it can be seen that $dV(\hat{u}(t))/dt = 0$ implies that

$$G(\hat{u})^T (\hat{u} - u^*) + b(\hat{u})^T \nabla G(\hat{u}) b(\hat{u}) = 0. \tag{4.7}$$

Since $\nabla G(u)$ is positive definite and $G(\hat{u})^T (\hat{u} - u^*) \geq 0$, relation (4.6) implies that $G(\hat{u})^T (\hat{u} - u^*) = 0$, $b(\hat{u})^T \nabla G(\hat{u}) b(\hat{u}) = 0$, and furthermore $\langle G(\hat{u}) - G(u^*), \hat{u} - u^* \rangle = 0$. Thus, $b(\hat{u})^T \nabla G(\hat{u}) b(\hat{u}) = (\nabla f(\hat{x}) + \nabla_x g(\hat{x})^T \hat{y} + \nabla_x h(\hat{x})^T \hat{z})^T J(\hat{x}) (\nabla f(\hat{x}) + \nabla_x g(\hat{x})^T \hat{y} + \nabla_x h(\hat{x})^T \hat{z}) = 0$, where $J(\hat{x}) = (\nabla_x^2 f(\hat{x}) + \nabla_x^2 g(\hat{x}) + \nabla_x^2 h(\hat{x})$.
Since $J(x)$ is positive definite, then

$$(\nabla f(\hat{x}) + \nabla_x g(\hat{x})^T \hat{y} + \nabla_x h(\hat{x})^T \hat{z}) = 0.$$

Thus, $d\hat{x}/dt = 0$. On the other hand,

$$\langle G(\hat{u}) - G(u^*), \hat{u} - u^* \rangle = \int_0^1 (\hat{u} - u^*)^T \nabla G(u^* + s(\hat{u} - u^*))(\hat{u} - u^*) ds$$

$$= \int_0^1 (\hat{u} - u^*)^T \nabla G(u_s)(\hat{u} - u^*) ds$$

$$= (\hat{x} - x^*)^T (\int_0^1 J(u_s) ds)(\hat{x} - x^*) = 0,$$

where $u_s = u^* + s(\hat{u} - u^*)$, $0 \le s \le 1$.

Note that $J(u_s)$ and $\int_0^1 J(u_s)ds$ are positive definite functionals. It follows that $\hat{x} = x^*$. Thus,

$$d\hat{z}/dt = \lambda h(\hat{x}) = \lambda h(x^*) = 0.$$

Now, consider that $G(\hat{u})^T(\hat{u} - u^*) = 0$. This gives:

$$g(\hat{x})^T(\hat{y} - y^*) + h(\hat{x})^T(\hat{z} - z^*) = g(\hat{x})^T(\hat{y} - y^*) + h(x^*)^T(\hat{z} - z^*) = g(\hat{x})^T(\hat{y} - y^*) = 0.$$

Then, $g(\hat{x})^T\hat{y} = g(\hat{x})^T y^* = g(x^*)^T y^* = 0$, which is equivalent to $\hat{y} = (\hat{y} - g(\hat{x}))^+$, i.e., $d\hat{y}/dt = 0$.
Therefore, the proposed neural network model (3.7) is globally convergent. □

**Corollary 4.4.** *If $\nabla_x^2 f(x)$ is positive definite and $\nabla_x^2 g(x)$ and $\nabla_x^2 h(x)$ are semi-positive definite, or if $\nabla_x^2 g(x)$ is positive definite and $\nabla_x^2 f(x)$ and $\nabla_x^2 h(x)$ are semi-positive definite, or if $\nabla_x^2 h(x)$ is positive definite and $\nabla_x^2 f(x)$ and $\nabla_x^2 g(x)$ are semi-positive definite, then the proposed neural network is Lyapunov stable and globally convergent to $u^* = ((x^*)^T, (y^*)^T, (z^*)^T)^T$, where $x^*$ is a solution of problem (3.1).*

