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LINEAR ESTIMATE OF THE NUMBER OF ZEROS OF ABELIAN INTEGRALS FOR A KIND OF QUINTIC HAMILTONIANS

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Abstract. We consider the number of zeros of the integral $I(h) = \oint_{\Gamma_h} \omega$ of real polynomial form ω of degree not greater than n over a family of vanishing cycles on curves $\Gamma_h : y^2 + 3x^2 - x^6 = h$, where the integral is considered as a function of the parameter h. We prove that the number of zeros of $I(h)$, for $0 < h < 2$, is bounded above by $2[\frac{n-1}{2}] + 1$.

1. Introduction

Let $H = H(x, y)$ be a polynomial in x, y of degree $m \geq 2$, and the level curves $\Gamma_h \subset \{(x, y) : H(x, y) = h\}$ from a continuous family of ovals $\{\Gamma_h\}$, for $\Sigma = \{h_1 < h < h_2\}$ (as a maximal interval of existence of Γ_h). Consider a polynomial 1-form $\omega = f(x, y)dx - g(x, y)dy$, where $\max\{\deg(f), \deg(g)\} = n \geq 2$. Arnold in [1, 2] proposed the following problem:

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For fixed integers m and n find the maximum number $Z(m,n)$ of (isolated) zeros of the Abelian integrals

$$I(h) = \oint_{\Gamma_h} \omega. \tag{1.1}$$

Recall that an Abelian integral is the integral of a rational 1-form along an algebraic oval. The general result of solving the weakened Hilbert 16th problem was achieved by Varchenko [14] and Khovanskii [4], who proved independently the existence of $Z(m,n)$, but no explicit expression of $Z(m,n)$ has been obtained. Ilyashenko, Yakovenko and Novikov proved in [5, 6, 7, 15] that for the set of "good" $H(x,y)$ there exists a constant $c(H) < +\infty$ such that the number of real isolated zeros of $I(h)$ in Σ does not exceed $\exp(c(H)n)$. Zhao and Zhang [16] gave an explicit upper bound $Z(4,n) \leq 7n+5$. Their proof exploited the properties of the Picard-Fuchs system satisfied by the four basic integrals, which are the generators of the module of complete Abelian integrals over polynomial rings. Petrov in [9, 10, 11] gave the explicit upper bounds $Z(4,n) \leq 2[(n-1)/2]+1$ and $Z(4,n) \leq n+[(n-1)/2]$. That proof exploits the argument principle by the three basic integrals which are the generators of the module of complete Abelian integrals over polynomial rings. The number of zeros of $I(h)$ was estimated in [9, 10, 11, 12], and most of these results were proved by Petrov's method.
In particular, here using Petrov's method in [9], we solve the problem of zeros of the integral $I(h) = \oint \omega$ of real polynomial form ω of degree not greater than n over a family of vanishing cycles on curves $y^2+3x^2-x^6 = h$, where the integral is considered to be a function of the parameter h. Our main result is the following.

Theorem 1.1. *The sum of the number of zeros of the elliptic integral (1.1) in the plane-region with cuts along rays $\{h \leq 0\}$ and $\{2 \leq h\}$, defined for a family of cycles vanishing for $h = 0$ (the minimal critical value of polynomial $y^2+3x^2-x^6$), and the number of zeros in this region of integral (1.1) of the same form ω, defined for a family of cycles vanishing for $h = 2$ (the maximal critical value of $y^2+3x^2-x^6$), is less than $2[(n-1)/2]+1$.*

The remainder of our work is organized as follows. In Section 2, $I(h)$ is expressed as a linear combination of three basic integrals I_0, I_1 and I_2 with polynomial coefficients. In Section 3, we derive a Picard-Fuchs system and asymptotic expansions of Abelian integrals of I_0 , I_1 and

I_2. Using these results, in Section 4 we reduce the initial problem to counting the number of zeros of certain Abelian integrals..

2. The algebraic structure of $I(h)$

In this section, we are going to express $I(h)$ as a combination of three basic integrals. But, first we need to introduce some notations.

The Hamiltonian $H(x,y) = y^2 + 3x^2 - x^6$ has a center at $P_0(0,0)$ and two saddles at $P_1(1,0), P_2(-1,0)$. The ovals around P_0 are defined for $h \in \sum = (0,2)$.
By partial integration, $I(h)$ can be expressed as

$$I(h) = \oint_{\Gamma_h} P(x,y)dx = \oint_{\Gamma_h} \sum_{k+j\leq n} a_{kj} x^{2k} y^j dx.$$

Hence, in the rest of this paper, we represent $I(h)$ as above instead of (1.1). Let we denote

$$(2.1)\quad I_{k,l}(h) = \oint_{\Gamma_h} x^{2k} y^l dx,$$

$$(2.2)\quad I_0(h) = I_{01}(h),\ \ I_1(h) = I_{11}(h),\ \ I_2(h) = I_{21}(h),\ \ h \in \Sigma.$$

Then, the following statement holds.

