*Special Issue of the Bulletin of the Iranian Mathematical Society*
*Vol. 37 No. 2 Part 2 (2011), pp 159-186.*

# ALGEBRAS WITH CYCLE-FINITE STRONGLY SIMPLY CONNECTED GALOIS COVERINGS

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Communicated by Lidia Angeleri Hügel

Abstract. Let $A$ be a finite dimensional $k$-algebra and $R$ be a locally bounded category such that $R \to R/G = A$ is a Galois covering defined by the action of a torsion-free group of automorphisms of $R$. Following [30], we provide criteria on the convex subcategories of a strongly simply connected category $R$ in order to be a cycle-finite category and describe the module category of $A$. We provide criteria for $A$ to be of polynomial growth.

## 1. Introduction

Throughout the article, algebras are finite dimensional associative $k$-algebras with identity over a fixed algebraically closed field $k$. By a module over an algebra $A$ we mean a left $A$-module of finite dimension over $k$, if not specified otherwise.

From Drozd's Tame and Wild Theorem [19] (see also [13]), the class of algebras are divided into two disjoint classes. On the one hand, we have tame algebras for which the indecomposable modules occur, in each dimension $d$, in a finite number of discrete and a finite number of one-parameter families. On the other hand, we have wild algebras whose representation theory includes the representation theories of all finite

MSC(2000): Primary: 16G60; Secondary: 16G70.
Keywords: Module category of an algebra, infinite radical, Galois coverings, cycles of modules.
Received: 30 March 2011, Accepted: 7 May 2011.

dimensional algebras over $k$ (see [35, Chapter XIX]). A well understood class of tame algebras is formed by the algebras of finite type which accept only finitely many isoclasses of indecomposable modules (see [4, 5, 8, 9]). In the more general situation, the representation theory of tame algebras is slowly emerging. Tame tilted algebras [24], domestic and tubular extensions of tame concealed algebras [33], coil algebras [3] and more generally, (tame) algebras of polynomial growth [37], for which there exists an integer $m$ such that the number of one-parameter families of indecomposable modules is bounded, in each dimension $d$, by $d^m$, are among the type of algebras studied in the past years.

The methods of the representation theory of algebras work best for triangular algebras $A = kQ_A/I$, where the Gabriel quiver $Q_A$ has no oriented cycles (see [1, 33, 34, 35]). To deal with arbitrary algebras, covering techniques were developed (see [8, 16, 18, 20, 28]). In many situations, an algebra $A$ admits a Galois covering $R \to R/G = A$, where $R$ is a triangular locally bounded category and $G$ is a torsion-free group acting freely on the objects of $R$, which allows to study the representation theory of $A$ by the consideration of finite dimensional algebras inside $R$. For instance, assume that $R$ is a strongly simply connected category (see [38]). Then, tameness of $A$ implies tameness of $R$, which happens exactly when $R$ does not accept convex subcategories which are hypercritical [11]. The converse is expected to hold. Moreover, under these assumptions, $A$ is of polynomial growth if and only if $R$ does not accept convex subcategories which are hypercritical or pg-critical; see [37].

An important role in the representation theory of algebras is played by cycles of modules. A cycle in the category mod $A$ of finite dimensional modules over an algebra $A$ is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

of non-zero non-isomorphisms between indecomposable modules in mod $A$, and the cycle is said to be finite if the homomorphisms $f_1, \dots, f_s$ do not belong to the infinite Jacobson radical of mod $A$. An algebra $A$ is said to be *cycle-finite* if all cycles in mod $A$ are finite [2]. Representation-finite algebras, tame tilted algebras, tame generalized multicoil algebras [26] are examples of cycle-finite algebras. In general, a cycle-finite algebra $A$ is of polynomial growth, while the converse holds if $A$ is a strongly simply connected algebra [40]. Recently, it was shown in [30] that every algebra $A$, which admits a Galois covering $R \to R/G = A$ with $R$ a cycle-finite

locally bounded category and $G$ a torsion-free group, is tame and the indecomposable finite dimensional $A$-modules were described. Moreover, for such a Galois covering, the algebra $A$ is of polynomial growth if and only if the number of $G$-orbits of isoclasses of indecomposable locally finite dimensional $R$-modules with non-trivial stabilizers is finite.

Here, we recall the main results and related techniques of the context discussed so far. Namely, we consider algebra $A$ and Galois covering $R \to R/G = A$ where $R$ is a "nice" locally bounded category and $G$ is a torsion-free group of automorphisms of $R$. The nicest situation corresponds to $R$ being a strongly simply connected and cycle-finite category. Assuming that $R$ is a strongly simply connected category, we show that $R$ is cycle-finite if and only if $R$ does not accept convex subcategories which are hypercritical, $pg-$critical or of type $(2, 2, \infty)$. Here, we say that a category is of type $(2, 2, \infty)$ if it is the direct limit of domestic extensions of type $(2, 2, n)$, for $0 \leq n \in \mathbb{N}$, of a fixed tame concealed algebra of type $(2, 2, s)$. These conditions are satisfied when there is a set of representatives $\mathcal{S}_0$ of the $G$-orbits in a separating family $\mathcal{S}$ of convex subcategories of $R$ with respect to $G$ which is formed by lines ${}_\infty\mathbb{A}_\infty$; see [30]. Moreover, if $R \to R/G = A$ is a covering in the nicest situation and $\mathcal{S}_0$ is not empty, then $A$ is of polynomial growth exactly when $G = \mathbb{Z}$.

The remainder of the paper is organized as follows. In Section 1, we recall basic facts on Galois coverings of algebras essential for further considerations. Section 2 contains results on cycle-finite strongly simply connected categories. Section 3 is devoted to the proof of the main result and its immediate consequences. In Section 4, we establish a criterion for polynomial growth. In the final Section 5, we exhibit a couple of examples illustrating our results.

For basic background on the representation theory of algebras, refer to the books [1, 33, 34, 35].

## 2. Galois coverings of algebras

Following [8], by a *locally bounded category* we mean a $k$-category $R$ which is isomorphic to a factor category $kQ_R/I$, where $Q_R$ is a locally finite quiver and $I$ is an admissible ideal of the path category $kQ_R$ of $Q_R$. An algebra $A$ will be considered as a *finite category*, that is, a locally bounded category given by a finite quiver. A full subcategory $C$ of a locally bounded category $R$ is said to be *convex* if any path in $Q_R$ with source and target in $Q_C$ lies entirely in $Q_C$.

Throughout this section, we denote by $R$ a fixed locally bounded category (over $k$). By an *$R$-module*, we mean a covariant functor $M$ from $R$ to the category $\operatorname{MOD} k$ of all vector spaces over $k$ [8]. An $R$-module $M$ is called *finite dimensional* (respectively, *locally finite dimensional*) if $\dim M = \sum_{x\in R} \dim_k M(x) < \infty$ (respectively, $\dim_k M(x) < \infty$ for any object $x$ of $R$). We denote by $\operatorname{MOD} R$, (respectively, $\operatorname{Mod} R$ or $\operatorname{mod} R$) the category of all (respectively, all locally finite dimensional or all finite dimensional) $R$-modules, and by $\operatorname{Ind} R$, (respectively, $\operatorname{ind} R$) the full subcategory of $\operatorname{Mod} R$ (respectively, $\operatorname{mod} R$) formed by all indecomposable modules. The *support* $\operatorname{supp} M$ of an $R$-module $M$ is the full subcategory of $R$ given by all objects $x$ such that $M(x) \neq 0$.

Let $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the objects of $R$. Then, following [20], we may consider the orbit category $R/G$ with objects being the $G$-orbits of the objects of $R$, and, for any two objects $a$ and $b$ of $R/G$, the morphism $k$-space $(R/G)(a,b)$ is defined as

$$(R/G)(a,b) = \left\{ (f_{y,x}) \in \prod_{(x,y)\in a\times b} R(x,y) \;\middle|\; g f_{y,x} = f_{gy,gx} \underset{g\in G, x\in a, y\in b}{\forall} \right\}$$

with the natural composition. Then, we have a canonical *Galois covering functor*

$$F : R \longrightarrow R/G$$

which assigns to any object $x$ of $R$ its $G$-orbit $Gx$ and maps a morphism $f \in R(x,y)$ onto the family $F(f) \in (R/G)(Gx, Gy)$ such that $F(f)_{hy,gx} = gf$ or $0$ in accordance with $h = g$ or $h \neq g$. Moreover, $F$ induces the $k$-linear isomorphisms

$$\bigoplus_{F(y)=a} R(x,y) \xrightarrow{\sim} (R/G)(F(x),a), \qquad \bigoplus_{F(y)=a} R(y,x) \xrightarrow{\sim} (R/G)(a,F(x)),$$

for all objects $x$ of $R$ and $a$ of $R/G$. For a full subcategory $D$ of $R$, we denote by $g(D)$ the full subcategory of $R$ formed by the objects $g(x)$, $x \in D$, and its stabilizer $G_D = \{g \in G \mid g(D) = D\}$. Then, we may consider the locally bounded category $D/G_D$. The group $G$ acts on $\operatorname{Mod} R$ by the translations $(-)^g$ which assign to each $R$-module $M$ the $R$-module $M^g = M \circ g$. For each $R$-module $M$, we denote by $G_M$ the stabilizer $\{g \in G \mid M^g \cong M\}$ of $M$. Following [18], a module $Y$ in $\operatorname{Ind} R$ is said to be *weakly $G$-periodic* if $\operatorname{supp} Y$ is infinite and $(\operatorname{supp} Y)/G_Y$ is a finite category. Observe that in such a case, $G_Y$ is infinite.

