ALGEBRAS WITH CYCLE-FINITE STRONGLY SIMPLY CONNECTED GALOIS COVERINGS

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Abstract. Let $A$ be a finite dimensional $k$-algebra and $R$ be a locally bounded category such that $R \rightarrow R/G = A$ is a Galois covering defined by the action of a torsion-free group of automorphisms of $R$. Following [30], we provide criteria on the convex subcategories of a strongly simply connected category $R$ in order to be a cycle-finite category and describe the module category of $A$. We provide criteria for $A$ to be of polynomial growth.

1. Introduction

Throughout the article, algebras are finite dimensional associative $k$-algebras with identity over a fixed algebraically closed field $k$. By a module over an algebra $A$ we mean a left $A$-module of finite dimension over $k$, if not specified otherwise.

From Drozd’s Tame and Wild Theorem [19] (see also [13]), the class of algebras are divided into two disjoint classes. On the one hand, we have tame algebras for which the indecomposable modules occur, in each dimension $d$, in a finite number of discrete and a finite number of one-parameter families. On the other hand, we have wild algebras whose representation theory includes the representation theories of all finite algebras.
dimensional algebras over $k$ (see [35, Chapter XIX]). A well understood class of tame algebras is formed by the algebras of finite type which accept only finitely many isoclasses of indecomposable modules (see [4, 5, 8, 9]). In the more general situation, the representation theory of tame algebras is slowly emerging. Tame tilted algebras [24], domestic and tubular extensions of tame concealed algebras [33], coil algebras [3] and more generally, (tame) algebras of polynomial growth [37], for which there exists an integer $m$ such that the number of one-parameter families of indecomposable modules is bounded, in each dimension $d$, by $d^m$, are among the type of algebras studied in the past years.

The methods of the representation theory of algebras work best for triangular algebras $A = kQ_A/I$, where the Gabriel quiver $Q_A$ has no oriented cycles (see [1, 33, 34, 35]). To deal with arbitrary algebras, covering techniques were developed (see [8, 16, 18, 20, 28]). In many situations, an algebra $A$ admits a Galois covering $R \to R/G = A$, where $R$ is a triangular locally bounded category and $G$ is a torsion-free group acting freely on the objects of $R$, which allows to study the representation theory of $A$ by the consideration of finite dimensional algebras inside $R$. For instance, assume that $R$ is a strongly simply connected category (see [38]). Then, tameness of $A$ implies tameness of $R$, which happens exactly when $R$ does not accept convex subcategories which are hypercritical [11]. The converse is expected to hold. Moreover, under these assumptions, $A$ is of polynomial growth if and only if $R$ does not accept convex subcategories which are hypercritical or pg-critical; see [37].

An important role in the representation theory of algebras is played by cycles of modules. A cycle in the category mod $A$ of finite dimensional modules over an algebra $A$ is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_s} X_s = X$$

of non-zero non-isomorphisms between indecomposable modules in mod $A$, and the cycle is said to be finite if the homomorphisms $f_1, \ldots, f_s$ do not belong to the infinite Jacobson radical of mod $A$. An algebra $A$ is said to be cycle-finite if all cycles in mod $A$ are finite [2]. Representation-finite algebras, tame tilted algebras, tame generalized multicoil algebras [26] are examples of cycle-finite algebras. In general, a cycle-finite algebra $A$ is of polynomial growth, while the converse holds if $A$ is a strongly simply connected algebra [40]. Recently, it was shown in [30] that every algebra $A$, which admits a Galois covering $R \to R/G = A$ with $R$ a cycle-finite
locally bounded category and $G$ a torsion-free group, is tame and the indecomposable finite dimensional $A$-modules were described. Moreover, for such a Galois covering, the algebra $A$ is of polynomial growth if and only if the number of $G$-orbits of isoclasses of indecomposable locally finite dimensional $R$-modules with non-trivial stabilizers is finite.

Here, we recall the main results and related techniques of the context discussed so far. Namely, we consider algebra $A$ and Galois covering $R \to R/G = A$ where $R$ is a "nice" locally bounded category and $G$ is a torsion-free group of automorphisms of $R$. The nicest situation corresponds to $R$ being a strongly simply connected and cycle-finite category. Assuming that $R$ is a strongly simply connected category, we show that $R$ is cycle-finite if and only if $R$ does not accept convex subcategories which are hypercritical, $pg$—critical or of type $(2,2,\infty)$. Here, we say that a category is of type $(2,2,\infty)$ if it is the direct limit of domestic extensions of type $(2,2,n)$, for $0 \leq n \in \mathbb{N}$, of a fixed tame concealed algebra of type $(2,2,s)$. These conditions are satisfied when there is a set of representatives $S_0$ of the $G$-orbits in a separating family $S$ of convex subcategories of $R$ with respect to $G$ which is formed by lines $\infty A_\infty$; see [30]. Moreover, if $R \to R/G = A$ is a covering in the nicest situation and $S_0$ is not empty, then $A$ is of polynomial growth exactly when $G = \mathbb{Z}$.