Lemma 2.1. *For $d \geq 3$ and $k+l = d$, $I_{k,l}$ can be expressed as the linear combination of I_{ij}(where, $i+j = d-1$) and hI_{ij} (where, $i+j = d-2$, $i = 0,1,2$).*

Proof. Multiplying $H(x,y) = y^2 + 3x^2 - x^6 = h$ by $x^{2k}y^{l-2}$ and integrating over Γ_h, yields

$$(2.3)\qquad I_{k,l} + 3I_{k+1,l-2} - I_{k+3,l-2} = hI_{k,l-2}.$$

By taking derivative of $H(x,y) = y^2 + 3x^2 - x^6 = h$ with respect to to x, we get

$$(2.4)\qquad ydy + (3x - 3x^5)dx = 0.$$

Multiplying (2.4) by $x^{2k-5}y^l$ and integrating over Γ_h gives by part

$$(2.5)\qquad I_{k,l} = -\frac{2k-5}{3(l+2)} I_{k-3,l+2} + I_{k-2,l}.$$

The equality (2.5) shows

$$(2.6)\qquad I_{k+3,l-2} = -\frac{2k+1}{3l} I_{k,l} + I_{k+1,l-2}.$$

Substituting (2.6) into (2.3), we get

$$\frac{3l+2k+1}{3l}I_{k,l}+2I_{k+1,l-2}=hI_{k,l-2}. \tag{2.7}$$

We use (2.7), with $k=0,1,2$, and (2.5), with $k=3,4,\ldots,d-1,d,$, respectively to obtain:

$$\begin{aligned}
\frac{3d+1}{3d}I_{0d} &= -2I_{1,d-2}+hI_{0,d-2},\\
\frac{3d}{3(d-1)}I_{1,d-1} &= -2I_{2,d-3}+hI_{1,d-3},\\
\frac{3d-1}{3(d-2)}I_{2,d-2} &= -2I_{3,d-4}+hI_{2,d-4},\\
I_{3,d-3} &= -\frac{1}{3(d-1)}I_{0,d-1}+I_{1,d-3},\\
I_{4,d-4} &= -\frac{3}{3(d-2)}I_{1,d-2}+I_{2,d-4},\\
&\vdots\\
I_{d,0} &= -\frac{2d-5}{6}I_{d-3,2}+I_{d-2,0}.
\end{aligned}$$

Then, we obtain a linear algebraic system of the form

$$AJ=B, A=\begin{pmatrix} A_3 & 0\\ 0 & E_{d-2}\end{pmatrix}, \tag{2.8}$$

where $J=col(I_{0,d},I_{1,d-1},I_{2,d-2},\ldots,I_{d,0})$, E_{d-2} is a unit matrix of order $d-2$, and

$$A_3=\begin{pmatrix} \frac{3d+1}{3d} & 0 & 0\\ 0 & \frac{d}{d-1} & 0\\ 0 & 0 & \frac{3d-1}{3(d-2)}\end{pmatrix}.$$

Since $\det A_3\neq 0$, for $d\geq 3$, and B contains only the integral I_{ij}, $i+j=d-1$ and hI_{ij}, $i+j=d-2, i=0,1,2$, the statement in the lemma is true. □

A straightforward computing by induction yields the following lemma.

Lemma 2.2. *For $n\geq 3$, the Abelian integral $I(h)$ can be expressed as:*

$$I(h)=\alpha(h)I_0(h)+\beta(h)I_1(h)+\gamma(h)I_2(h), \tag{2.9}$$

where $\alpha(h), \beta(h)$, *and* $\gamma(h)$ *are polynomials of* h, *deg* $\alpha(h) \leq [\frac{n-1}{2}]$, *deg* $\beta(h) \leq [\frac{n-2}{2}], deg\ \gamma(h) \leq [\frac{n-3}{2}]$, *and* $[s]$ *denotes the largest integer less than* s.

