

AN OPERATOR EXTENSION OF BOHR'S INEQUALITY

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ABSTRACT. We establish an operator extension of the following generalization of Bohr's inequality, due to M.P. Vasić and D.J. Kečkić:

$$\left| \sum_{i=1}^n z_i \right|^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i |z_i|^r$$

$(r > 1, z_i \in \mathbb{C}, \alpha_i > 0, 1 \leq i \leq n).$

We also present some inequalities related to our noncommutative generalization of Bohr's inequality.

1. Introduction

Let \mathfrak{A} be a C^* -algebra of Hilbert space operators and let T be a locally compact Hausdorff space. A field $(A_t)_{t \in T}$ of operators in \mathfrak{A} is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T . If μ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, then one can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in \mathfrak{A} such that

$$\varphi \left(\int_T A_t d\mu(t) \right) = \int_T \varphi(A_t) d\mu(t)$$

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for every linear functional φ in the norm dual \mathfrak{A}^* of \mathfrak{A} ; cf. [3, Section 4.1].

Furthermore, a field $(\varphi_t)_{t \in T}$ of positive linear mappings $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ between C^* -algebras of operators is called continuous if the function $t \mapsto \varphi_t(A)$ is continuous for every $A \in \mathfrak{A}$. If the C^* -algebras include the identity operators, denoted by the same I , and the field $t \mapsto \varphi_t(I)$ is integrable with integral I , then we say that $(\varphi_t)_{t \in T}$ is unital.

The classical Bohr's inequality states that for any $z, w \in \mathbb{C}$ and any positive real numbers r, s with $\frac{1}{r} + \frac{1}{s} = 1$,

$$|z + w|^2 \leq r|z|^2 + s|w|^2.$$

This inequality admits the operator extension,

$$|A + B|^2 \leq r|A|^2 + s|B|^2$$

for operators A, B in the algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space \mathcal{H} (to see this, use the Cauchy–Schwarz inequality and the fact that the operator C is positive if and only if $\langle Cx, x \rangle \geq 0$).

Over the years, interesting generalizations of this inequality have been obtained in various settings; cf. [2, 6, 7, 8, 9, 11]. There is one of special interest given by M.P. Vasić and D.J. Kečkić [10]:

If z_1, \dots, z_n are complex numbers, $r > 1$ and $\alpha_i > 0$ ($i = 1, 2, \dots, n$), then

$$(1.1) \quad \left| \sum_{i=1}^n z_i \right|^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i |z_i|^r.$$

This is indeed an immediate consequence of the Hölder inequality. Here, we establish the operator version of inequality (1.1) and apply the obtained operator inequalities to obtain some inequalities related to our operator extension of Bohr's inequality.

2. Main results

Recall that a continuous real function f defined on a real interval J of any type is said to be operator convex if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ holds for all $\lambda \in [0, 1]$ and all self-adjoint operators A, B acting on a Hilbert space with spectra in J . For instance, $f(x) = x^r$, where, $1 \leq r \leq 2$ is operator convex on $[0, \infty)$; see [1, p. 123]. In [5], the authors gave a general formulation of Jensen's inequality for unital fields

of positive linear mappings in which they dealt with operator convex functions.

We need the main result [5, Theorem 2.1]. We state it for the sake of convenience.

Theorem 2.1. *Let f be an operator convex function on an interval J and let \mathfrak{A} and \mathfrak{B} be unital C^* -algebras. If $(\varphi_t)_{t \in T}$ is a unital field of positive linear mappings $\varphi_t : \mathfrak{A} \rightarrow \mathfrak{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ , then the inequality*

$$f\left(\int_T \varphi_t(A_t) d\mu(t)\right) \leq \int_T \varphi_t(f(A_t)) d\mu(t).$$

holds for every bounded continuous field $(A_t)_{t \in T}$ of self-adjoint elements of \mathfrak{A} with spectra contained in J .

Utilizing the above theorem we prove our main result.

Theorem 2.2. *Let \mathfrak{A} and \mathfrak{B} be C^* -algebras of operators containing I , T be a locally compact Hausdorff space equipped with a bounded Radon measure μ , (α_t) a bounded continuous nonnegative function such that $(\alpha_t) \in L^{\frac{1}{1-r}}(T, \mu)$ and $1 < r \leq 2$. Also, let $(A_t)_{t \in T}$ be a bounded continuous field of positive elements in \mathfrak{A} and $(\varphi_t)_{t \in T}$ be a field of positive linear mappings $\varphi_t : \mathfrak{A} \rightarrow \mathfrak{B}$ defined on T satisfying*

$$\int_T \alpha_t^{1/(1-r)} \varphi_t(I) d\mu(t) \leq \int_T \alpha_t^{1/(1-r)} d\mu(t) I.$$