The remainder of the paper is organized as follows. In Section 1, we recall basic facts on Galois coverings of algebras essential for further considerations. Section 2 contains results on cycle-finite strongly simply connected categories. Section 3 is devoted to the proof of the main result and its immediate consequences. In Section 4, we establish a criterion for polynomial growth. In the final Section 5, we exhibit a couple of examples illustrating our results.

For basic background on the representation theory of algebras, refer to the books [1, 33, 34, 35].

2. Galois coverings of algebras

Following [8], by a locally bounded category we mean a $k$-category $R$ which is isomorphic to a factor category $kQ_R/I$, where $Q_R$ is a locally finite quiver and $I$ is an admissible ideal of the path category $kQ_R$ of $Q_R$. An algebra $A$ will be considered as a finite category, that is, a locally bounded category given by a finite quiver. A full subcategory $C$ of a locally bounded category $R$ is said to be convex if any path in $Q_R$ with source and target in $Q_C$ lies entirely in $Q_C$. 
Throughout this section, we denote by $R$ a fixed locally bounded category (over $k$). By an $R$-module, we mean a covariant functor $M$ from $R$ to the category $\mathrm{MOD} k$ of all vector spaces over $k$ [8]. An $R$-module $M$ is called finite dimensional (respectively, locally finite dimensional) if $\dim M = \sum_{x \in R} \dim_k M(x) < \infty$ (respectively, $\dim_k M(x) < \infty$ for any object $x$ of $R$). We denote by $\mathrm{MOD} R$, (respectively, $\mathrm{Mod} R$ or $\mathrm{mod} R$) the category of all (respectively, all locally finite dimensional or all finite dimensional) $R$-modules, and by $\mathrm{Ind} R$, (respectively, $\mathrm{ind} R$) the full subcategory of $\mathrm{Mod} R$ (respectively, $\mathrm{mod} R$) formed by all indecomposable modules. The support $\mathrm{supp} M$ of an $R$-module $M$ is the full subcategory of $R$ given by all objects $x$ such that $M(x) \neq 0$.

Let $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the objects of $R$. Then, following [20], we may consider the orbit category $R/G$ with objects being the $G$-orbits of the objects of $R$, and, for any two objects $a$ and $b$ of $R/G$, the morphism $k$-space $(R/G)(a,b)$ is defined as

$$ (R/G)(a,b) = \left\{ (f_{y,x}) \in \prod_{(x,y) \in a \times b} R(x,y) \Big| g f_{y,x} = f_{gy,gx} \forall g \in G, x \in a, y \in b \right\} $$

with the natural composition. Then, we have a canonical Galois covering functor

$$ F : R \longrightarrow R/G $$

which assigns to any object $x$ of $R$ its $G$-orbit $Gx$ and maps a morphism $f \in R(x,y)$ onto the family $F(f) \in (R/G)(Gx,Gy)$ such that $F(f)_{hy,gx} = gf$ or $0$ in accordance with $h = g$ or $h \neq g$. Moreover, $F$ induces the $k$-linear isomorphisms

$$ \bigoplus_{F(y)=a} R(x,y) \xrightarrow{\sim} (R/G)(F(x),a), \quad \bigoplus_{F(y)=a} R(y,x) \xrightarrow{\sim} (R/G)(a,F(x)), $$