Assume now that $G$ is a group of $k$-linear automorphisms of $R$ acting freely on the isoclasses of modules in $\operatorname{ind} R$. Clearly, then $G$ acts freely on the objects of $R$, since $G$ acts freely on the isoclasses of indecomposable projective $R$-modules $R(x,-)$, $x \in R$. Consider the associated Galois covering functor $F : R \to R/G$. We denote by $F_\bullet : \operatorname{MOD} R/G \to \operatorname{MOD} R$ the *pull-up functor*, which assigns to an $R/G$-module $M$ the $R$-module $M \circ F$, and by $F_\lambda : \operatorname{MOD} R \to \operatorname{MOD} R/G$ the *push-down functor*, left adjoint to $F_\bullet$ (see [8, (3.2)]). Since $G$ acts freely on the isoclasses in $\operatorname{ind} R$, $F_\lambda$ induces an injection from the set $(\operatorname{ind} R/\cong)/G$ of $G$-orbits of isoclasses in $\operatorname{ind} R$ into the set $(\operatorname{ind} R/G)/\cong$ of isoclasses in $\operatorname{ind} R/G$ [20, (3.5)]. We denote by $\operatorname{mod}_1 R/G$ the full subcategory of $\operatorname{mod} R/G$ consisting of all modules isomorphic to $F_\lambda(M)$ for some module $M$ in $\operatorname{mod} R$, and by $\operatorname{mod}_2 R/G$ the full subcategory of $\operatorname{mod} R/G$ formed by all modules without nonzero direct summands from $\operatorname{mod}_1 R/G$. It was shown in [18, (2.2) and (2.3)] that a module $X$ from $\operatorname{mod} R/G$ belongs to $\operatorname{mod}_1 R/G$ (respectively, $\operatorname{mod}_2 R/G$) if and only if $F_\bullet(X)$ is a direct sum of finite dimensional $R$-modules (respectively, weakly $G$-periodic $R$-modules). We denote by $\operatorname{ind}_1 R/G$ (respectively, $\operatorname{ind}_2 R/G$) the full subcategory of $\operatorname{mod}_1 R/G$ (respectively, $\operatorname{mod}_2 R/G$) formed by the indecomposable modules. Following [18], the modules from $\operatorname{ind}_1 R/G$ (respectively, $\operatorname{ind}_2 R/G$) are called *indecomposable modules of the first kind* (respectively, *indecomposable modules of the second kind*). The category $R$ is said to be *$G$-exhaustive* if $\operatorname{mod} R/G = \operatorname{mod}_1 R/G$ [18].

Assume that $R$ is not $G$-exhaustive. Following [18, (3.1)], a family $\mathcal{S}$ of full subcategories of $R$ is called *separating* (with respect to $G$) if $\mathcal{S}$ satisfies the following conditions:

(i) for each $L \in \mathcal{S}$ and $g \in G$, $gL \in \mathcal{S}$;
(ii) for each $L \in \mathcal{S}$ and each $G$-orbit $\mathcal{O}$ of $R$, $\mathcal{O} \cap L$ is contained in finitely many $G_L$-orbits;
(iii) for any two different $L, L' \in \mathcal{S}$, $L \cap L'$ is locally support-finite;
(iv) for each weakly $G$-periodic $R$-module $Y$, there exists an $L \in \mathcal{S}$ such that $\operatorname{supp} Y \subseteq L$.

The following theorem is the main result in [18, Theorem 3.1].

**Theorem 2.1.** *Let $R$ be a locally bounded $k$-category and $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the isoclasses in* $\operatorname{ind} R$. *Let $\mathcal{S}$ be a separating family of convex subcategories of $R$ with respect to $G$ and $\mathcal{S}_0$ be a fixed set of representatives of $G$-orbits in $\mathcal{S}$. There are*

*natural embedding functors* $E^L_\lambda : \operatorname{mod} L/G_L \to \operatorname{mod} R/G$, $L \in \mathcal{S}_0$ *which induce a natural* $k$*-linear equivalence of categories*

$$E : \coprod_{L\in\mathcal{S}_0} (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L] \longrightarrow (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

*In particular, the Auslander-Reiten quiver* $\Gamma_{R/G}$ *of* $R/G$ *is the disjoint union of the translation quivers*

$$\Gamma_{R/G} = (\Gamma_R/G) \sqcup \left( \coprod_{L\in\mathcal{S}_0} \left(\Gamma_{L/G_L}\right)_2 \right),$$

*where* $\left(\Gamma_{L/G_L}\right)_2$ *is the union of all connected components of* $\Gamma_{L/G_L}$ *formed by the indecomposable* $L/G_L$*-modules of the second kind.*

For a convex subcategory $L$ of a locally bounded category $R$, the canonical embedding $E^L : \operatorname{MOD} L \to \operatorname{MOD} R$ is defined for a module $N$ in $\operatorname{MOD} L$, $E^L(N)$ as an $R$-module such that $E^L(N)(x) = N(x)$ for any object $x$ of $L$, $E^L(N)(f) = N(f)$ for any morphism $f$ in $L$, and $E^L(N)(y) = 0$ for any object $y$ of $R$ which is not in $L$. Moreover, we have a commutative diagram of functors

$$\begin{tikzcd}
\operatorname{MOD} L \arrow[r, "E^L"] \arrow[d, "F^L_\lambda"] & \operatorname{MOD} R \arrow[d, "F_\lambda"] \\
\operatorname{MOD} L/G_L \arrow[r, "E^L_\lambda"] & \operatorname{MOD} R/G
\end{tikzcd}$$

where $F^L_\lambda$ is the push-down functor associated to the Galois covering $F^L : L \to L/G_L$, $F_\lambda$ is the push-down functor associated to the Galois covering $F : R \to R/G$, and $E^L_\lambda$ assigns to a module $X$ in $\operatorname{MOD} L/G_L$ the module $E^L_\lambda(X)$ in $\operatorname{MOD} R/G$ such that $F_\bullet E^L_\lambda(X) = \bigoplus_{g\in U_L} F^L_\bullet(X)^g$, where $F_\bullet : \operatorname{MOD} R/G \to \operatorname{MOD} R$ and $F^L_\bullet : \operatorname{MOD} L/G_L \to \operatorname{MOD} L$ are the pull-up functors associated to $F$ and $F^L$, and $U_L$ is a fixed set of representatives of the cosets of $G$ modulo $G_L$ (see [18], (2.4) and (3.2)).

The following is an important special case of the last Theorem; see [18].

**Proposition 2.2.** *Let* $R$ *be a tame locally bounded* $k$*-category,* $G$ *be a group of* $k$*-linear automorphisms of* $R$ *acting freely on the objects of* $R$*, and* $Y$ *be a weakly* $G$*-periodic* $R$*-module. Then, the followings hold:*

(1) *the stabilizer* $G_Y$ *is an infinite cyclic group;*

(2) *the push-down module $F_\lambda(Y)$ carries a canonical structure of a $kG_Y$-$R/G$-bimodule which is a free module of finite rank as left module over the group algebra $kG_Y$ of $G_Y$. In particular, we have a canonical functor*

$$\Phi^Y = - \otimes_{kG_Y} F_\lambda(Y) : \operatorname{mod} kG_Y \longrightarrow \operatorname{mod} R/G,$$

*whose image is contained in* $\operatorname{mod}_2 R/G$.

Let $R$ be a locally bounded $k$-category and $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the objects of $R$. A *line* in $R$ is a convex subcategory $L$ of $R$ which is isomorphic to the path category $kQ$ of a linear quiver $Q$ of type $\mathbb{A}_n$, $\mathbb{A}_\infty$ or ${}_\infty A_\infty$. A line $L$ in $R$ is said to be *$G$-periodic* if its stabilizer $G_L$ is nontrivial. Clearly, in this case, the quiver $Q_L$ of $L$ is of type

$${}_\infty\mathbb{A}_\infty : \quad \ldots \text{——} \bullet \text{——} \bullet \text{——} \bullet \text{——} \bullet \text{——} \bullet \text{——} \ldots$$

and has a $G_L$-periodic orientation. With each $G$-periodic line $L$ of $R$ we may associate a canonical weakly $G$-periodic $R$-module $M_L$ by setting $M_L(x) = k$ for any vertex $x$ of $Q_L$, $M_L(y) = 0$ for all vertices $y$ of $Q_R \backslash Q_L$, and $M_L(\gamma) = \mathrm{id}_k$ for each arrow $\gamma$ of $Q_L$. Since $G_{M_L} = G_L = \mathbb{Z}$, we then obtain a canonical functor

$$\Phi^L = - \otimes_{k[T,T^{-1}]} F_\lambda(M_L) : \operatorname{mod} k[T,T^{-1}] \longrightarrow \operatorname{mod} R/G$$

where $\operatorname{mod} k[T,T^{-1}]$ denotes the category of finite dimensional modules over $k[T,T^{-1}]$.