Proof. By substituting $k = 0, l = 3$, $k = 3, l = 1$ and $k = 1, l = 3$ in (2.5), (2.7), we have

$$\begin{aligned} I_{03} &= \frac{9}{10}hI_0 - \frac{9}{5}I_1 \\ I_{13} &= -\frac{3}{2}I_2 + \frac{3}{4}hI_1 \\ I_{31} &= -\frac{1}{9}I_{03} + I_{11} = -\frac{1}{10}hI_0 + \frac{6}{5}I_1. \end{aligned} \tag{2.10}$$

Then, for $n = 3$, we have

$$\begin{aligned} I(h) &= \oint_{\Gamma_h} \sum_{k+j\leq n} a_{kj}x^{2k}y^j dx \\ &= (\frac{9}{10}a_{03}h + a_{01})I_0 + (a_{11} - \frac{9}{5}a_{03})I_1 + a_{21}I_2, \end{aligned}$$

and, for $n = 4$, we have

$$\begin{aligned} I(h) &= \oint_{\Gamma_h} \sum_{k+j\leq n} a_{kj}x^{2k}y^j dx \\ &= (\frac{9}{10}a_{03}h + a_{01} - \frac{1}{10}a_{31}h)I_0 + (a_{11} - \frac{9}{5}a_{03} + \frac{6}{5}a_{31} + \frac{3}{4}a_{13}h)I_1 \\ &+ (a_{21} - \frac{3}{2}a_{13})I_2, \end{aligned}$$

which implies that the result holds for $n = 3$ and $n = 4$.
Suppose for $n \leq d-1$, $\deg P(x, y) = n$ in (1.1) and $I^n(h) = \oint_{\Gamma_h} P(x, y)dx$ can be expressed as

$$I^n(h) = \alpha^n(h)I_0 + \beta^n(h)I_1 + \gamma^n(h)I_2, \tag{2.11}$$

where $\deg \alpha^n(h) \leq \left[\frac{n-1}{2}\right]$, $\deg \beta^n(h) \leq \left[\frac{n-2}{2}\right]$ and $\deg \gamma^n(h) \leq \left[\frac{n-3}{2}\right]$.
For $n = d$,

$$I(h) = \oint_{\Gamma_h} \sum_{k+j\leq n} a_{kj}x^{2k}y^j dx = \sum_{k+j\leq d-1} a_{kj}I_{kj} + \sum_{k+j=d} a_{kj}I_{kj}.$$

By the equality (2.11), we have

$$\sum_{k+j\leq d-1} a_{kj}I_{kj} = \alpha^{d-1}(h)I_0 + \beta^{d-1}(h)I_1 + \gamma^{d-1}(h)I_2,$$

and by Lemma 2.1, we obtain:

$$\begin{aligned}\sum_{k+j=d} a_{kj}I_{kj} &= \sum_{k+j=d-1} b_{kj}I_{kj} + \sum_{k+j=d-2} c_{kj}I_{kj} + \sum_{k+j=d-2} e_{kj}hI_{kj}\\ &= \Big(b_0^{d-1}(h)I_0 + b_1^{d-1}(h)I_1 + b_2^{d-1}(h)I_2\Big)\\ &+ \Big(c_0^{d-2}(h)I_0 + c_1^{d-2}(h)I_1 + c_2^{d-2}(h)I_2\Big)\\ &+ h\Big(e_0^{d-2}(h)I_0 + e_1^{d-2}(h)I_1 + e_2^{d-2}(h)I_2\Big).\end{aligned}$$

Then, $I(h) \equiv \alpha(h)I_0 + \beta(h)I_1 + \gamma(h)I_2$. Therefore,

$$\begin{aligned}\deg\alpha(h) &\le \max\{\deg\alpha^{d-1}(h), \deg b_0^{d-1}(h), \deg c_0^{d-1}(h), 1+\deg e_0^{d-1}(h)\}\\ &\le \max\left\{\left[\frac{d-1-1}{2}\right], \left[\frac{d-1-1}{2}\right], \left[\frac{d-2-1}{2}\right],\right.\\ &\qquad \left. 1+\left[\frac{d-2-1}{2}\right]\right\}\\ &= \left[\frac{d-1}{2}\right],\end{aligned}$$

which implies $\deg\alpha(h) \le [\frac{n-1}{2}]$, for arbitrary n. Similarly, $\deg\beta(h) \le [\frac{n-2}{2}]$ and $\deg\gamma(h) \le [\frac{n-3}{2}]$. □

3. The Picard-Fuchs system and asymptotic expansions of the Abelian integrals

In this section, we first derive the Picard-Fuchs equations satisfied by I_0(h), $I_1(h)$ and $I_2(h)$. Further more, we obtain the asymptotic expansion of the Abelian integrals I_0(h), $I_1(h)$ and $I_2(h)$ near $h = 0$, $h = 2$ and $h = \infty$.
Let $I_k(h)$ be defined as before by

$$I_k(h) = \oint_{\Gamma_h} x^{2k} y dx, \quad k = 0, 1, 2, \ldots. \tag{3.1}$$

Consider the following Hamiltonian of degree 6:

$$H(x, y) = y^2 + 3x^2 - x^6, \tag{3.2}$$

which is associated with a Newtonian mechanical system. Taking $y = y(x,h)$ on Γ_h, the equation $H(x,y) = h$ implies that $\frac{\partial y}{\partial h} = \frac{1}{2y}$. Thus,

$$I_k(h) = \oint_{\Gamma_h} x^{2k} \frac{\partial y}{\partial h} dx = \oint_{\Gamma_h} \frac{x^{2k}}{y} dx. \tag{3.3}$$