for all objects $x$ of $R$ and $a$ of $R/G$. For a full subcategory $D$ of $R$, we denote by $g(D)$ the full subcategory of $R$ formed by the objects $g(x)$, $x \in D$, and its stabilizer $G_D = \{ g \in G \mid g(D) = D \}$. Then, we may consider the locally bounded category $D/G_D$. The group $G$ acts on $\mathrm{Mod} R$ by the translations $(-)^g$ which assign to each $R$-module $M$ the $R$-module $M^g = M \circ g$. For each $R$-module $M$, we denote by $G_M$ the stabilizer $\{ g \in G \mid M^g \cong M \}$ of $M$. Following [18], a module $Y$ in $\mathrm{Ind} R$ is said to be weakly $G$-periodic if $\mathrm{supp} Y$ is infinite and $(\mathrm{supp} Y)/G_Y$ is a finite category. Observe that in such a case, $G_Y$ is infinite.
Assume now that $G$ is a group of $k$-linear automorphisms of $R$ acting freely on the isoclasses of modules in $\text{ind} R$. Clearly, then $G$ acts freely on the objects of $R$, since $G$ acts freely on the isoclasses of indecomposable projective $R$-modules $R(x,-)$, $x \in R$. Consider the associated Galois covering functor $F : R \to R/G$. We denote by $F_* : \text{MOD} R/G \to \text{MOD} R$ the pull-up functor, which assigns to an $R/G$-module $M$ the $R$-module $M \circ F$, and by $F_\lambda : \text{MOD} R \to \text{MOD} R/G$ the push-down functor, left adjoint to $F_*$ (see [8, (3.2)]). Since $G$ acts freely on the isoclasses in $\text{ind} R$, $F_\lambda$ induces an injection from the set $(\text{ind} R/\sim)/G$ of $G$-orbits of isoclasses in $\text{ind} R$ into the set $(\text{ind} R/G)/\sim$ of isoclasses in $\text{ind} R/G$ [20, (3.5)]. We denote by $\text{mod}_1 R/G$ the full subcategory of $\text{mod} R/G$ consisting of all modules isomorphic to $F_* \lambda(M)$ for some module $M$ in $\text{mod} R$, and by $\text{mod}_2 R/G$ the full subcategory of $\text{mod} R/G$ formed by all modules without nonzero direct summands from $\text{mod}_1 R/G$. It was shown in [18, (2.2) and (2.3)] that a module $X$ from $\text{mod} R/G$ belongs to $\text{mod}_1 R/G$ (respectively, $\text{mod}_2 R/G$) if and only if $F_*(X)$ is a direct sum of finite dimensional $R$-modules (respectively, weakly $G$-periodic $R$-modules). We denote by $\text{ind}_1 R/G$ (respectively, $\text{ind}_2 R/G$) the full subcategory of $\text{mod}_1 R/G$ (respectively, $\text{mod}_2 R/G$) formed by the indecomposable modules. Following [18], the modules from $\text{ind}_1 R/G$ (respectively, $\text{ind}_2 R/G$) are called indecomposable modules of the first kind (respectively, indecomposable modules of the second kind). The category $R$ is said to be $G$-exhaustive if $\text{mod} R/G = \text{mod}_1 R/G$ [18].

Assume that $R$ is not $G$-exhaustive. Following [18, (3.1)], a family $S$ of full subcategories of $R$ is called separating (with respect to $G$) if $S$ satisfies the following conditions:

(i) for each $L \in S$ and $g \in G$, $gL \in S$;
(ii) for each $L \in S$ and each $G$-orbit $O$ of $R$, $O \cap L$ is contained in finitely many $G_L$-orbits;
(iii) for any two different $L, L' \in S$, $L \cap L'$ is locally support-finite;
(iv) for each weakly $G$-periodic $R$-module $Y$, there exists an $L \in S$ such that $\text{supp} Y \subseteq L$.

The following theorem is the main result in [18, Theorem 3.1].

**Theorem 2.1.** Let $R$ be a locally bounded $k$-category and $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the isoclasses in $\text{ind} R$. Let $S$ be a separating family of convex subcategories of $R$ with respect to $G$ and $S_0$ be a fixed set of representatives of $G$-orbits in $S$. There are
natural embedding functors $E^L_\lambda : \operatorname{mod} L/G_L \to \operatorname{mod} R/G, \ L \in S_0$ which induce a natural $k$-linear equivalence of categories

$$E : \prod_{L \in S_0} (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L] \to (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

In particular, the Auslander-Reiten quiver $\Gamma_{R/G}$ of $R/G$ is the disjoint union of the translation quivers

$$\Gamma_{R/G} = (\Gamma_{R/G})_0 \cup \left( \bigcup_{L \in S_0} (\Gamma_{L/G_L})_2 \right),$$

where $(\Gamma_{L/G_L})_2$ is the union of all connected components of $\Gamma_{L/G_L}$ formed by the indecomposable $L/G_L$-modules of the second kind.

For a convex subcategory $L$ of a locally bounded category $R$, the canonical embedding $E^L : \operatorname{MOD} L \to \operatorname{MOD} R$ is defined for a module $N$ in $\operatorname{MOD} L$, $E^L(N)$ as an $R$-module such that $E^L(N)(x) = N(x)$ for any object $x$ of $L$, $E^L(N)(f) = N(f)$ for any morphism $f$ in $L$, and $E^L(N)(y) = 0$ for any object $y$ of $R$ which is not in $L$. Moreover, we have a commutative diagram of functors

$$\begin{array}{ccc}
\operatorname{MOD} L & \xrightarrow{E^L} & \operatorname{MOD} R \\
\downarrow F^L_\lambda & & \downarrow F_\lambda \\
\operatorname{MOD} L/G_L & \xrightarrow{E^L_\lambda} & \operatorname{MOD} R/G
\end{array}$$

where $F^L_\lambda$ is the push-down functor associated to the Galois covering $F^L : L \to L/G_L$, $F_\lambda$ is the push-down functor associated to the Galois covering $F : R \to R/G$, and $E^L_\lambda$ assigns to a module $X$ in $\operatorname{MOD} L/G_L$ the module $E^L_\lambda(X)$ in $\operatorname{MOD} R/G$ such that $F_\bullet E^L_\lambda(X) = \bigoplus_{g \in U_L} F^L_\bullet(X)^g$, where $F_\bullet : \operatorname{MOD} R/G \to \operatorname{MOD} R$ and $F^L_\bullet : \operatorname{MOD} L/G_L \to \operatorname{MOD} L$ are the pull-up functors associated to $F$ and $F^L$, and $U_L$ is a fixed set of representatives of the cosets of $G$ modulo $G_L$ (see [18], (2.4) and (3.2)).