## 5. **Extension**

Consider the following nonlinear optimization problem:

$$(5.1) \qquad Min\ f(x) \qquad s.t \qquad g(x) \le 0, \qquad h(x) = 0, \quad x \in \Omega_1,$$

where $\Omega_1 = \{x \in R^n | l_i \le x_i \le h_i, \forall i = 1, ..., n\}$.
In contrast with (3.7), we propose the neural network model for solving (5.1) in the form:

$$(5.2) \quad \frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} P_{\Omega_1}(x - (\nabla_x f(x) + \nabla_x g(x)^T y + \nabla_x h(x)^T z)) - x \\ (y - g(x))^+ - y \\ h(x) \end{pmatrix},$$

where,

$$P_{\Omega_1} = \begin{cases} l_i & x_i < l_i \\ x_i & l_i \le x_i \le h_i \\ h_i & x_i > h_i. \end{cases}$$

Similar to Theorem 4.3, we can get the following stability and convergent results for the neural network model (5.2).

**Theorem 5.1.** *If $\nabla_x^2 f(x) + \nabla_x^2 g(x)^T y + \nabla_x^2 h(x)^T z$ is positive definite on*

$$s_1 = \{u = (x^T, y^T, z^T)^T \in R^{n+m+r} | x \in \Omega_1, y \ge 0\},$$

*then the proposed neural network model (5.2) is stable in the Lyapunov sense and globally convergent to a stationary point $u^* = ((x^*)^T, (y^*)^T, (z^*)^T)^T$, where $x^*$ is a solution of problem (5.1).*

## 6. **Illustrative examples**

In this section, we give several examples to illustrate the effectiveness of the proposed neural networks (3.7) and (5.2) in solving optimization problems with linear or nonlinear constraints. The simulations are conducted in MATLAB 7.1.

**Example 1.** *Consider the following nonlinear programming problem:*

$$\min f(x) = \frac{1}{4}x_1^4 + 0.5x_1^2 + \frac{1}{4}x_2^4 + 0.5x_2^2 - 0.9x_1x_2$$

$$s.t. \qquad g_i(x) \le 0, \quad i = 1, 2, 3,$$

*where,*

$$\begin{cases} g_1(x) = x_1^2 + x_2^2 - 64, \\ g_2(x) = (x_1+3)^2 + (x_2+4)^2 - 36, \\ g_3(x) = (x_1-3)^2 + (x_2+4)^2 - 36. \end{cases}$$

This problem has an optimal solution $x^* = (0, 0)^T$. Simulation results show that the trajectories of (3.7) with any initial point will converge successfully to $u^* = ((x^*)^T, (y^*)^T)^T$. For example, Fig. 1 displays the trajectory of (3.7) with seven initial points.

**Example 2.** [5]. *We now use the projection neural network to solve a pseudoconvex optimization problem. Consider the following quadratic*

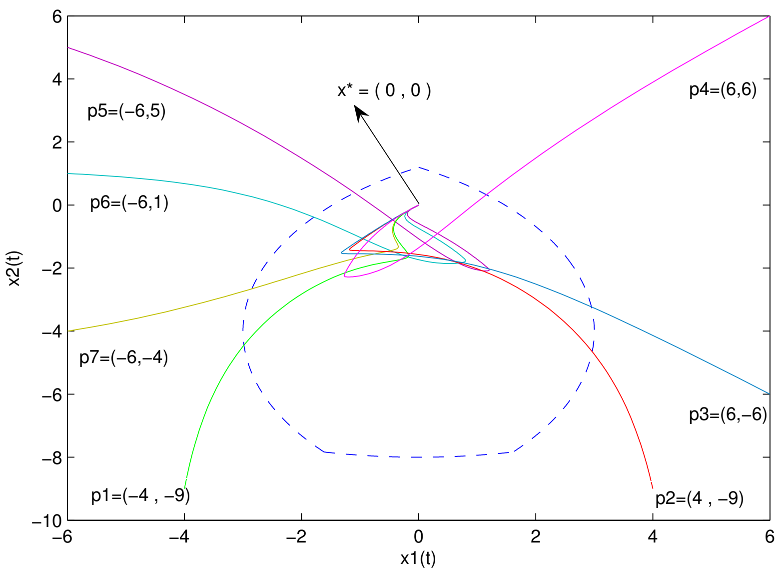


FIGURE 1. Transient behavior for the neural network model (3.7) with seven initial points in Example 1