**Proposition 2.3.** *Let $R$ be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion-free group $G$. Let $\mathcal{S}_0$ be a set of representatives of the $G$-orbits in a separating family $\mathcal{S}$ of convex subcategories of $R$ with respect to $G$. The followings hold:*

(i) *each $L \in \mathcal{S}$ is a convex subcategory of $R$ which is a line $L$ in $R$, that is, the quiver $Q_L$ of $L$ is of type*

$${}_\infty\mathbb{A}_\infty : \quad \ldots \text{——} \bullet \text{——} \bullet \text{——} \bullet \text{——} \bullet \text{——} \bullet \text{——} \ldots$$

*and $L = kQ_L$;*

(ii) *for any two different $L, L' \in \mathcal{S}$, the intersection $L \cap L'$ is a connected finite linear quiver.*

*Proof.* $(i)$: In [30] (3.1), without assuming that $R$ is strongly simply connected, it was shown that $L$ is a convex subcategory of $R$ admitting a simply connected Galois covering $F' : \tilde{L} \to \tilde{L}/H = L$ determined

by the action of a torsion free group $H$ and $\tilde{L}$ is a line of type ${}_{\infty}\mathbb{A}_{\infty}$. Assuming that $R$ is strongly simply connected, then $\tilde{L} = L$ as desired.

Assume $(ii)$ fails. Since $L \cap L'$ is locally suport finite, then $L \cap L'$ is formed by at least two disconnected finite intervals of the line $L$. Thus, we get a convex segment $x_1 \text{———} x_2 \text{———} \cdots x_{s-1} \text{———} x_s$ in $L$ with $x_1, x_s \in L \cap L'$ and $x_i \notin L'$, $2 \leq i \leq s-1$. Then, the convex closure of $x_1, x_s$ in $R$ is of the shape

$$\begin{tikzcd}
 & & & x_p \arrow[dl] \arrow[dr] & & & \\
 & & x_{p-1} \arrow[dl] & & x_{p+1} \arrow[dr] & & \\
 & \cdots \arrow[dl] & & & & \cdots \arrow[dr] & \\
x_1 = y_1 \arrow[dr] & & & & & & x_s = y_t \arrow[dl] \\
 & y_2 \arrow[dr] & & & & y_{t-1} \arrow[dl] & \\
 & & \cdots \arrow[dr] & & \cdots \arrow[dl] & & \\
 & & & y_q & & &
\end{tikzcd}$$

where all $y_i \in L'$ but $y_j \notin L$, for $2 \leq j \leq t-1$, and there is a commutativity relation from $x_p$ to $y_q$. Since the stabilizer $G_L$ acts on $L$, we get a weakly $G$-periodic convex subcategory of $R$ which is not a line, contradicting $(i)$. $\square$

## 3. Cycle-finite strongly simply connected categories

By a *tame concealed algebra*, we mean a tilted algebra $C = \mathrm{End}_H(T)$, where $H$ is the path algebra $kQ$ of a quiver $Q$ of Euclidean type $\widetilde{\mathbb{A}}_m (m \geq 1)$, $\widetilde{\mathbb{D}}_n (n \geq 4)$, or $\widetilde{\mathbb{E}}_p (6 \leq p \leq 8)$, and $T$ is a (multiplicity-free) preprojective tilting $H$-module. Recall that the Auslander-Reiten quiver $\Gamma_C$ of a tame concealed algebra $C$ is of the form

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{I}^C,$$

where $\mathcal{P}^C$ is a preprojective component containing all indecomposable projective $C$-modules, $\mathcal{I}^C$ is a preinjective component containing all indecomposable injective $C$-modules, and $\mathcal{T}^C$ is a $\mathbb{P}_1(k)$-family $\mathcal{T}^C_\lambda, \lambda \in \mathbb{P}_1(k)$, of pairwise orthogonal standard stable tubes, all but finite number of them of rank one (see [33, Chapter 4]) and [34]).

By a *tubular algebra*, we mean a tubular extension (equivalently, tubular coextension) of a tame concealed algebra of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$, as defined in [33]. Recall that a tubular algebra $B$ admits two different tame concealed convex subcategories $C_0$

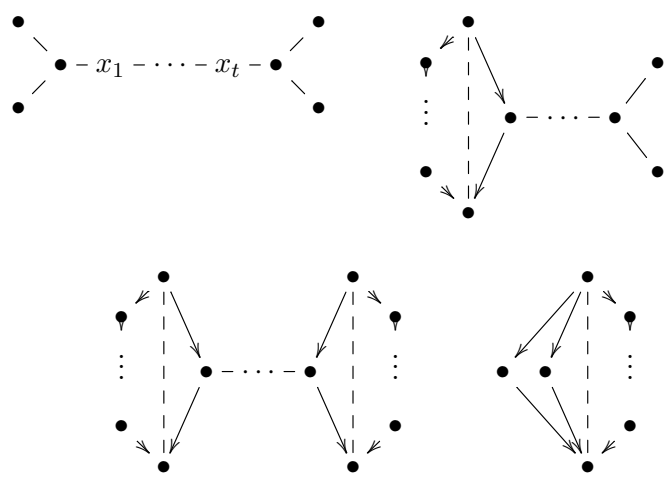
and $C_\infty$ such that the Auslander-Reiten quiver $\Gamma_B$ of $B$ is of the form

$$\Gamma_B = \mathcal{P}_0^B \vee \mathcal{T}_0^B \vee \left( \bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \right) \vee \mathcal{T}_\infty^B \vee I_\infty^B,$$

where $\mathcal{P}_0^B$ is the preprojective component $\mathcal{P}^{C_0}$ of $\Gamma_{C_0}$, $\mathcal{T}_0^B$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard ray tubes, obtained from the stable tubes of $\mathcal{T}^{C_0}$ by ray insertions, $I_\infty^B$ is the preinjective component $I^{C_\infty}$ of $\Gamma_{C_\infty}$, $\mathcal{T}_\infty^B$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard coray tubes, obtained from the stable tubes of $\mathcal{T}^{C_\infty}$ by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers), $\mathcal{T}_q^B$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard stable tubes; see [33].

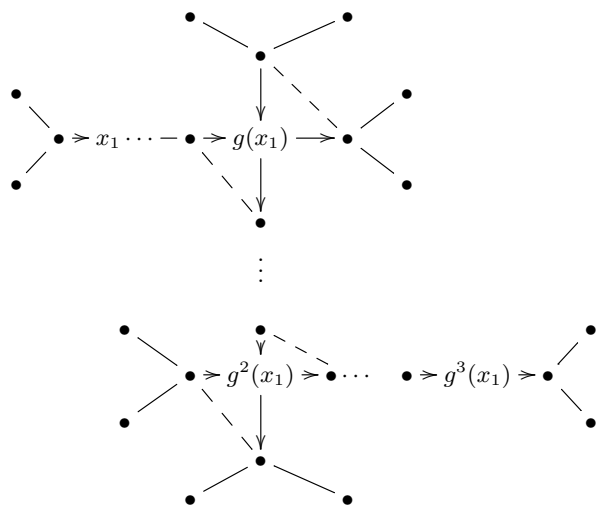
**Lemma 3.1.** *Let $R$ be a tame strongly simply connected locally bounded category and $G$ be a group acting freely on $R$. Let $C$ be a tame concealed algebra of type $\tilde{\mathbb{D}}_n$ which is a convex subcategory of $R$. Assume $x_1$ is a vertex of $C$ in a convex line $y - x_1 - x_2 - \cdots - x_t - y'$ such that each $x_i$ has exactly two neighbors in the quiver of $C$ and $x_t = g(x_1)$, for some $g \in G$. Then, for every number $s$ there are indecomposable $R$-modules $Y_s$ containing at least $s$ convex tame concealed subcategories in the support supp $Y_s$.*

*Proof.* Tame concealed algebras of type $\tilde{\mathbb{D}}_s$ are given by the following frames:




with all commutativity relations. For the sake of simplicity, we assume that $x_i$, for $1 \le i \le t$, are given as in the first frame. Then, in $R$ we get

a convex subcategory $B_3$ of the shape




up to change of some arrow orientations. Clearly, $B_3$ accepts an indecomposable sincere module $Y_3$ whose support contains 6 tame concealed convex subcategories. Similarly, we may construct the desired indecomposable $R$-modules $Y_s$, for $s \geq 4$. $\square$

Let $B$ be an algebra, $\mathcal{C}$ be a standard component of $\Gamma_B$ and $X$ be an indecomposable module in $\mathcal{C}$. In [3], three *admissible operations* (ad 1), (ad 2) and (ad 3) were defined depending on the shape of the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ in order to obtain a new algebra $B'$.