The following is an important special case of the last Theorem; see [18].

**Proposition 2.2.** Let $R$ be a tame locally bounded $k$-category, $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the objects of $R$, and $Y$ be a weakly $G$-periodic $R$-module. Then, the followings hold:

1. the stabilizer $G_Y$ is an infinite cyclic group;
(2) the push-down module $F_\lambda(Y)$ carries a canonical structure of a $kG_Y$-$R/G$-bimodule which is a free module of finite rank as left module over the group algebra $kG_Y$ of $G_Y$. In particular, we have a canonical functor

$$\Phi^Y = - \otimes_{kG_Y} F_\lambda(Y) : \text{mod } kG_Y \longrightarrow \text{mod } R/G,$$

whose image is contained in $\text{mod}_2 R/G$.

Let $R$ be a locally bounded $k$-category and $G$ be a group of $k$-linear automorphisms of $R$ acting freely on the objects of $R$. A line in $R$ is a convex subcategory $L$ of $R$ which is isomorphic to the path category $kQ$ of a linear quiver $Q$ of type $\infty A_\infty$, $A_\infty$, or $\infty A_\infty$. A line $L$ in $R$ is said to be $G$-periodic if its stabilizer $G_L$ is nontrivial. Clearly, in this case, the quiver $Q_L$ of $L$ is of type

$$\infty A_\infty : \ldots \bullet \bullet \bullet \bullet \bullet \bullet \ldots$$

and has a $G_L$-periodic orientation. With each $G$-periodic line $L$ of $R$ we may associate a canonical weakly $G$-periodic $R$-module $M_L$ by setting $M_L(x) = k$ for any vertex $x$ of $Q_L$, $M_L(y) = 0$ for all vertices $y$ of $Q_R \setminus Q_L$, and $M_L(\gamma) = \text{id}_k$ for each arrow $\gamma$ of $Q_L$. Since $G_{M_L} = G_L = \mathbb{Z}$, we then obtain a canonical functor

$$\Phi^L = - \otimes_{k[T,T^{-1}]} F_\lambda(M_L) : \text{mod } k[T,T^{-1}] \longrightarrow \text{mod } R/G$$

where $\text{mod } k[T,T^{-1}]$ denotes the category of finite dimensional modules over $k[T,T^{-1}]$.

**Proposition 2.3.** Let $R$ be a cycle-finite strongly simply connected category and $F : R \rightarrow A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion-free group $G$. Let $S_0$ be a set of representatives of the $G$-orbits in a separating family $S$ of convex subcategories of $R$ with respect to $G$. The followings hold:

(i) each $L \in S$ is a convex subcategory of $R$ which is a line $L$ in $R$, that is, the quiver $Q_L$ of $L$ is of type

$$\infty A_\infty : \ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet \ldots$$

and $L = kQ_L$;

(ii) for any two different $L,L' \in S$, the intersection $L \cap L'$ is a connected finite linear quiver.

**Proof.** (i): In [30] (3.1), without assuming that $R$ is strongly simply connected, it was shown that $L$ is a convex subcategory of $R$ admitting a simply connected Galois covering $F' : \tilde{L} \rightarrow \tilde{L}/H = L$ determined
by the action of a torsion free group $H$ and $\tilde{L}$ is a line of type $\infty \tilde{A}_\infty$. Assuming that $R$ is strongly simply connected, then $\tilde{L} = L$ as desired.

Assume (ii) fails. Since $L \cap L'$ is locally support finite, then $L \cap L'$ is formed by at least two disconnected finite intervals of the line $L$. Thus, we get a convex segment $x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{s-1} \longrightarrow x_s$ in $L$ with $x_1, x_s \in L \cap L'$ and $x_i \notin L'$, $2 \leq i \leq s - 1$. Then, the convex closure of $x_1, x_s$ in $R$ is of the shape

$$
x_1 = y_1 \rightarrow \cdots \rightarrow x_s = y_t \rightarrow \cdots \rightarrow y_1 \rightarrow \cdots \rightarrow y_{t-1} \rightarrow \cdots \rightarrow y_q \rightarrow \cdots \rightarrow y_1 \rightarrow \cdots \rightarrow y_{t-1} \rightarrow \cdots \rightarrow y_q$$

where all $y_i \in L'$ but $y_j \notin L$, for $2 \leq j \leq t - 1$, and there is a commutativity relation from $x_p$ to $y_q$. Since the stabilizer $G_L$ acts on $L$, we get a weakly $G$-periodic convex subcategory of $R$ which is not a line, contradicting (i).