Along the curve Γ_h, we have $y^2 = h - 3x^2 + x^6$. Therefore,

$$\begin{aligned} I_k(h) &= \oint_{\Gamma_h} x^{2k} y dx \\ &= 2 \oint_{\Gamma_h} \frac{x^{2k}}{2y} (h - 3x^2 + x^6) dx \\ &= 2h \oint_{\Gamma_h} \frac{x^{2k}}{2y} dx - 6 \oint_{\Gamma_h} x^{2k+2} y dx + 2 \oint_{\Gamma_h} x^{2k+6} y dx \\ &= 2hI_k'(h) - 6I_{k+1}'(h) + 2I_{k+3}'(h). \end{aligned} \tag{3.4}$$

On the other hand, by integrating by parts, we have

$$\begin{aligned} I_k(h) &= \oint_{\Gamma_h} x^{2k} y dx \\ &= \frac{1}{2k+1} \oint_{\Gamma_h} y dx^{2k+1} = \frac{-1}{2k+1} \oint_{\Gamma_h} x^{2k+1} dy \\ &= \frac{-1}{2k+1} \oint_{\Gamma_h} x^{2k+1} (\frac{-6x + 6x^5}{2y}) dx \\ &= \frac{1}{2k+1} (6I_{k+1}' - 6I_{k+3}'). \end{aligned} \tag{3.5}$$

Eliminating $I_{k+3}'(h)$ from (3.4) and (3.5), we find that

$$18(k+2)I_k(h) = 54hI_k'(h) - 108I_{k+1}'. \tag{3.6}$$

Taking $k = 0, 1, 2$ in (3.5), we have

$$36I_0(h) = 54hI_0' - 108I_1'(h), \tag{3.7}$$

$$54I_1(h) = 54hI_1' - 108I_2'(h), \tag{3.8}$$

$$72I_2(h) = 54hI_2' - 108I_3'(h). \tag{3.9}$$

Taking $k = 0$ in (3.5), we have

$$I_3'(h) = I_1'(h) - \frac{1}{6} I_0(h). \tag{3.10}$$

Substituting 3.10 into (3.9) leads to

$$72I_2(h) = -108I_1'(h) + 54hI_2'(h) - 18hI_3'(h).$$

If we set $J(h) = column(I_0(h), I_1(h), I_2(h))$, then (3.7), (3.8) and (3.10) can be written in the matrix form

$$AJ(h) = (54hE + B)J'(h), \tag{3.11}$$

where E is the identity matrix and

$$A = \begin{pmatrix} 36 & 0 & 0 \\ 0 & 54 & 0 \\ -18 & 0 & 72 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -108 & 0 \\ 0 & 0 & -108 \\ 0 & -108 & 0 \end{pmatrix}, \tag{3.12}$$

yielding the following Picard-Fuchs equation:

$$G(h)J'(h) = \begin{pmatrix} 36(h^2-6) & 108h & 288 \\ -36h & 54h^2 & 144h \\ -18h^2 & 108h & 72h^2 \end{pmatrix} J(h), \tag{3.13}$$

where $G(h) = 54h(h^2 - 4)$.

Lemma 3.1. *$I_0(h)$, $I_1(h)$ and $I_2(h)$ have the following asymptotic expansion, as $h \to +\infty$,*

$$I_0(h) = 2C_1h^{2/3} - 3C_2 + 2C_3h^{-2/3} + O(h^{-4/3}), \tag{3.14}$$

$$I_1(h) = C_2h - 4C_3h^{1/3} + O(h^{-1/3}), \tag{3.15}$$

$$I_2(h) = C_3h^{4/3} + C_1h^{2/3} - \frac{9}{4}C_2 + \frac{5}{2}C_3h^{-2/3} + O(h^{-4/3}), \tag{3.16}$$

where C_1 and C_2 are real constant.

Proof. We find asymptotic expansions of (3.14)-(3.16) using the methods developed in [8]. We consider the system (3.13) which can be reduced to the following system

$$\frac{dJ}{dh} = h^q A(h)J(h), \tag{3.17}$$

where $q = -1$ and

$$A(h) = \begin{pmatrix} \frac{2}{3}(1 - \frac{2}{h^2-4}) & \frac{2h}{h^2-4} & \frac{16}{3(h^2-4)} \\ \frac{-2h}{3(h^2-4)} & 1 + \frac{4}{h^2-4} & \frac{8}{3(h^2-4)} \\ -\frac{1}{3}(1 + \frac{4}{h^2-4}) & \frac{2h}{h^2-4} & \frac{4}{3}(1 + \frac{4}{h^2-4}) \end{pmatrix}.$$