(ad 1) If the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ is of the form

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

then we set $B' = (B \times D)[X \oplus Y_1]$, where $D$ is the full $t \times t$ lower triangular matrix algebra and $Y_1$ is the indecomposable projective-injective $D$-module.

(ad 2) If the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ is of the form

$$Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

with $t \geq 1$, so that $X$ is injective, then we set $B' = B[X]$.

(ad 3) If the support of $\mathrm{Hom}_B(X,-)|_{\mathcal{C}}$ is of the form

$$\begin{array}{ccccccccccc} & Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & & \\ & \uparrow & & \uparrow & & & & \uparrow & & & \\ X = & X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t & \to \cdots \end{array}$$

with $t \geq 2$, so that $X_{t-1}$ is injective, then we set $B' = B[X]$.

In each case, the module $X$ and the integer $t$ are called, respectively, the *pivot* and the *parameter of the admissible operation*. The dual operations are denoted by (ad 1*), (ad 2*) and (ad 3*).

Following [3], an algebra $A$ is a *coil enlargement* of the critical algebra $C$ if there is a sequence of algebras $C = A_0, A_1, \ldots, A_m = A$ such that for $0 \leq i < m$, $A_{i+1}$ is obtained from $A_i$ by an admissible operation with pivot in a stable tube of $\Gamma_C$ or in a component (coil) of $\Gamma_{A_i}$ obtained from a stable tube of $\Gamma_C$ by means of the admissible operations done so far. When $A$ is tame, then we call $A$ a *coil algebra*.

If $A$ is a coil enlargement of a critical algebra $C$, then there is a maximal branch coextension $A^-$ of $C$ inside $A$ which is full and convex in $A$, and such that $A$ is obtained from $A^-$ by a sequence of admissible operations of types (ad 1), (ad 2) and (ad 3). Dually, there is a maximal branch extension $A^+$ of $C$ inside $A$ which is full and convex in $A$, and such that $A$ is obtained from $A^+$ by a sequence of admissible operations of types (ad 1*), (ad 2*) and (ad 3*).

For a coil enlargement $A$ of a critical algebra $C$, we consider the type $r(A)$ of $A$ as follows: Let $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(k)}$ be the separating tubular family of $\mathrm{mod}\, C$. For each $\lambda \in \mathbb{P}_1(k)$, let $n_\lambda$ be the rank of $\mathcal{T}_\lambda$ and $r_\lambda^+ - n_\lambda$ (respectively, $r_\lambda^- - n_\lambda$) be the number of rays (respectively, corays) inserted in $\mathcal{T}_\lambda$ by the sequence of admissible operations that leads from $C$ to $A$. Finally, let $r(A) = (r_\lambda^+, r_\lambda^-)_{\lambda \in \mathbb{P}_1(k)}$, where we write down only those numbers greater or equal to 1.

**Proposition 3.2.** *Let $B$ be a coil enlargement of a tame concealed algebra $C$. The following conditions are equivalent.*

(*a*) *$B$ is tame.*
(*b*) *$B^+$ and $B^-$ are tame.*
(*c*) *Every cycle*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_s} X_s = X$$

*of non-zero non-isomorphisms between indecomposable modules in* $\mathrm{mod}\, B$. *belongs to a standard coil in* $\Gamma_B$.

(*d*) *$B$ is of polynomial growth.*
(*e*) *$B$ is of linear growth.*
(*f*) *$B$ is cycle-finite.*
(*g*) *Each of $r^+(B)$ and $r^-(B)$ is one of the following: $(p,q)$ where $1 \leq p \leq q$, $(2,2,r)$ with $r \geq 2$, $(2,3,3),(2,3,4),(2,3,5),(3,3,3)$, $(2,4,4),(2,3,6),(2,2,2,2)$.*

Essential for our considerations is the following theorem which is the main result of [42].

**Theorem 3.3.** *Let $A$ be a strongly simply connected algebra. The following conditions are equivalent.*

(*a*) *$A$ is of polynomial growth.*
(*b*) *$A$ is of linear growth.*
(*c*) *$A$ is cycle-finite.*
(*d*) *$A$ does not contain a convex subcategory which is pg-critical or hypercritical.*
(*e*) $\operatorname{rad}^\infty(\operatorname{mod} A)$ *is locally nilpotent.*
(*f*) *The component quiver $C(A)$, whose vertices are components of the Auslander-Reiten quiver $\Gamma_A$ and arrows $\mathcal{C} \to \mathcal{C}'$ are set when there are modules $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$ with $\operatorname{rad}^\infty(X,X') \neq 0$, has no oriented cycles.*
(*g*) *Every connected component of $\Gamma_A$ is standard.*

A special situation of the above Theorem is the following.

**Lemma 3.4.** *Let $B$ be a strongly simply connected cycle-finite algebra and $M$ be an indecomposable $B$-module. Assume that*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

*is a cycle of non-zero non-isomorphisms between pairwise different indecomposable modules in* $\operatorname{mod} B$*, such that $6 \leq s$ and $f_1$ factorizes nontrivially in* $\operatorname{mod} B[M]$*. Then, one of the following two situations occur:*

(*i*) *$B$ contains a convex subcategory $B'$ which is a coil extension such that one of the two $r^+(B')$ or $r^-(B')$ is $(2,2,s)$.*
(*ii*) *$B[M]$ is of wild type.*

*Proof.* Indeed, by [42] (2.3), the algebra $B$ is multicoil and the given cycle belongs to a standard coil $\mathcal{T}$ of a multicoil of $\Gamma_B$. Let $C$ be a tame concealed algebra such that $B'$ is a convex subcategory of $B$ and coil extension of a tame concealed algebra $C$. Assume $(i)$ does not hold,

that is, $B'$ is of type $(r_1, r_2, r_3)$, with $r_1 \leq r_2 \leq r_3$ and $3 \leq r_2$, or of type $(2, 2, 2, 2)$.

Let $\mathcal{T}'$ be the component of $\Gamma_{B[M]}$ where $X$ belongs. Observe that $\mathrm{Hom}_B(M, \mathcal{T}) \neq 0$ and since $f_1$ is factorized there is a cycle of non-zero non-isomorphisms between $6 < s + 1$ pairwise different indecomposable modules in $\mathrm{mod}\, B[M]$. If $M$ belongs to $\mathcal{T}'$, then either $M$ is not a pivot module or the extension type of $B'[M]$ is not tame. In the latter case, $B[M]$ is wild. Moreover, if $M$ is not a pivot module, according to [29], the one-point extension $B'[M]$ is tame only when $B'$ is of type $(2, 2, s)$. Since this is forbidden, then $B[M]$ is wild.

If $M$ does not belong to $\mathcal{T}'$, then there is a regular $C$-module $Y$ such that $\mathrm{Hom}_B(M, Y) \neq 0$, and $B[M]$ contains a convex subcategory of the form $C[N]$ for a preprojective $C$-module $N$. The extension $C[N]$ being wild implies that $B[M]$ is wild. □

The following theorem is the main result of [30].

**Theorem 3.5.** *Let $R$ be a connected cycle-finite locally bounded $k$-category over an algebraically closed field $k$, $G$ be a torsion-free admissible group of $k$-linear automorphisms of $R$, and $A = R/G$. Let $\mathcal{S}$ be a separating family of convex subcategories of $R$ with respect to $G$ and $\mathcal{S}_0$ be a fixed set of representatives of $G$-orbits in $\mathcal{S}$. Then, the functors $\Phi^Y = F_\lambda(Y) \otimes_{k[T,T^{-1}]} - : \mathrm{mod}\, k[T, T^{-1}] \to \mathrm{mod}\, A$, $Y \in \mathcal{S}_0$, induce a $k$-linear equivalence of categories*

$$\Phi : \coprod_{\mathcal{S}_0} \mathrm{mod}\, k[T, T^{-1}] \xrightarrow{\sim} \mathrm{mod}\, A/[\mathrm{mod}_1 A].$$

*Moreover, the following statements hold.*

(*i*) *$A$ is tame.*

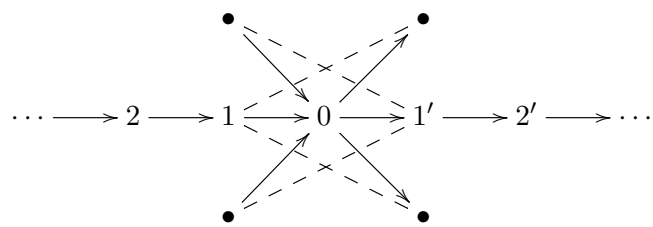
(*ii*) *Every indecomposable finite dimensional $A$-module $X$ is isomorphic either to $F_\lambda(M)$ for some indecomposable finite dimensional $R$-module $M$ or to $\Phi^Y(V)$ for some $Y \in \mathcal{S}_0$ and some indecomposable finite dimensional $k[T, T^{-1}]$-module $V$.*