\[\square\]

3. Cycle-finite strongly simply connected categories

By a tame concealed algebra, we mean a tilted algebra $C = \text{End}_H(T)$, where $H$ is the path algebra $kQ$ of a quiver $Q$ of Euclidean type $\tilde{A}_m$ ($m \geq 1$), $\tilde{D}_n$ ($n \geq 4$), or $\tilde{E}_p$ ($6 \leq p \leq 8$), and $T$ is a (multiplicity-free) preprojective tilting $H$-module. Recall that the Auslander-Reiten quiver $\Gamma_C$ of a tame concealed algebra $C$ is of the form

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{I}^C,$$

where $\mathcal{P}^C$ is a preprojective component containing all indecomposable projective $C$-modules, $\mathcal{I}^C$ is a preinjective component containing all indecomposable injective $C$-modules, and $\mathcal{T}^C$ is a $\mathbb{P}_1(k)$-family $\mathcal{T}_\lambda^C$, $\lambda \in \mathbb{P}_1(k)$, of pairwise orthogonal standard stable tubes, all but finite number of them of rank one (see [33, Chapter 4]) and [34]).

By a tubular algebra, we mean a tubular extension (equivalently, tubular coextension) of a tame concealed algebra of tubular type $(2, 2, 2, 2, 3, 3, 3), (2, 4, 4), (2, 3, 6)$, as defined in [33]. Recall that a tubular algebra $B$ admits two different tame concealed convex subcategories $C_0$
and $C_\infty$ such that the Auslander-Reiten quiver $\Gamma_B$ of $B$ is of the form

$$\Gamma_B = P^B_0 \vee T^B_0 \vee \left( \bigvee_{q \in \mathbb{Q}^+} T^B_q \right) \vee T^B_\infty \vee I^B_\infty,$$

where $P^B_0$ is the preprojective component $P^{C_0}$ of $\Gamma_{C_0}$, $T^B_0$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard ray tubes, obtained from the stable tubes of $T^{C_0}$ by ray insertions, $I^B_\infty$ is the preinjective component $I^{C_\infty}$ of $\Gamma_{C_\infty}$, $T^B_\infty$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard coray tubes, obtained from the stable tubes of $T^{C_\infty}$ by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers), $T^B_q$ is a $\mathbb{P}_1(k)$-family of pairwise orthogonal standard stable tubes; see [33].

**Lemma 3.1.** Let $R$ be a tame strongly simply connected locally bounded category and $G$ be a group acting freely on $R$. Let $C$ be a tame concealed algebra of type $\tilde{D}_n$ which is a convex subcategory of $R$. Assume $x_1$ is a vertex of $C$ in a convex line $y - x_1 - x_2 - \cdots - x_t - y'$ such that each $x_i$ has exactly two neighbors in the quiver of $C$ and $x_t = g(x_1)$, for some $g \in G$. Then, for every number $s$ there are indecomposable $R$-modules $Y_s$ containing at least $s$ convex tame concealed subcategories in the support $\text{supp} Y_s$.

**Proof.** Tame concealed algebras of type $\tilde{D}_n$ are given by the following frames:

with all commutativity relations. For the sake of simplicity, we assume that $x_i$, for $1 \leq i \leq t$, are given as in the first frame. Then, in $R$ we get
a convex subcategory \( B_3 \) of the shape

\[
\begin{array}{c}
\bullet \rightarrow x_1 \rightarrow \cdots \rightarrow g(x_1) \\
\downarrow & \downarrow & \downarrow \\
\bullet & \cdots & \bullet \\
\end{array}
\]

up to change of some arrow orientations. Clearly, \( B_3 \) accepts an indecomposable sincere module \( Y_3 \) whose support contains 6 tame concealed convex subcategories. Similarly, we may construct the desired indecomposable \( R \)-modules \( Y_s \), for \( s \geq 4 \).

Let \( B \) be an algebra, \( C \) be a standard component of \( \Gamma_B \) and \( X \) be an indecomposable module in \( C \). In [3], three admissible operations (ad 1), (ad 2) and (ad 3) were defined depending on the shape of the support of \( \text{Hom}_B(X,-)|_C \) in order to obtain a new algebra \( B' \).

(\text{ad 1}) If the support of \( \text{Hom}_B(X,-)|_C \) is of the form

\[
X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots
\]

then we set \( B' = (B \times D)[X \oplus Y_1] \), where \( D \) is the full \( t \times t \) lower triangular matrix algebra and \( Y_1 \) is the indecomposable projective-injective \( D \)-module.