*programming problem:*

$$\min f(x) = \frac{x^T Q x + a^T x + a_0 x}{b^T x + b_0},$$

*with*

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, b_0 = 4.$$

It is easily verified that $Q$ is symmetric and positive-definite in $R^4$ and consequently [1] $f$ is pseudoconvex on $X = \{x \in R^4 | b^T x + b_0 > 0\}$. This problem has an optimal solution $x^* = (-0.4370, 0.3286, 0.5979, -0.2972)^T$. All simulation results show that the projection neural network model (3.7) is Lyapunov stable at $u^* = (x^*, y^*)$, where $y^* = 0$. For instance, Fig. 2 shows that the trajectories of (3.7) with $\lambda = 10$ and the initial point $u_0 = (0, 3, 6, 10, 0)^T$ converge to $u^*$.
Now, we minimize $f$ over $\Omega = \{x \in R^4 | 1 \leq x_i \leq 10, i = 1, ..., 4\} \subset X$, by

using the neural network of (5.2). This problem has a unique solution $x^* = (1,1,1,1)^T$ in $\Omega$. Fig. 3 shows the trajectories of the proposed neural network with $u_0 = (0,3,6,10,0)^T$.

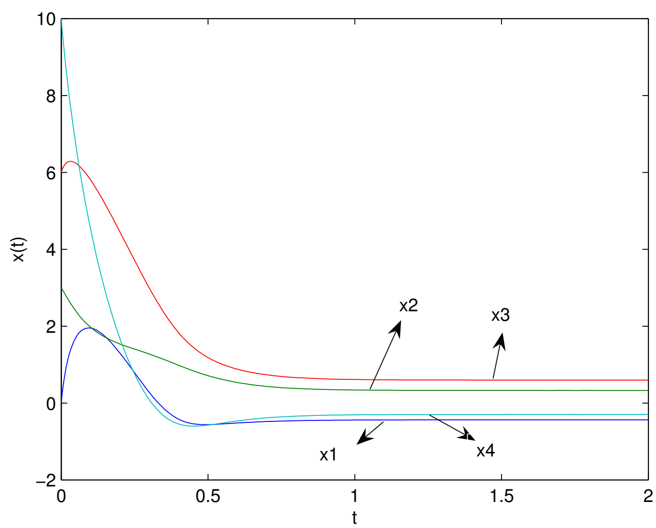


FIGURE 2. Simulation results based on the proposed neural network model (3.7) for Example 2