(*iii*) *The Auslander-Reiten quiver $\Gamma_A$ of $A$ has the disjoint union decomposition*

$$\Gamma_A = (\Gamma_R/G) \sqcup \left( \coprod_{\mathcal{S}_0} \Gamma_{k[T,T^{-1}]} \right)$$

*where $\Gamma_{k[T,T^{-1}]}$ is the Auslander-Reiten quiver of the category of finite dimensional $k[T, T^{-1}]$-modules.*

There are strongly simply connected categories $R$ of polynomial growth which are not cycle-finite, as the following example shows. Consider the category $R$ given by the following quiver with relations as indicated by the dotted edges:




Since $R$ has tame coil enlargements $R_s$ of a hereditary algebra $C$ of Euclidean type $\tilde{\mathbb{D}}_4$ of type $(2,2,s)$, for arbitrary $s \geq 1$, then $\operatorname{mod} R$ accepts cycles of non-zero morphisms between indecomposable $R$-modules of arbitrary length. We may build a non-trivial infinite cycle in $\operatorname{mod} R'$, where $R'$ is the quotient of $R$ obtained by adding a zero-relation from 1 to $1'$, of the form

$$S_0 \to P_1 \to P_s \to \begin{matrix} 0 & & 1 & & 1 & & 1 & & 1 & & 0 \\ & 1 & & \to & & 2 & & \to & & 1 & \\ 0 & & 1 & & 1 & & 1 & & 1 & & 0 \end{matrix} \to I_{s'} \to I_{1'} \to S_0$$

where $S_0$ is the simple module at 0, $P_j$ (respectively, $I_j$) is the indecomposable projective cover (respectively, injective envelope) of $S_j$ in $\operatorname{mod} R'$ and the dimension vectors correspond to indecomposable $C$-modules $X_i$, $i = 1,2,3$. Observe that the composition of maps $S_0 \to X_1$ is non-zero in $\operatorname{rad}^\infty(\operatorname{mod} R)$.

We say that the category $R$ is of **type** $(2,2,\infty)$ if for every $m$ it contains a convex subcategory $B_m$ which is a coil enlargement of type $(2,2,m)$, $B_m$ is a subcategory of $B_{m+1}$ and $R = \bigcup_m B_m$.

The next result is preparatory for the main theorem of our work.

**Lemma 3.6.** *Let $R$ be a strongly simply connected cycle-finite category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion-free group $G$. Assume that $R$ is of polynomial growth. Then, the followings hold:*

(*i*) *there is a number $s_0$ such that, for any finite convex subcategory $B$ of $R$, any periodic $B$-module has period at most $s_0$;*

(*ii*) *for any cycle*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} X_s = X$$

*of length $s \geq s_0$, there is a convex subcategory $B$ of $R$ and a coil $\mathcal{T}$ in $\operatorname{mod} B$ containing all modules $X_i$, $1 \leq i \leq s$, and at least $s - s_0$ indecomposable projective modules.*

*Proof.* $(i)$ : Let $s_0 = 2n + 4$, where $n$ is the number of vertices in the quiver of $A$. Consider a convex subcategory $B$ of $R$ with a periodic module $X$ of period $p > s_0$. Since $B$ is a multicoil algebra, then $X$ lies in a stable tube. By [3], the support of $X$ is a tame concealed or a tubular algebra. Without lost of generality, we may asume that $B$ is tame concealed or a tubular algebra.

Since $p > 6$, then $B$ is tame concealed of type $\tilde{\mathbb{D}}_{p-2}$. From the structure of the frames of the tame concealed algebras, we get a linear convex subcategory of $B$ of the shape $y - x_1 - x_2 - \cdots - x_t - y'$ such that each $x_i$ has exactly two neighbors in the quiver of $B$ and $x_t = g(x_1)$, for some $g \in G$. By Lemma 3.1, there is an indecomposable $R$-module whose support contains at least 4 convex tame concealed subcategories. This contadicts with the result in [25].

$(ii)$ : is a consequence of $(i)$ and the structure of multicoil components of the Auslander-Reiten quiver of multicoil algebras. □

## 4. The main results

**Theorem 4.1.** *Let $R$ be a strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion free group $G$. The followings are equivalent.*

- $(a)$ *$R$ is of polynomial growth and does not contain a convex subcategory of type $(2, 2, \infty)$.*
- $(b)$ *$R$ is of linear growth and does not contain a convex subcategory of type $(2, 2, \infty)$.*
- $(c)$ *$R$ is cycle-finite.*
- $(d)$ *$R$ does not contain a convex subcategory which is of type $(2, 2, \infty)$, pg-critical or hypercritical.*
- $(e)$ *$R$ does not contain a convex subcategory which is pg-critical or hypercritical and there exists a set of representatives $\mathcal{S}_0$ of the $G$-orbits in a separating family $\mathcal{S}$ of convex subcategories of $R$ with respect to $G$ formed by lines.*

  *Moreover, if any of the above holds, then the following holds:*
- $(f)$ $\operatorname{rad}^{\infty}(\operatorname{mod}_1 A)$ *is locally nilpotent.*

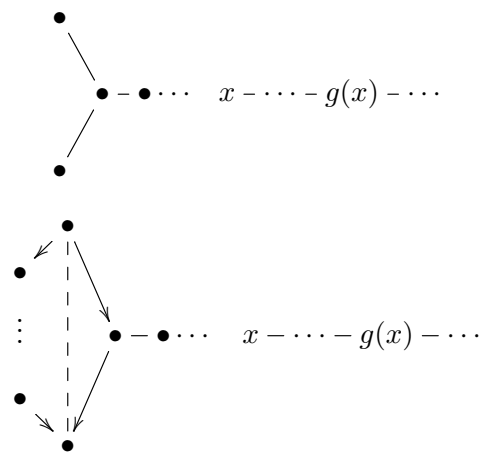
*Proof.* The equivalence of $(a)$, $(b)$ and $(d)$ follows obviously from Theorem (4.1) in [42]. If $(c)$ is satisfied, then clearly $(a)$ is satisfied. Assume that $(a)$ holds, that is, $R$ is of polynomial growth not accepting convex subcategories of type $(2,2,\infty)$. We shall show that there is a number $s$ such that the maximal length of a cycle in $\operatorname{mod} R$ is $s$ and therefore $R$ is cycle finite.

Suppose, to get a contradiction, that for every number $s$ there is a cycle

$$\eta_s : X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_{t(s)}} X_{t(s)} = X$$

of length $t(s) \geq s$. As in Lemma 3.6, there is a number $s_0$ such that, for any finite convex subcategory $B$ of $R$, any periodic $B$-module has period at most $s_0$. In particular, any tame concealed convex subcategory $C$ of $R$ is of Euclidean type $(2,3,3),(2,3,4),(2,3,5)$ or $(2,2,r)$ with $2 \leq r \leq s_0$. Moreover, each cycle $\eta_s$ lies in a coil $\mathcal{T}_s$ in $\operatorname{mod} B_s$ containing at least $t(s) - s_0$ indecomposable projective modules, where $B_s$ is a convex subcategory of $R$ which is a coil extension of a tame concealed algebra $C_s$. Moreover, without lost of generality, we may assume that $B_s = B'_s[M_s]$ is a one-point extension of a coil algebra $B'_s$ by a module in $\mathcal{T}_s$. Since there are only finitely many orbits of the action of $G$ on $R$, there is a finite set $F$ of numbers such that for every number $s$ there is an element $g_s \in G$ such that $g_s(C_s) = C_{f(s)}$, for some $f(s) \in F$. Replacing $\eta_s$ by $g_s(\eta_s)$ and choosing some $s' \in F$ with an infinite preimage $f^{-1}(s')$, we may assume, without lost of generality, that every $B_s$ is a coil extension of the tame concealed algebra $C$. By Lemma 3.6 and $s \geq 7$, $C$ is of type $(2,2,t_0)$ with $t_0 \leq s_0$ and therefore, for $t_0 \leq s$, the cycle $\eta_s$ lies in a coil $\mathcal{T}_s$ with at least $t(s) - t_0$ projective modules. Moreover, $\mathcal{T}_{s'}$ is a coil extension of the coil $\mathcal{T}_s$, for any $s' \geq s$. Clearly, $R$ contains a convex subcategory of type $(2,2,\infty)$, a contradiction showing $(c)$.