Then,

(2.1)

$$\left(\int_T \varphi_t(A_t) d\mu(t)\right)^r \leq \left(\int_T \alpha_t^{1/(1-r)} d\mu(t)\right)^{r-1} \int_T \alpha_t \varphi_t(A_t^r) d\mu(t).$$

Proof. Let ∞ be an object not belonging to T . Consider $T_\infty := T \cup \{\infty\}$ as a locally compact topological space by equipping $\{\infty\}$ with the discrete topology and extend μ on T_∞ by $\mu(\{\infty\}) = 1$. Set $f(x) = x^r$ ($x \in [0, \infty)$), $\tilde{\varphi}_t := \frac{P_t}{Q} \varphi_t$ ($t \in T_\infty$), where $P_t := \alpha_t^{1/(1-r)}$, $P_\infty := 1$, $Q := \int_T \alpha_t^{1/(1-r)} d\mu(t)$ and

$$\varphi_\infty(A) := \langle Ae, e \rangle \left(\int_T P_t d\mu(t) I - \int_T P_t \varphi_t(I) d\mu(t) \right),$$

in which A belongs to the C^* -algebra \mathfrak{A} acting on a Hilbert space \mathcal{H} and $e \in \mathcal{H}$ is a fixed unit vector. Then,

$$\begin{aligned} \int_{T_\infty} \tilde{\varphi}_t(I) d\mu(t) &= \frac{1}{Q} \int_T P_t \varphi_t(I) d\mu(t) + \tilde{\varphi}_\infty(I) \\ &= \frac{1}{Q} \int_T P_t \varphi_t(I) d\mu(t) \\ &\quad + \frac{1}{Q} \left(\int_T P_t d\mu(t) I - \int_T P_t \varphi_t(I) d\mu(t) \right) \\ &= I. \end{aligned}$$

It follows from Theorem 2.1 that

$$\left(\int_{T_\infty} \tilde{\varphi}_t(\tilde{A}_t) d\mu(t) \right)^r \leq \int_{T_\infty} \tilde{\varphi}_t(\tilde{A}_t^r) d\mu(t),$$

namely,

$$(2.2) \quad \left(\int_{T_\infty} P_t \varphi_t(\tilde{A}_t) d\mu(t) \right)^r \leq Q^{r-1} \int_{T_\infty} P_t \varphi_t(\tilde{A}_t^r) d\mu(t),$$

for all bounded continuous fields $(\tilde{A}_t)_{t \in T_\infty}$.

Put $\tilde{A}_t = A_t/P_t$ ($t \in T$) and $\tilde{A}_\infty = 0$ in (2.2) to obtain

$$(2.3) \quad \left(\int_T \varphi_t(A_t) d\mu(t) \right)^r \leq Q^{r-1} \int_T P_t^{1-r} \varphi_t(A_t^r) d\mu(t).$$

By the definitions of P_t and Q , (2.3) becomes (2.1). \square

Remark 2.3. Using an analogous argument as in the proof of Theorem 2.2, one can prove the following Jensen's inequality. Suppose that (φ_t) is a continuous field of positive linear mappings $\varphi_t : \mathfrak{A} \rightarrow \mathfrak{B}$, \mathfrak{A} and \mathfrak{B} are unital C^* -algebras, (A_t) is a bounded continuous field of self-adjoint elements in \mathfrak{A} with spectra in an interval J such that $0 \in J$, (β_t) is a continuous nonnegative function such that $\int_T \beta_t d\mu(t) > 0$ and

$$(2.4) \quad \int_T \beta_t \varphi_t(I) d\mu(t) \leq \int_T \beta_t d\mu(t) I.$$

If f is an operator convex function on an interval J such that $f(0) \leq 0$, then,

$$(2.5) \quad f \left(\frac{1}{\int_T \beta_t d\mu(t)} \int_T \beta_t \varphi_t(A_t) d\mu(t) \right) \leq \frac{1}{\int_T \beta_t d\mu(t)} \int_T \beta_t \varphi_t(f(A_t)) d\mu(t).$$

If equality holds in (2.4), then it is not necessary to assume that $0 \in J$ and $f(0) \leq 0$.

A discrete version of the above theorem is the following result obtained by taking $T = \{1, \dots, n\}$.

Corollary 2.4. *Let $1 < r \leq 2$, $\alpha_i > 0$ ($i = 1, \dots, n$), A_1, \dots, A_n be positive operators acting on a Hilbert space \mathcal{H} and φ_i ($i = 1, \dots, n$) be positive linear mappings on $\mathbb{B}(\mathcal{H})$ satisfying*

$$(2.6) \quad \sum_{i=1}^n \alpha_i^{1/(1-r)} \varphi_i(I) \leq \sum_{i=1}^n \alpha_i^{1/(1-r)} I.$$

Then,

$$\left(\sum_{i=1}^n \varphi_i(A_i) \right)^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i \varphi_i(A_i^r).$$

By setting $\varphi_i(A) = X_i^* A X_i$ in Corollary 2.4, we find the following result.