(\text{ad 2}) If the support of \( \text{Hom}_B(X,-)|_C \) is of the form

\[
Y_t \leftarrow \cdots \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots
\]

with \( t \geq 1 \), so that \( X \) is injective, then we set \( B' = B[X] \).
(ad 3) If the support of Hom$_B(X, -)|_C$ is of the form

\[
Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_t
\]

\[
\uparrow \quad \uparrow \quad \uparrow
\]

\[
X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{t-1} \rightarrow X_t \rightarrow \cdots
\]

with $t \geq 2$, so that $X_{t-1}$ is injective, then we set $B' = B[X]$.

In each case, the module $X$ and the integer $t$ are called, respectively, the pivot and the parameter of the admissible operation. The dual operations are denoted by (ad 1*), (ad 2*) and (ad 3*).

Following [3], an algebra $A$ is a coil enlargement of the critical algebra $C$ if there is a sequence of algebras $C = A_0, A_1, \ldots, A_m = A$ such that for $0 \leq i < m$, $A_{i+1}$ is obtained from $A_i$ by an admissible operation with pivot in a stable tube of $\Gamma_C$ or in a component (coil) of $\Gamma_{A_i}$ obtained from a stable tube of $\Gamma_C$ by means of the admissible operations done so far. When $A$ is tame, then we call $A$ a coil algebra.

If $A$ is a coil enlargement of a critical algebra $C$, then there is a maximal branch coextension $A^-$ of $C$ inside $A$ which is full and convex in $A$, and such that $A$ is obtained from $A^-$ by a sequence of admissible operations of types (ad 1), (ad 2) and (ad 3). Dually, there is a maximal branch extension $A^+$ of $C$ inside $A$ which is full and convex in $A$, and such that $A$ is obtained from $A^+$ by a sequence of admissible operations of types (ad 1*), (ad 2*) and (ad 3*).

For a coil enlargement $A$ of a critical algebra $C$, we consider the type $r(A)$ of $A$ as follows: Let $T = (T_{\lambda})_{\lambda \in \mathbb{P}_1(k)}$ be the separating tubular family of mod $C$. For each $\lambda \in \mathbb{P}_1(k)$, let $n_{\lambda}$ be the rank of $T_{\lambda}$ and $r^+_{\lambda} - n_{\lambda}$ (respectively, $r^-_{\lambda} - n_{\lambda}$) be the number of rays (respectively, corays) inserted in $T_{\lambda}$ by the sequence of admissible operations that leads from $C$ to $A$. Finally, let $r(A) = (r^+_{\lambda}, r^-_{\lambda})_{\lambda \in \mathbb{P}_1(k)}$, where we write down only those numbers greater or equal to 1.

**Proposition 3.2.** Let $B$ be a coil enlargement of a tame concealed algebra $C$. The following conditions are equivalent.

(a) $B$ is tame.
(b) $B^+$ and $B^-$ are tame.
(c) Every cycle

\[
X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} X_s = X
\]

of non-zero non-isomorphisms between indecomposable modules in mod $B$, belongs to a standard coil in $\Gamma_B$. 


(d) \( B \) is of polynomial growth.
(e) \( B \) is of linear growth.
(f) \( B \) is cycle-finite.
(g) Each of \( r^+(B) \) and \( r^-(B) \) is one of the following: \( (p, q) \) where \( 1 \leq p \leq q, (2, 2, r) \) with \( r \geq 2 \), \( (2, 3, 3), (2, 3, 4), (2, 3, 5), (3, 3, 3), (2, 4, 4), (2, 3, 6), (2, 2, 2, 2) \).

Essential for our considerations is the following theorem which is the main result of [42].

**Theorem 3.3.** Let \( A \) be a strongly simply connected algebra. The following conditions are equivalent.

(a) \( A \) is of polynomial growth.
(b) \( A \) is of linear growth.
(c) \( A \) is cycle-finite.
(d) \( A \) does not contain a convex subcategory which is pg-critical or hypercritical.
(e) rad \( \infty \) (mod \( A \)) is locally nilpotent.
(f) The component quiver \( C(A) \), whose vertices are components of the Auslander-Reiten quiver \( \Gamma_A \) and arrows \( C \to C' \) are set when there are modules \( X \in C \) and \( X' \in C' \) with \( \text{rad} \infty (X, X') \neq 0 \), has no oriented cycles.
(g) Every connected component of \( \Gamma_A \) is standard.

A special situation of the above Theorem is the following.

**Lemma 3.4.** Let \( B \) be a strongly simply connected cycle-finite algebra and \( M \) be an indecomposable \( B \)-module. Assume that

\[
X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} X_s = X
\]

is a cycle of non-zero non-isomorphisms between pairwise different indecomposable modules in mod \( B \), such that \( 6 \leq s \) and \( f_1 \) factorizes non-trivially in mod \( B[M] \). Then, one of the following two situations occur:

(i) \( B \) contains a convex subcategory \( B' \) which is a coil extension such that one of the two \( r^+(B') \) or \( r^-(B') \) is \( (2, 2, s) \).