$(c)$ is equivalent to $(e)$: we already observed that weakly $G$-periodic subcategories of a strongly simply connected cycle-finite category $R$ are lines. For the converse, assume that $(e)$ is satisfied. By theorem 3.3, every finite convex subcategory of $R$ is of polynomial growth, that is, $R$ is of polynomial growth. Assume, to get a contradiction, that $B$ is a convex subcategory of $R$ of type $(2,2,\infty)$; in particular, there is a convex subcategory $D$ of $R$ tilted of type $\mathbb{D}_s$ with $s > n+2$ for $n$, the number of vertices of the quiver $Q_A$, given by a quiver with relations corresponding to one of the following frames of categories:




for some $x \in Q_D$ and some $g \in G$. Clearly, this yields a convex subcategory $D'$ of $R$ which is tame concealed of type $\widetilde{\mathbb{D}}_t$ and a convex line $x - x_1 - x_2 - \cdots - x_t - g(x)$ such that each $x_i$ has exactly two neighbors in the quiver of $C$. Applying Lemma 3.1, we get indecomposable $R$-modules $Y$ whose support contain at least 4 different tame concealed algebras. This contradicts the main result in [25].

$(c)$ implies $(f)$: Assume $(c)$ holds. Consider $M$ an indecomposable $A$-module of the first kind and a linear map $f : M \to M$ in $\operatorname{rad}^\infty(\operatorname{mod}_1 A)$. Suppose that $F_\lambda(X) = M$, for some indecomposable $R$-module $X$. Then, there are maps $f_g \in \operatorname{Hom}_R(X, X^g)$, almost all $f_g = 0$, such that $\sum\limits_{g\in G} F(f_g)$
$= f$. Since $f \in \operatorname{rad}^s(\operatorname{mod}_1 A)$ then, $f^g \in \operatorname{rad}^s(\operatorname{mod} R)$, for any $s \geq 1$. Suppose $0 \neq f = f_1 \cdots f_r$, for some $f_i \in \operatorname{rad}^\infty(M, M)$, there exist maps $f_{(i,g)} \in \operatorname{rad}^\infty(X, X^g)$, for $1 \leq i \leq r$, with almost all $f_{(i,g)} = 0$, such that $\sum\limits_{g\in G} F(f_{(i,g)}) = f_i$. We get

$$f_g = \sum_{g=g_r\cdots g_1} f_{(r,g_r)}^{g_{r-1}\cdots g_1} \cdots f_{(2,g_2)}^{g_1} f_{(1,g_1)}.$$

Call $X_0 = X, X_1 = X^{g_1}, X_2 = X^{g_2 g_1}, \cdots, X_r = X^{g_r\cdots g_2 g_1}$ and consider a non-zero composition of maps $0 \neq f_r \cdots f_2 f_1$ with $f_i \in \operatorname{rad}^\infty(X_{i-1}, X_i)$, $1 \leq i \leq r$. Since $R$ is cycle-finite and therefore $\operatorname{rad}^\infty(Y, Y) = 0$ for any indecomposable $R$-module $Y$, then the modules $X_i$, $0 \leq i \leq r$, are pairwise non-isomorphic indecomposable $R$-modules with the same

dimension $d = \dim_k M$. The Harada-Sai lemma yields a contradiction, in case $r \geq 2^d$. This shows that $\mathrm{rad}^\infty(\mathrm{mod}_1 A)$ is locally nilpotent. $\square$

Given a Galois covering $R \to R/G = A$ of a finite dimensional $k$-algebra $A$, we observe that a component $\mathcal{C}'$ of $\Gamma_A$ is either of the *first kind*, that is formed by the modules $F_\lambda(X)$, for $X \in \mathcal{C}$ for a component $\mathcal{C}$ in $\Gamma_R$, or of the *second kind*, that is formed by the modules $\Phi^Y(V)$, for $Y$ a fixed weakly $G$-periodic module and $V$ an indecomposable $k[T, T^{-1}]$-module. The following consequence for the structure of components of the Auslander-Reiten quiver $\Gamma_A$ is obtained.

**Proposition 4.2.** *Let $R$ be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion free group $G$. Let $\mathcal{C}$ be a component of the Auslander-Reiten quiver $\Gamma_R$. The followings hold:*

- $(a)$ *the set of vertices $a$ such that $X(a) \neq 0$, for some indecomposable $X \in \mathcal{C}$, form a convex subcategory $B(\mathcal{C})$ of $R$;*
- $(b)$ *the stabilizer $G' = G_{\mathcal{C}}$ of $\mathcal{C}$ is a normal subgroup of $G$;*
- $(c)$ *the category $B(\mathcal{C})$ is strongly simply connected and cycle-finite, the induced functor $F' : B(\mathcal{C}) \to A'$ is a Galois covering defined by the action of a torsion free group $G'$, and $\mathcal{C}$ is a component of $\Gamma_{B(\mathcal{C})}$ with stabilizer $G'_{\mathcal{C}} = G'$;*
- $(d)$ *every component of the Auslander-Reiten quiver $\Gamma(\mathrm{mod}_1 A)$ is generalized standard.*

*Proof.* $(a)$ : Assume $a_1 \to a_2 \to \cdots \to a_r$ is a path in the quiver $Q_R$ such that $X(a_1) \neq 0 \neq Y(a_r)$, for indecomposable modules $X, Y \in \mathcal{C}$ and $Z(a_i) = 0$, for $2 \leq i \leq r-1$, and all $Z \in \mathcal{C}$. We shall construct a cycle in the componental quiver $C(R)$. This contradicts [42](4.1).

Indeed, consider the quotient $R'$ of $R$ obtained by adding relations $a_1 \to a_2 \to b$ and $c \to a_{r-1} \to a_r$, for all arrows $a_2 \to b$ and $c \to a_{r-1}$. Consider $I_x$ to be the injective envelope and $P_x$ to be the projective cover of the simple module $S_x$ corresponding to a vertex $x$ in the category $\mathrm{mod}\, R'$. We get a path of morphisms in $\mathrm{mod}\, R$ to be

$$Y \to I_{a_r} \to S_{a_{r-1}} \to F(a_{r-1}, a_{r-2}) \to S_{a_{r-2}} \to \cdots \to F(a_3, a_2) \to S_{a_2} \\ \to P_{a_1} \to X$$

where for any arrow $y \to x$ in $Q_R$, the $R$-module $F(x, y)$ is the unique indecomposable whose composition factors are $S_x$ and $S_y$. Since $S_{a_i}$ does not belong to $\mathcal{C}$, for $2 \leq i \leq r-1$, we get a cycle through $\mathcal{C}$ in the componental quiver $C(R)$.

$(b)$ and $(c)$ are obvious.

$(d)$: Let $M$ and $N$ be two modules in $\mathcal{C}'$ and $0 \neq f \in \operatorname{rad}_A^\infty(M,N)$. Assuming that $\mathcal{C}'$ is of the first kind implies that there exists a component $\mathcal{C}$ in $\Gamma_R$ and indecomposable $R$-modules $X, Y \in \mathcal{C}$ such that $F_\lambda(X) = M$ and $F_\lambda(Y) = N$. Lifting the morphism $f$ provides morphisms $f_g \in \operatorname{rad}_R^\infty(X, Y^g)$, for $g \in G'$, almost all zero, such that $\sum_{g \in G'} F_\lambda(f_g) = f$. We remark that, for the algebra $A'$, we have $\operatorname{rad}_{A'}^\infty(M,N) = 0$. Indeed, a morphism $f' \in \operatorname{rad}_{A'}^\infty(M,N)$ yields the existence of morphisms $f'_g \in \operatorname{rad}_R^\infty(X, Y^g)$, for $g \in G'$, almost all zero, such that $\sum_{g \in G'} F_\lambda(f'_g) = f'$. Since for $g \in G'$ the module $Y^g \in \mathcal{C}$, then [42](4.1) implies that $\operatorname{rad}_R^\infty(X, Y^g) = 0$. Hence $f'_g = 0$ and $f' = 0$.

Since $\operatorname{rad}_{A'}^\infty(M,N) = 0$ then, for every $g \in G$ such that $f_g \neq 0$, we have $\operatorname{rad}_{B(\mathcal{C})}^\infty(X, Y^g) = 0$ and there is a chain of irreducible maps connecting $X$ and $Y^g$, that is, $Y^g \in \mathcal{C}$ and $g \in G_{\mathcal{C}}$. Up to a change of orientation, we may assume that there is an indecomposable projective $R$-module $P_a \notin \mathcal{C}$ such that $\operatorname{rad} P_a = L$ and the one-point extension category $B' = B(\mathcal{C})[L]$ is convex in $R$. Moreover, $f_g \in \operatorname{rad}_{B'}^\infty(X, Y^g)$ factorizes through a module $Z \in \operatorname{mod} B'$ satisfying $Z(a) \neq 0$. Therefore, there is a direct summand $Z'$ of $Z$ satisfying $Z' \notin \mathcal{C}$ and there is a cycle in the componental quiver $C(R)$ of the form $\mathcal{C} = [X] \to [Z'] \to [Y^g] = \mathcal{C}$, where $[Z']$ denotes the component in $\Gamma_R$ containing $Z'$. $\square$

## 5. Criteria for polynomial growth

The aim of this section is to establish a criterion for an algebra with a cycle-finite Galois covering to be of polynomial growth (respectively, domestic type). We start by recalling a criterion in [30].