Now, it is clear that, as $h \to +\infty$, we have

$$\begin{aligned} \frac{1}{h^2-4} &= \frac{1}{4}\Big(\frac{1}{h-2} - \frac{1}{h+2}\Big) \\ &= \frac{1}{4h}\Big(\sum_{n=0}^{\infty}(\frac{2}{h})^n - \sum_{n=0}^{\infty}((-1)^n\frac{2}{h})^n\Big) \end{aligned}$$

$$= \sum_{n=0}^{\infty}(1-(-1)^n)2^{n-2}h^{-n-1}.$$

Thus, $A(h) = \sum_{m=0}^{\infty} A_m h^{-m}$, as $h \to \infty$, where

$$A_0 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{4}{3} \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 2 & 0 \\ -\frac{2}{3} & 0 & \frac{8}{3} \\ 0 & 2 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -\frac{4}{3} & 0 & \frac{16}{3} \\ 0 & 4 & 0 \\ -\frac{4}{3} & 0 & \frac{16}{3} \end{pmatrix}.$$

The eigenvalues of the matrix A_0 are $\sigma_1 = 2/3$, $\sigma_2 = 1$ and $\sigma_3 = 4/3$, with the eigenvectors

$$u_{01} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, u_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_{02} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now, by taking $J = h^{\sigma}\Sigma_{m=0}^{\infty} u_m h^{-m}$ in (3.17) and equating the coefficients of the terms with equal powers, we find that the scalar σ and the column vectors U_m satisfy the following recursive formulas:

(3.18) $A_0 u_0 = \sigma u_0, \quad (\sigma - m)u_m = \Sigma_{i=0}^{\infty} A_i u_{m-i}; \quad m = 1, 2, 3, \ldots.$

In other words, σ is an eigenvalue and u_0 is an eigenvector associated with σ for the matrix A_0. Since A_0 has three distinct eigenvalues, the linear combination of corresponding solutions of the system (3.18) gives the general solution of (3.17), as $h \to +\infty$.

By taking $\sigma = 2/3$ in (3.18), we obtain

$$u_0 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, u_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} -1/3 \\ 0 \\ -25/4 \end{pmatrix}, \ldots,$$

and by taking $\sigma = 1$ in (3.18), we obtain

$$u_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_1 = \begin{pmatrix} -3 \\ 0 \\ -9/4 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ldots.$$

Also, by taking $\sigma = 4/3$ in (3.18), we obtain

$$u_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, u_1 = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 0 \\ 5/3 \end{pmatrix}, \ldots.$$

Finally, we derive three linearly independent asymptotic solutions for the system (3.17), as $h \to +\infty$, as follows:

$$J_1(h) = h^{2/3}\begin{pmatrix}2\\0\\1\end{pmatrix} + h^{-1/3}\begin{pmatrix}0\\-1\\0\end{pmatrix} + h^{-4/3}\begin{pmatrix}-1/3\\0\\-25/4\end{pmatrix} + O(h^{-7/3}),$$

$$J_2(h) = h\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}-3\\0\\-9/4\end{pmatrix} + h^{-1}\begin{pmatrix}0\\0\\0\end{pmatrix} + O(h^{-2}),$$

$$J_2(h) = h^{4/3}\begin{pmatrix}0\\0\\1\end{pmatrix} + h^{1/3}\begin{pmatrix}0\\-4\\0\end{pmatrix} + h^{-2/3}\begin{pmatrix}2\\0\\5/3\end{pmatrix} + O(h^{-5/3}).$$

Thus, as $h \to +\infty$, the general solution of the system (3.17) has the form

$$J(h) = C_1J_1(h) + C_2J_2(h) + C_3J_3(h). \tag{3.19}$$

Now, from (3.19), we get the result. □

Lemma 3.2. *The Abelian integral I_0, I_1 and I_2 have the following asymptotic expansions, as $h \to 2^-$:*

$$\begin{aligned}
I_0(h) &= 3.95087 + (3.4641\ln(n) - 15.4056)n + 0.96225n^{3/2}\\
&+ (2.88675\ln(n) + 0.585364)n^2 + O(n^{5/2}),\\
I_1(h) &= 0.830186 + (3.4641\ln(n) + 0.397857)n - v0.19245n^{3/2}\\
&+ (4.73266 - 0.57735\ln(n))n^2 + O(n^{5/2}),\\
I_2(h) &= 0.365079 + (3.4641\ln(n) + 5.37897)n - 1.34715n^{3/2}\\
&- (0.57735\ln(n) + 5.65964)n^2 + O(n^{5/2}),
\end{aligned}$$

where $n = \frac{2-h}{12} \to 0^+$.