(ii) \( B[M] \) is of wild type.

**Proof.** Indeed, by [42] (2.3), the algebra \( B \) is multicoil and the given cycle belongs to a standard coil \( \mathcal{T} \) of a multicoil of \( \Gamma_B \). Let \( C \) be a tame concealed algebra such that \( B' \) is a convex subcategory of \( B \) and coil extension of a tame concealed algebra \( C \). Assume (i) does not hold,
that is, $B'$ is of type $(r_1, r_2, r_3)$, with $r_1 \leq r_2 \leq r_3$ and $3 \leq r_2$, or of type $(2, 2, 2, 2)$.

Let $\mathcal{T}'$ be the component of $\Gamma_{B'[M]}$ where $X$ belongs. Observe that $\text{Hom}_{B}(M, \mathcal{T}) \neq 0$ and since $f_1$ is factorized there is a cycle of non-zero non-isomorphisms between $6 < s + 1$ pairwise different indecomposable modules in $\text{mod} B[M]$. If $M$ belongs to $\mathcal{T}'$, then either $M$ is not a pivot module or the extension type of $B'[M]$ is not tame. In the latter case, $B[M]$ is wild. Moreover, if $M$ is not a pivot module, according to [29], the one-point extension $B'[M]$ is tame only when $B'$ is of type $(2, 2, s)$. Since this is forbidden, then $B[M]$ is wild.

If $M$ does not belong to $\mathcal{T}'$, then there is a regular $C$-module $Y$ such that $\text{Hom}_{B}(M, Y) \neq 0$, and $B[M]$ contains a convex subcategory of the form $C[N]$ for a preprojective $C$-module $N$. The extension $C[N]$ being wild implies that $B[M]$ is wild.

The following theorem is the main result of [30].

**Theorem 3.5.** Let $R$ be a connected cycle-finite locally bounded $k$-category over an algebraically closed field $k$, $G$ be a torsion-free admissible group of $k$-linear automorphisms of $R$, and $A = R/G$. Let $\mathcal{S}$ be a separating family of convex subcategories of $R$ with respect to $G$ and $\mathcal{S}_0$ be a fixed set of representatives of $G$-orbits in $\mathcal{S}$. Then, the functors $\Phi^Y = F_\lambda(Y) \otimes_{k[T,T^{-1}]} - : \text{mod} k[T,T^{-1}] \rightarrow \text{mod} A$, $Y \in \mathcal{S}_0$, induce a $k$-linear equivalence of categories

$$\Phi : \coprod_{\mathcal{S}_0} \text{mod} k[T,T^{-1}] \sim \rightarrow \text{mod} A/[\text{mod}_1 A].$$

Moreover, the following statements hold.

(i) $A$ is tame.

(ii) Every indecomposable finite dimensional $A$-module $X$ is isomorphic either to $F_\lambda(M)$ for some indecomposable finite dimensional $R$-module $M$ or to $\Phi^Y(V)$ for some $Y \in \mathcal{S}_0$ and some indecomposable finite dimensional $k[T,T^{-1}]$-module $V$.

(iii) The Auslander-Reiten quiver $\Gamma_A$ of $A$ has the disjoint union decomposition

$$\Gamma_A = (\Gamma_R/G) \sqcup \left( \coprod_{\mathcal{S}_0} \Gamma_{k[T,T^{-1}]} \right)$$

where $\Gamma_{k[T,T^{-1}]}$ is the Auslander-Reiten quiver of the category of finite dimensional $k[T,T^{-1}]$-modules.
There are strongly simply connected categories $R$ of polynomial growth which are not cycle-finite, as the following example shows. Consider the category $R$ given by the following quiver with relations as indicated by the dotted edges:

\[
\begin{array}{cccccc}
0 & 1 & 2 & 1' & 2' & \cdots \\
\end{array}
\]

Since $R$ has tame coil enlargements $R_s$ of a hereditary algebra $C$ of Euclidean type $\tilde{D}_4$ of type $(2,2,s)$, for arbitrary $s \geq 1$, then $\text{mod } R$ accepts cycles of non-zero morphisms between indecomposable $R$-modules of arbitrary length. We may build a non-trivial infinite cycle in $\text{mod } R'$, where $R'$ is the quotient of $R$ obtained by adding a zero-relation from 1 to 1', of the form

\[
S_0 \to P_1 \to \cdots \to 0 \to 1 \to 2 \to 1' \to 1' \to S_0
\]

where $S_0$ is the simple module at 0, $P_j$ (respectively, $I_j$) is the indecomposable projective cover (respectively, injective envelope) of $S_j$ in $\text{mod } R'$ and the dimension vectors correspond to indecomposable $C$-modules $X_i$, $i = 1,2,3$. Observe that the composition of maps $S_0 \to X_1$ is non-zero in $\text{rad}^\infty(\text{mod } R)$.