**Theorem 5.1.** *Let $R$ be a connected cycle-finite locally bounded $k$-category, $G$ be a torsion-free admissible group of $k$-linear automorphisms of $R$, and $A = R/G$. Then the followings hold.*

- $(i)$ *$A$ is of polynomial growth if and only if the number of $G$-orbits of isoclasses of weakly $G$-periodic $R$-modules is finite.*
- $(ii)$ *$A$ is domestic if and only if $R$ does not contain a convex subcategory which is tubular and the number of $G$-orbits of isoclasses of weakly $G$-periodic $R$-modules is finite.*

Part of the following result is explicit in [30].

**Theorem 5.2.** *Let $R$ be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion free group $G$. Let $\mathcal{S}_0$ be a set of representatives of the $G$-orbits in a separating family $\mathcal{S}$ of convex subcategories of $R$ with respect to $G$. Then the followings hold.*

(1) *The category* $\operatorname{mod}_1 A$ *of modules of the first type is of polynomial growth.*
(2) *The category* $\operatorname{mod}_2 A$ *of modules of the second kind is of polynomial growth if and only if $\mathcal{S}_0$ is a finite set.*
(3) *The algebra $A$ is of polynomial growth if and only if the cardinality of $\mathcal{S}_0$ is bounded by the number $n$ of vertices in $Q_A$.*

*Proof.* (1): Since every convex subcategory of $R$ is of polynomial growth, by [16], Lemma 3, the category of modules of the first kind $\operatorname{mod}_1 A$ is of polynomial type.

(2): This results from Theorem 4.1 in [30].

(3): Assume there are different lines $L_1, \dots, L_s \in \mathcal{S}_0$ for any $s > n$. Obviously, not all sets of vertices $F(L_i)$ are disjoint. We may suppose $x$ is a vertex in $L_1 \cap L_2$. Let $1 \neq g \in G_{L_1}$ and observe that $g(x) \notin L_2$, since otherwise, by Proposition 2.3, we would have $L_1 = L_2$. Consider the line $L'_s$, for $s \in \mathbb{N}$ formed by the vertices

$$\cdots - y_{-2} - y_{-1} - x - x_1 - \cdots - x_t = g(x) \quad \cdots - g^2(x) - \cdots - g^s(x) -$$

$$- g(y_1) - g(y_2) - \cdots$$

where $x - x_1 - \cdots x_t = g(x) - \cdots - x_{2t} = g^2(x) - \cdots - x_{st} = g^s(x)$ is the convex segment of $L_1$ connecting $x$ and $g^s(x)$, and $\cdots - y_{-2} - y_{-1} - x - y_1 - y_2 - \cdots$

is the line $L_2$. We may assume that $y_{-1}$ and $g(y_1)$ are not in the line $L_1$. We claim that the lines $L'_s$ determine pairwise different elements in $\mathcal{S}_0$. Indeed, assume that $h(L'_p) = L'_q$, for some $p \leq q$, and $h \in G$. Then $h$ sends infinite segments of $L_2$ to $L_2$, and hence $h \in G_{L_2}$. Moreover, $L'_p$ contains exactly $tp$ vertices of $L_1$, which yields $p = q$. □

The structure of $G$ is sometimes a source of information on the families of second kind modules, and hence on the representation type of $R/G$. Namely, we show the following proposition.

**Proposition 5.3.** *Let $R$ be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional*

*algebra $A$ defined by the action of a torsion-free group $G$. If $G$ is cyclic, then $A$ is of polynomial growth.*

*Proof.* Assume that $\mathcal{S}_0$ is not empty and assume that $G$ is cyclic. Take lines $L_1, \ldots, L_s \in \mathcal{S}_0$, for any $s > n$, where $n$ is the number of vertices in $Q_A$. Obviously, not all the sets of vertices $F(L_i)$ are disjoint. We may suppose $x$ to be a vertex in $L_1 \cap L_2$. Since $G_{L_1}$ and $G_{L_2}$ are non-trivial cyclic subgroups of $G$, then $G/(G_{L_1} \cap G_{L_2})$ is a finite group. Let $1 \neq g \in G_{L_1} \cap G_{L_2}$ and observe that $x$ and $g(x)$ belong to $L_1 \cap L_2$ which, according to Proposition 2.3, is formed by a unique connected segment. This yields $L_1 = L_2$, and the cardinality of $\mathcal{S}_0$ is at most $n$. □

## 6. **Examples**

Here we illustrate some results of our work in four parts.

(1) We start by giving an example (see [12]) of the relation between structural properties of the category $R$ and the group $G$ defining the Galois covering.

**Theorem 6.1.** *Let $R$ be a strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional triangular algebra $A$ defined by the action of a torsion free group $G$. Then, $G$ is a free (non-abelian) group.*

**Sketch of proof:** (i) Assume $A = B[M]$ to be a one-point extension of an algebra $B$ by a module $M$. Let $a$ be a source vertex in $Q_A$ such that $\operatorname{rad} P_a = M$ and $x$ be a vertex in $Q_R$ such that $F(x) = a$. Consider $R'$ the convex subcategory formed by those vertices at the preimage $F^{-1}(B)$ and choose a connected component $R^B$ of $R'$. The stabilizer $G^B$ of $R^B$ is a normal subgroup of $G$. Consider $F^B : R^B \to B$ to be the functor obtained as the restriction of $F$. We get that $R_B$ is a strongly simply connected category and $F^B : R^B \to B$ is a Galois covering functor of a finite dimensional triangular algebra $B$ defined by the action of a torsion free group $G^B$.

By induction on the $\dim_k A$, we may assume that $G^B$ is a free group.

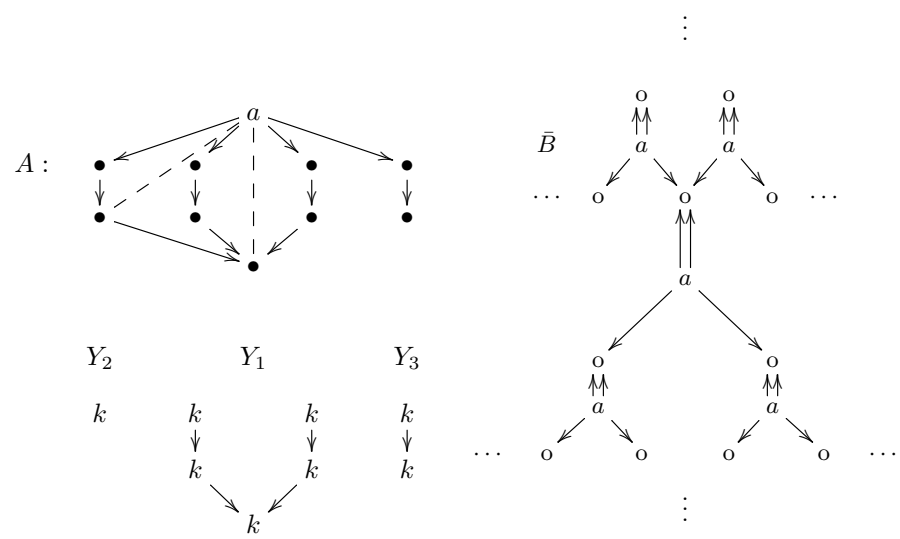
(ii) We show that $F$ is a covering of the first kind, that is, if $\operatorname{rad} P_a = M_1 \oplus \cdots \oplus M_t$ is an indecomposable decomposition in $\operatorname{mod} B$, then there is an indecomposable decomposition $\operatorname{rad} P_x = Y_1 \oplus \cdots \oplus Y_s$ in $\operatorname{mod} R'$ such that $s = t$ and a permutation $\sigma$ satisfying $F_\lambda(Y_i) = M_{\sigma(i)}$, for $1 \leq i \leq t$.

Indeed, since $R$ is strongly simply connected, then the source $x$ *separates* $R$, that is, there are connected components $R_1, \cdots, R_s$ of $R'$ such that the support of $Y_i$ is contained in $R_i$, for $1 \leq i \leq s$. Therefore, $\operatorname{rad} P_a = \operatorname{rad} F_\lambda(Px) = \bigoplus_{i=1}^{s} F_\lambda(Y_i)$ is an indecomposable decomposition and the claim follows.

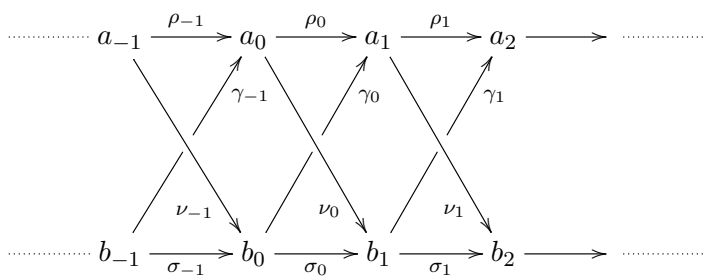
(iii) The group $G/G^B$ is a free group $F(t-1)$ of rank $t-1$.

(iv) $G$ is isomorphic to the free product $G^B * F(t-1)$ and it is therefore a free group.