**Example 3.** *Consider the following nonlinear programming problem:*

$$\min f(x) = x_1^2 + x_2^2 + 0.5x_3^2 + x_1(x_2 - x_4) - 3x_2 - 2x_3$$

$$s.t.$$

$$3x_1 - 9x_2 + 9x_3 = 1$$

$$x_1 + x_2 + 2x_3 = 3$$

$$x_1 - x_2 + x_3 \le 1$$

$$-x_1 + x_2 + x_3 \le -\frac{1}{9}$$

$$0 \le x_i \le \frac{4}{3}, \quad i = 1,2,3.$$

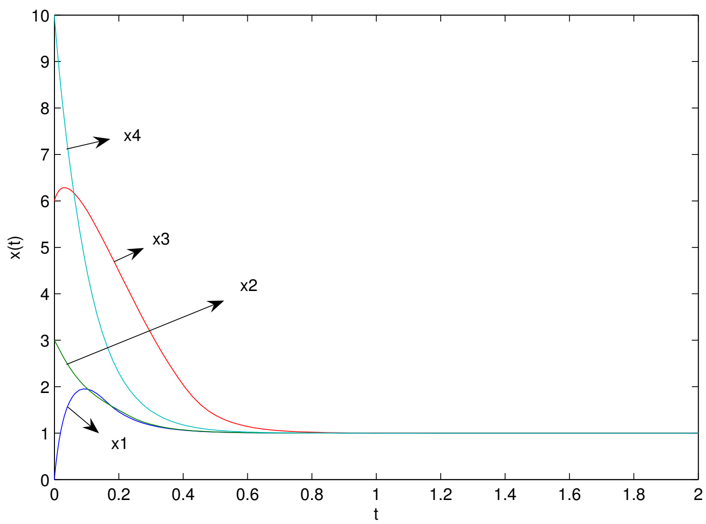


FIGURE 3. Simulation result based on the proposed neural network model (5.2) for Example 2

This problem has an optimal solution $x^* = (1.333, 0.7777, 0.4445)^T$. Simulation results show that all the trajectories of (5.2) with any initial point converge to $u^* = ((x^*)^T, (y^*)^T, (z^*)^T)^T$ successfully. For example, Fig. 4 displays the trajectory of (5.2) with five random initial points.

## 7. Conclusion

We proposed a continuous-time recurrent neural network for solving general nonlinear pseudoconvex programming problems subject to linear and nonlinear constraints. By using the projection technique, a relationship between the pseudoconvex optimization problem and globally projected dynamical system was established. Based on the projection formulations, the equilibrium points of the proposed neural network model were found. It was shown that these equilibrium points corresponded to the optimal solution of the nonlinear pseudoconvex programming problems. The globally convergence and Lyapunov stability of the proposed neural network werer proved under a specific condition. Compared with other existing neural network models for solving such

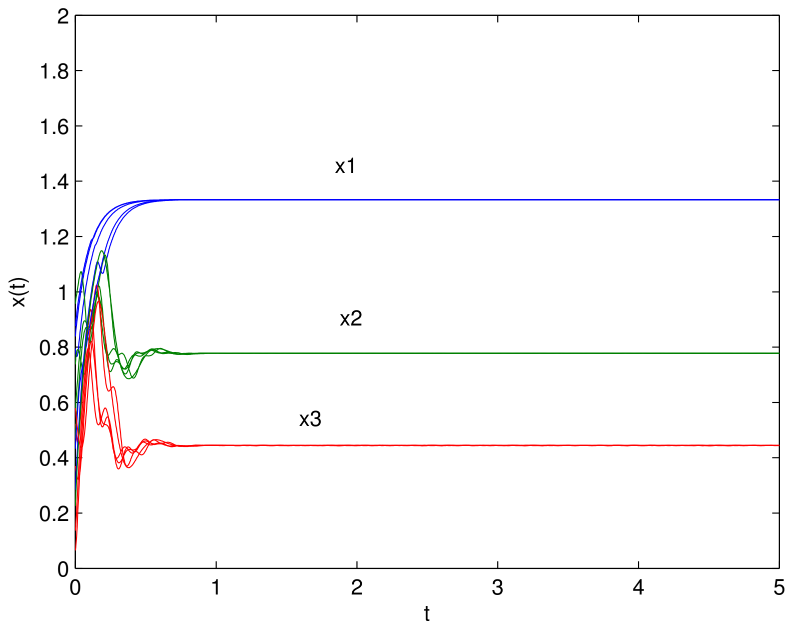


FIGURE 4. Transient behavior of the neural network model (5.2) with five random initial points between $[0, 1]$ in Example 3

problems, the proposed model was lasily implemented. The simulation results demonstrated the good global convergence behaviors and characteristics of the proposed neural network for solving several nonlinear programming problems.

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