Proof. Let $h = 2 - 12n$ and $x = 1 - z$. Then, $y^2 + 3x^2 - x^6 = h$ implies that $z^2 - \frac{5}{3}z^3 + \frac{5}{4}z^4 - \frac{1}{2}z^5 + \frac{1}{12}z^6 = n$, and hence, as $z \to 0^+$, we have

$$z^2 = n(1 + \frac{5}{3}z + \frac{55}{36}z^2 + \frac{26}{27}z^3 + O(z^4)). \tag{3.20}$$

To find the positive solution of (3.20), first we consider the successive equations $z^2 = n$, $z^2 = n(1 + \frac{5}{3}z + \frac{55}{36}z^2)$, and so on, and then solve the first equation and substitute the solution in the right hand side of the

second equation and continue this process to obtain

$$z = 1 - a(n) = \sqrt{n}(1 + \frac{5}{6}\sqrt{n} + \frac{10}{9}n + \frac{379}{216}n\sqrt{n} + O(n^2)). \tag{3.21}$$

Now, by using (3.21) and for $0 < 2 - h \ll 1$, we can write

$$\begin{aligned}
I_0(h) &= \oint_{H=h} y dx = 4\int_0^{a(n)} \sqrt{(1-x^2)^2(x^2+2) - 12n} dx \\
&= 4\int_0^{a(n)} (1-x^2)\sqrt{x^2+2}\sqrt{1 - \frac{12n}{(x^2-1)^2(x^2+2)}} dx \\
&= 4\int_0^{a(n)} (1-x^2)\sqrt{x^2+2}(1 - \frac{6n}{(x^2-1)^2(x^2+2)} \\
&- \frac{18n^2}{(x^2-1)^4(x^2+2)^2} + O(n^3))dx \\
&= 4\int_0^{a(n)} (1-x^2)\sqrt{x^2+2}dx - 24n\int_0^{a(n)} \frac{dx}{(1-x^2)\sqrt{x^2+2}} \\
&- 72n^2\int_0^{a(n)} \frac{dx}{(1-x^2)^3(x^2+2)^{3/2}} + O(n^3) \\
&= \Big(x(1-x^2)\sqrt{x^2+2} + 6\sinh^{-1}(\frac{x}{\sqrt{2}}) - \frac{24n}{\sqrt{3}}\tanh^{-1}(\frac{x\sqrt{3}}{\sqrt{2+x^2}}) \\
&+ \frac{2n^2x(2x^4+3x^2-8)}{(x^2-1)^2\sqrt{x^2+2}} - \frac{20n^2}{\sqrt{3}}\tanh^{-1}(\frac{x\sqrt{3}}{\sqrt{2+x^2}})\Big)\Big|_0^{a(n)} + O(n^3) \\
&= 6\sinh^{-1}(\frac{1}{\sqrt{2}}) - \Big(\frac{9\sqrt{3}}{2} + 4\ln(3) - 2\ln(n)\Big)n + (\frac{20}{3} - \frac{55}{3\sqrt{3}})n^{3/2} \\
&+ (\frac{20}{3} - \frac{8}{\sqrt{3}} - \frac{10}{\sqrt{3}}\ln(3) + \frac{5}{\sqrt{3}}\ln(n))n^2 + O(n^{5/2}).
\end{aligned}$$

Similarly, we can obtain the result for $I_1(h)$ and $I_2(h)$. □

Lemma 3.3. *$I_0(h)$, $I_1(h)$ and $I_2(h)$, for $H(x,y) = y^2 + 3x^2 - x^6$, have the following asymptotic expansions, as $h \to 0^+$,*

$$\begin{aligned}
I_0(h) &= I_0'(0)[h + O(h^3)], \\
I_1(h) &= I_0'(0)[\frac{1}{12}h^2 + O(h^3)], \\
I_2(h) &= I_0'(0)[\frac{1}{72}h^3 + O(h^4)].
\end{aligned}$$

Proof. Since $I_k(0) = 0$ and $I_k(h)$, for $k = 0, 1, 2$, is analytic by [13] at $h = 0$, then by putting

$$I_0(h) = \sum_{n=1}^{\infty} a_n h^n, \quad I_1(h) = \sum_{n=1}^{\infty} b_n h^n, \quad I_2(h) = \sum_{n=1}^{\infty} c_n h^n,$$

into (3.13) (the Picard-Fuchs equation) and equating the coefficients of the terms with the same degree, we get the result. □