To illustrate the idea of the proof, assume that $t = 3$. Construct the category $\bar{B}$ as a model for a covering $F' : \bar{B} \to A$ defined by the action of $F(t-1) = F(2)$. Sustitute each o in the diagram by the category $R_B$ in such a way that, for every vertex $a$, the radical $\operatorname{rad} P_a = Y_1 \oplus Y_2 \oplus Y_3$ (in the representations $Y_i$ the arrows stand for identity maps; observe that the vertical arrows in $\bar{B}$ contribute $Y_1$ to the radical of $P_a$). The functor $F : R \to A$ factorizes as $F = \bar{F}F'$ by a Galois covering functor $\bar{F} : R \to \bar{B}$ defined by the action of $G^B$ (in the example $G^B = \mathbb{Z}$):

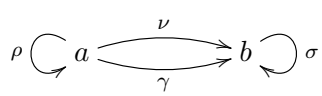



(2) As another series of examples, consider the categories $R_{\alpha,\beta}$ given by the quiver with relations




$$\left.\begin{aligned}
\sigma_{i+1}\sigma_i &= \alpha\nu_{i+1}\gamma_i \\
\nu_{i+1}\rho_i &= \beta\sigma_{i+1}\nu_i \\
\gamma_{i+1}\nu_i &= \rho_{i+1}\rho_i \\
\rho_{i+1}\gamma_i &= \gamma_{i+1}\sigma_i
\end{aligned}\right\}$$

and $(\alpha,\beta) \neq (1,1)$, to be locally support finite; it is simply connected but not strongly simply connected. Moreover, the group $\mathbb{Z}$ generated by the action $(a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1})$ acts freely on $R_{\alpha,\beta}$ and on $\operatorname{ind}_{R_{\alpha,\beta}} / \cong$. Hence, the Galois covering $F\colon R_{\alpha,\beta} \to A_{\alpha,\beta}/\mathbb{Z}$ yields a bijection $F_\lambda\colon (\operatorname{ind} R_{\alpha,\beta}/\cong)/\mathbb{Z} \to (\operatorname{ind} A_{\alpha,\beta})/\cong$. The algebra $A_{\alpha,\beta}$ is given by the quiver with relations




$$\left.\begin{aligned}
\sigma^2 &= \alpha\gamma\nu \\
\rho\nu &= \beta\nu\sigma \\
\nu\gamma &= \rho^2 \\
\gamma\rho &= \sigma\gamma
\end{aligned}\right\}$$

Since $R_{\alpha,\beta}$ is tame (respectively polynomial growth for $\alpha\beta \neq 1$), so is $A_{\alpha,\beta}$.

(3) Consider the Galois covering

$$F\colon A = A^{(2)}_{1,1} \to \bar{A} = A^{(2)}_{1,1}/\mathbb{Z}_2$$

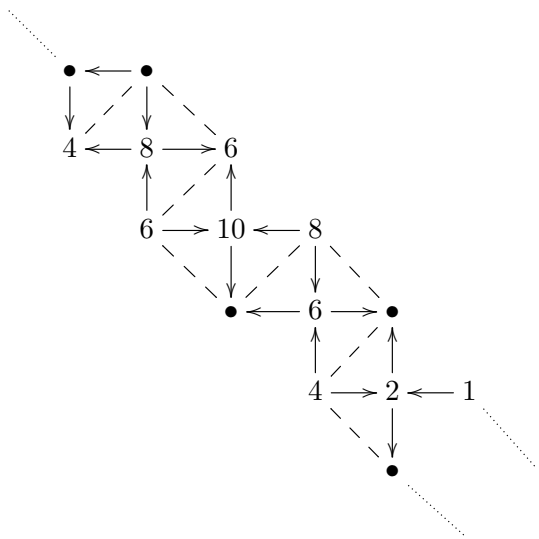
and assume that the characteristic of $k$ is 2. As a tubular algebra, we know that $A$ is tame. We show that $\bar{A}$ is a wild algebra.

Set $x_0 = \alpha_0 + \beta_0$, $y_0 = \beta_0$, $x_1 = \alpha_1 + \beta_1$, $y_1 = \beta_1$. Then, $\bar{A}$ is isomorphic to the algebra $A'$ given by the quiver with relations

$$A' : \bullet \underset{y_0}{\overset{x_0}{\rightrightarrows}} \bullet \underset{y_1}{\overset{x_1}{\rightrightarrows}} \bullet$$

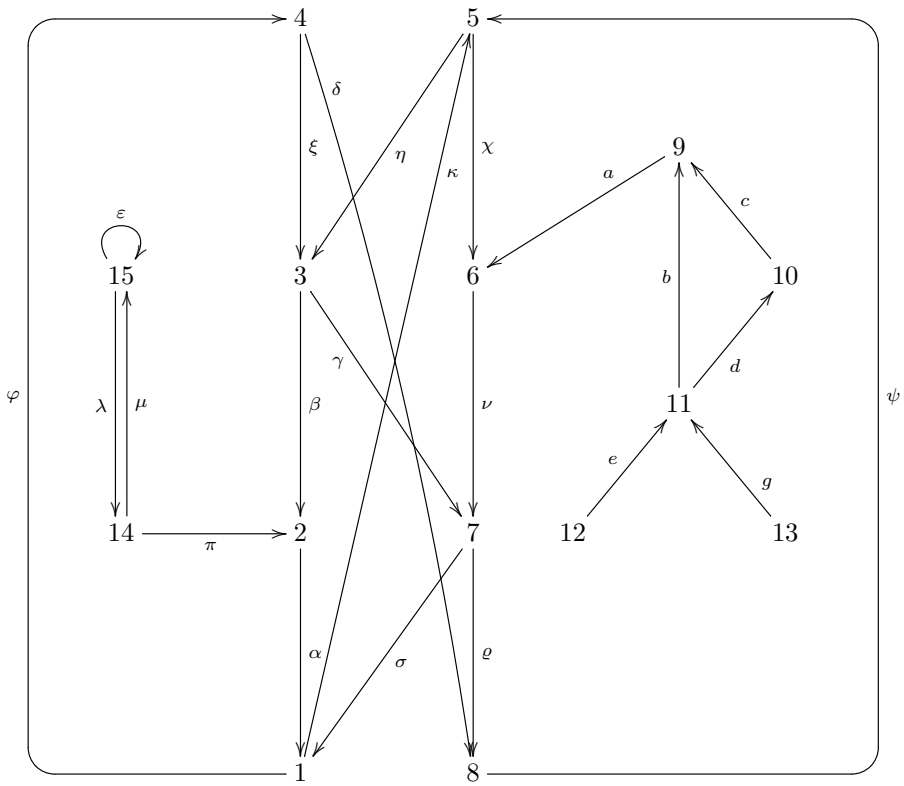
$$\left.\begin{array}{rcl} x_1 x_0 & = & 0 \\ y_1 x_0 & = & x_1 y_0 \end{array}\right\}$$

Observe that $A'$ accepts a Galois covering $R \to R/\mathbb{Z} = A'$, given by the category




which is strongly simply connected. Therefore, $R$ is tame if and only if the Tits form $q_R$ is weakly non-negative. Observe that the vector $y$ marked on the vertices of $R$ determines a convex subcategory $B$ of $R$ whose Tits form takes value $q_B(y) = -1$. Therefore, $R$ is a wild category and $A'$ a wild algebra.

(4) Our last example is similar to an example given in [30]. Let $A$ be the bound quiver algebra $kQ/I$ given by the quiver




and the ideal $I$ of the path algebra $kQ$ of $Q$ be generated by the elements $\alpha\varphi$, $\alpha\kappa$, $\sigma\varphi$, $\sigma\kappa$, $\gamma\sigma$, $\gamma\varrho$, $\nu\varrho$, $\nu\sigma$, $\eta\gamma$, $\eta\beta$, $\xi\beta$, $\xi\gamma$, $\varphi\delta$, $\delta\psi$, $\psi\eta$, $\psi\chi$, $\kappa\chi$, $\kappa\eta$, $a\nu$, $ba$, $dca$, $eb$, $gb - gdc$, $\pi\alpha$, $\lambda\pi$, $\mu\lambda\mu$, $\varepsilon^2 - \lambda\mu$, $\mu\lambda - \mu\varepsilon\lambda$.

For $k$ of characteristic 2, the convex subcategory $B$ of $A$ given by the objects 14 and 15 is a penny-farthing, and hence is a non-standard representation-finite algebra. Hence, for $k$ of characteristic 2, the algebra $A$ does not admit a simply connected (even triangular) Galois covering. For characteristic different from 2, $A$ is isomorphic to $kQ/I'$, where $I'$ is obtained by sustituting the last given relation by $\mu\lambda$. For this presentation, the algebra $A$ accepts a covering $R \to R/G = A$, where $R$ is strongly simply connected and $G$ is a torsion-free group.

The convex subcategory $C$ given by the vertices $9, 10, 11, 12, 13$ determines a convex subcategory of $R$ which contains subcategories of type $(2, 2, \infty)$; that is, $R$ is not cycle-finite.

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