Corollary 2.5. *Let $1 < r \leq 2$, $\alpha_i > 0$, for $i = 1, \dots, n$, A_1, \dots, A_n be bounded operators acting on a Hilbert space \mathcal{H} with $A_i \geq 0$ and $X_1, \dots, X_n \in \mathbb{B}(\mathcal{H})$ satisfying*

$$(2.7) \quad \sum_{i=1}^n \alpha_i^{1/(1-r)} X_i^* X_i \leq \sum_{i=1}^n \alpha_i^{1/(1-r)} I.$$

Then,

$$\left(\sum_{i=1}^n X_i^* A_i X_i \right)^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i.$$

Condition (2.7) trivially holds if $X_i^* X_i \leq I$, for all $i = 1, \dots, n$. In fact, we can give another proof of the result for this case.

Corollary 2.6. *Let $A_1, \dots, A_n, X_1, \dots, X_n \in \mathbb{B}(\mathcal{H})$ with $A_i \geq 0$, $X_i^* X_i \leq I$, for $i = 1, \dots, n$, and $1 < r \leq 2$, $\alpha_i > 0$, for $i = 1, \dots, n$. Then,*

$$\left(\sum_{i=1}^n X_i^* A_i X_i \right)^r \leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i.$$

Proof. First note that $0 \leq (X_i^* A_i X_i)^r \leq X_i^* A_i^r X_i$, for each i ; cf. [4, Theorem 2.1]. For $i = 1, \dots, n$, set $\beta_i = \alpha_i^{1/(1-r)}$ and $B_i = X_i^* A_i X_i / \beta_i$. We have,

$$\begin{aligned}
\left(\sum_{i=1}^n X_i^* A_i X_i \right)^r &= \left(\sum_{i=1}^n \beta_i B_i \right)^r \\
&= \left(\sum_{i=1}^n \beta_i \sum_{j=1}^n \frac{\beta_j}{\sum_{k=1}^n \beta_k} B_j \right)^r \\
&= \left(\sum_{i=1}^n \beta_i \right)^r \left(\sum_{i=1}^n \frac{\beta_i}{\sum_{k=1}^n \beta_k} B_i \right)^r \\
&\leq \left(\sum_{i=1}^n \beta_i \right)^r \frac{\sum_{i=1}^n \beta_i B_i^r}{\sum_{i=1}^n \beta_i} \\
&\quad \text{(by the operator convexity of } f(t) = t^r \text{)} \\
&= \left(\sum_{i=1}^n \beta_i \right)^{r-1} \sum_{i=1}^n \beta_i^{1-r} (X_i^* A_i X_i)^r \\
&= \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i (X_i^* A_i X_i)^r \\
&\leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i.
\end{aligned}$$

□

Proposition 2.7. Let $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ with $A_i^* A_j = 0$, for $1 \leq i \neq j \leq n$, and $2 < r \leq 4$, $\alpha_i > 0$, for $i = 1, \dots, n$. Then,

$$\left\| \sum_{i=1}^n A_i \right\|^r \leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i \|A_i\|^r$$

and

$$(2.8) \quad \left\| \sum_{i=1}^n A_i \right\|^r \leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i \|A_i\|^r.$$

Proof.

$$\begin{aligned}
\left| \sum_{i=1}^n A_i \right|^r &= \left(\left| \sum_{i=1}^n A_i \right|^2 \right)^{r/2} \\
&= \left(\sum_{i,j=1}^n A_i^* A_j \right)^{r/2} \\
&= \left(\sum_{i=1}^n |A_i|^2 \right)^{r/2} \\
&\leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i (|A_i|^2)^{r/2} \\
&\hspace{15em} \text{(by Corollary 2.6 with } X_i = I) \\
&= \left(\sum_{i=1}^n \alpha_i^{2/(2-r)} \right)^{(r-2)/2} \sum_{i=1}^n \alpha_i |A_i|^r .
\end{aligned}$$

Inequality (2.8) is easily deduced from the fact that $\|Z\|^r = \||Z|^r\|$ for each $Z \in \mathbb{B}(\mathcal{H})$. \square

Remark 2.8. It is clear that $A_1, \dots, A_n \in \mathbb{B}(\mathcal{H})$ have orthogonal ranges if and only if $A_i^* A_j = 0$. An example of such operators is obtained by considering an orthogonal family $(e_i)_{1 \leq i \leq n}$ and a vector x in \mathcal{H} and defining the rank one operators $A_i : \mathcal{H} \rightarrow \mathcal{H}$ by $A_i = e_i \otimes x$, $1 \leq i \leq n$. Then, $A_i^* A_j = \langle e_j, e_i \rangle x \otimes x$, for all $1 \leq i, j \leq n$.

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