4. Number of zeros of the Abelian integrals

In this section, by the Petrov method, we solve the problem of zeros of the integral $I_\omega = \oint \omega$ of real polynomial form ω of degree not greater than n over a family of vanishing cycles on curves $y^2 + 3x^2 - x^6 = h$, where the integral is considered as a function of the parameter h.
By using the methods developed in [3], we define the vanishing cycles. Consider a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}, z \mapsto t$. Inside the ball U, f has exactly μ critical points $a_1, \ldots, a_\mu$, with pairwise distinct critical values $\alpha_1 = f(a_1), \cdots, \alpha_\mu = f(a_\mu)$; moreover, all the α_i lie inside a disk $T \subset \mathbb{C}$ and the level sets $V_t = f_\varepsilon^{-1}(t) \bigcap U$ are transverse to the boundary ∂U of U. Take a noncritical value α_* on the boundary of the disk T and the corresponding nonsingular level manifold $V_* = V_{\alpha_*}$. In disk T, consider a path $\varphi(\tau), \tau \in [0, 1]$, connecting the marked point $\alpha_* = \varphi(0) \in \partial T$ with one of the critical values $\alpha_i = \varphi(1)$; assume that, for $\tau < 1$, the path φ avoids all the critical values of f. By the Morse lemma, in a neighborhood U of the critical point a_i there is a coordinate system in which

$$f(z) = \alpha_i + z_1^2 + \cdots + z_n^2.$$

These coordinates allow us to single out in the nonsingular fibre $V_{\varphi(\tau)}$, for τ close to 1, the sphere

$$S_\tau = \sqrt{\varphi(\tau) - \alpha_i} S^{n-1}.$$

As $\tau \to 1$, S_τ degenerates to a point.

Definition 4.1. *The homology class $\triangle_\varphi \in H_{n-1}(V_*)$, defined by the sphere S_0 in the nonsingular fibre V_*, is called a vanishing cycle (along the path φ).*

First, we shall define a continuous family of cycles on curves $y^2 + 3x^2 - x^6 = h$, vanishing for $h = 0$ (respectively, $h = 2$), and forming a basis in the space of homologies of families of cycles, vanishing for this value of parameter h.

Now, we consider the consecutive real roots of the polynomial $3x^2 - x^6 - 1$, indexed in increasing order: $x_0 < x_1 < x_2 < x_3$. For $i = 0, 1, 2$, we define a cycle γ_i on the curve $y^2 + 3x^2 - x^6 = 1$ in $\mathbb{C}^2 = (x, y)$ as a cycle whose natural projection on the x-plane circles points x_i and x_{i+1} once, and such that the normalization condition is fulfilled: the integral of the form ydx over this cycle is positive. For each value of h in the plane-region with cuts along rays $\{h \leq 0\}$ and $\{2 \leq h\}$, we consider the path lying in this region and connecting the value of h with the point 2. A continuous change of parameter τ along this path gives a continuous deformation of the curves $y^2 + 3x^2 - x^6 = \tau$ and defines a homotopy of curves $y^2 + 3x^2 - x^6 = 1$ and $y^2 + 3x^2 - x^6 = h$ (not depending on the choice of the path in the region). The image of the cycle γ_i for this homotopy is, by definition, the cycle $\gamma_i(h)$. It is clear that a continuous family of cycles $\gamma_1(h)$ (respectively, families of cycles $\gamma_0(h)$ and $\gamma_2(h)$) is a family (are families) of cycles vanishing at $h = 0$ (at $h = 2$). In this connection, each family of cycles $\gamma(h)$, continuously depending on a parameter h which varies on the plane with cuts $\{h \leq 0\}$ and $\{2 \leq h\}$, vanishing for $h = 2$ ($h = 0$), is homologous, for each fixed h, to a linear combination $\alpha\gamma_0(h) + \beta\gamma_2(h)$ (cycle $\rho\gamma_1(h)$) with constants α and β (constant ρ) not depending on h.

We introduce some notations. For the integral $I_\omega = \oint \omega$, over cycle $\gamma_i(h)$, we denote $I_{\omega^i}(h)$.
Finally, we denote

$$\begin{aligned} D &= \mathbb{C}\backslash\Big(\{h \in \mathbb{R}; h \leq 0\} \cup \{h \in \mathbb{R}; h \geq 2\}\Big), \\ D^+ &= \mathbb{C}\backslash\{h \in \mathbb{R}; h \geq 2\}, \\ D^- &= \mathbb{C}\backslash\{h \in \mathbb{R}; h \leq 0\}. \end{aligned}$$

We consider the integrals I_{ω^0}, I_{ω^1} and I_{ω^2} of form ω of degree not greater than n (see definitions) and denote by α the number equal to $[(n-1)/2] + 2/3$.
In order to use the Argument principle to $I(h)$, we define $G = G_{R,\epsilon} \subset D$ (a simply connected region) with $\partial G = C = C_{R,\epsilon}$, a simple closed curve,

$$C_{R,\epsilon} = C_R \cup C^1_\epsilon \cup C^2_\epsilon \cup L^1_\pm(R, \epsilon) \cup L^2_\pm(R, \epsilon),$$

where $C_R = \{z \in ; |z| = R >> 1\}$, $C^1_\epsilon = \{z \in ; |z - 2| = \epsilon << 1\}$, $C^2_\epsilon = \{z \in ; |z| = \epsilon << 1\}$, $L^1_\pm(R, \epsilon) = \{z \in \mathbb{R}; 2 + \epsilon \leq z \leq R\}$ and $L^2_\pm(R, \epsilon) = \{z \in \mathbb{R}; -R \leq z \leq -\epsilon\}$.

Lemma 4.2. *The quotient $I_{\omega^1}(h)/h^\alpha$ (quotients $I_{\omega^0}(h)/h^\alpha$ and $I_{\omega^2}(h)/h^\alpha$) uniformly converges to a constant (constants), as $h \to \infty$, in region D^+ (D^-).*

Proof. By Lemma 2.2, we have

$$I_{\omega^1}(h) = \alpha(h)I_0(h) + \beta(h)I_1(h) + \gamma(h)I_2(h), \tag{4.1}$$

where $\alpha(h), \beta(h)$, and $\gamma(h)$ are polynomials of h, *deg* $\alpha(h) \leq [\frac{n-1}{2}]$, *deg* $\beta(h) \leq [\frac{n-2}{2}]$ and *deg* $\gamma(h) \leq [\frac{n-3}{2}]$. And by Lemma 3.1 we have $I_0(h) = O(h^{2/3})$, $I_1(h) = O(h)$, $I_2(h) = O(h^{4/3})$. □

Lemma 4.3. *Integral I_{ω^1} (integrals I_{ω^0} and I_{ω^2}) is holomorphic in D^+ (D^-) and continuously extends to the upper and lower sides of cut $L^1_\pm$ ($L^2_\pm$). In this connection, on the cut $L^1_\pm$ ($L^2_\pm$), the functions ImI_{ω^1} and I_{ω^0} (ImI_{ω^0} and I_{ω^1}) are proportional to a non-null coefficient.*

Proof. Family of cycles $\gamma_1(h)$ (families $\gamma_0(h)$ and $\gamma_2(h)$) is a family (families) of cycles not vanishing at $h = 0 (h = 2)$. Therefore the, integral I_{ω^1} (integrals I_{ω^0} and I_{ω^2}) is (are) holomorphic to a neighborhood of the point $h = 0$ ($h = 2$), and therefore holomorphically extends to region D^+ (D^-). We shall prove the second part of the lemma. The function I_{ω^0} is complex-conjugate and therefore on any side of the real cut $L^2_\pm$ the function ImI_{ω^0} is proportional to the difference of the values of the integral I_{ω^0} on its upper and lower sides. But, the difference of these values at a point on the cut $L^2_\pm$ is the integral of form ω over the difference of cycles $\gamma_0(h)$ defining the values of the integral at this point on the upper and lower sides of the cut. This difference of cycles is homologous to the cycle $\gamma_1(h)$. From this, the assertion of the lemma follows for the integral I_{ω^1}. The assertion for the integral I_{ω^0} is proved analogously. □

Proof of the Theorem: We denote by $r_+(r_-)$ the number of zeros of integral I_{ω^1} (integral I_{ω^0}) on D, by $p_+(p_-)$ the number of zeros of I_{ω^1} (I_{ω^0}) on $L^2_\pm \cup C^2_\epsilon$ ($L^1_\pm \cup C^1_\epsilon$), and finally, by $q_+(q_-)$ the number of zeros of the function ImI_{ω^1} (function ImI_{ω^0}) on any of the sides of $L^1_\pm$ ($L^1_\pm$). First, let us compute the rotation number of I_{ω^1} on D^+. By Lemma 3.2, the number of complete turns of I_{ω^1} on C^1_ϵ, when ϵ goes to 0, tends to zero. By the previous notation, the number of zeros of ImI_{ω^1}, for $h \in L^1_\pm$, is at most $2q_+$. Since each complete turn of I_{ω^1} forces at least two zeros of ImI_{ω^1}, we get that the number of complete turns on this two segments is at most $q_+ + 1$ (we add less than one half turn on

each bank). Finally, from Lemma 4.2, the number of complete turns on C_R is at most α. Putting all the results together, we obtain that the number of turns is at most $\alpha + q^+ + 1$. Since ImI_{ω^1} always has a zero for $h = 0$, then we have the inequality $p_+ + r_+ + 1 \leq q_+ + \alpha + 1$. Completely analogously for the integral I_{ω^0} and by Lemma 3.3, we have $p_- + r_- + 1 \leq q_- + \alpha + 1$. But, $p_- = q_+, p_+ = q_-$ (Lemma 4.3), from which after adding he inequalities, we get $r_+ + r_- \leq 2\alpha$. □

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