We say that the category $R$ is of type $(2,2,\infty)$ if for every $m$ it contains a convex subcategory $B_m$ which is a coil enlargement of type $(2,2,m)$, $B_m$ is a subcategory of $B_{m+1}$ and $R = \bigcup_m B_m$.

The next result is preparatory for the main theorem of our work.

**Lemma 3.6.** Let $R$ be a strongly simply connected cycle-finite category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion-free group $G$. Assume that $R$ is of polynomial growth. Then, the followings hold:

(i) there is a number $s_0$ such that, for any finite convex subcategory $B$ of $R$, any periodic $B$-module has period at most $s_0$;

(ii) for any cycle

\[ X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} X_s = X \]
of length \( s \geq s_0 \), there is a convex subcategory \( B \) of \( R \) and a coil \( T \) in \( \mod B \) containing all modules \( X_i \), \( 1 \leq i \leq s \), and at least \( s - s_0 \) indecomposable projective modules.

**Proof.** (i) : Let \( s_0 = 2n + 4 \), where \( n \) is the number of vertices in the quiver of \( A \). Consider a convex subcategory \( B \) of \( R \) with a periodic module \( X \) of period \( p > s_0 \). Since \( B \) is a multicoil algebra, then \( X \) lies in a stable tube. By [3], the support of \( X \) is a tame concealed or a tubular algebra. Without lost of generality, we may assume that \( B \) is tame concealed or a tubular algebra.

Since \( p > 6 \), then \( B \) is tame concealed of type \( \tilde{D}_{p-2} \). From the structure of the frames of the tame concealed algebras, we get a linear convex subcategory of \( B \) of the shape \( y - x_1 - x_2 - \cdots - x_t - y' \) such that each \( x_i \) has exactly two neighbors in the quiver of \( B \) and \( x_t = g(x_1) \), for some \( g \in G \). By Lemma 3.1, there is an indecomposable \( R \)-module whose support contains at least 4 convex tame concealed subcategories. This contradicts with the result in [25].

(ii) : is a consequence of (i) and the structure of multicoil components of the Auslander-Reiten quiver of multicoil algebras. □

### 4. The main results

**Theorem 4.1.** Let \( R \) be a strongly simply connected category and \( F : R \to A \) be a Galois covering functor of a finite dimensional algebra \( A \) defined by the action of a torsion free group \( G \). The followings are equivalent.

(a) \( R \) is of polynomial growth and does not contain a convex subcategory of type \((2,2,\infty)\).

(b) \( R \) is of linear growth and does not contain a convex subcategory of type \((2,2,\infty)\).

(c) \( R \) is cycle-finite.

(d) \( R \) does not contain a convex subcategory which is of type \((2,2,\infty)\), \( pg \)-critical or hypercritical.

(e) \( R \) does not contain a convex subcategory which is \( pg \)-critical or hypercritical and there exists a set of representatives \( S_0 \) of the \( G \)-orbits in a separating family \( S \) of convex subcategories of \( R \) with respect to \( G \) formed by lines.

Moreover, if any of the above holds, then the following holds:

(f) \( \text{rad}^\infty(\mod_1 A) \) is locally nilpotent.
Proof. The equivalence of (a), (b) and (d) follows obviously from Theorem (4.1) in [42]. If (c) is satisfied, then clearly (a) is satisfied. Assume that (a) holds, that is, \( R \) is of polynomial growth not accepting convex subcategories of type \((2, 2, \infty)\). We shall show that there is a number \( s \) such that the maximal length of a cycle in \( \text{mod} R \) is \( s \) and therefore \( R \) is cycle finite.

Suppose, to get a contradiction, that for every number \( s \) there is a cycle \( \eta_s : X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_{t(s)}} X_{t(s)} = X \) of length \( t(s) \geq s \). As in Lemma 3.6, there is a number \( s_0 \) such that, for any finite convex subcategory \( B \) of \( R \), any periodic \( B \)-module has period at most \( s_0 \). In particular, any tame concealed convex subcategory \( C \) of \( R \) is of Euclidean type \((2, 3, 3), (2, 3, 4), (2, 3, 5) \) or \((2, 2, r)\) with \( 2 \leq r \leq s_0 \). Moreover, each cycle \( \eta_s \) lies in a coil \( T_s \) in \( \text{mod} B_s \) containing at least \( t(s) - s_0 \) indecomposable projective modules, where \( B_s \) is a convex subcategory of \( R \) which is a coil extension of a tame concealed algebra \( C_s \). Moreover, without lost of generality, we may assume that every \( B_s \) is a coil extension of the tame concealed algebra \( C_s \).

(c) is equivalent to (e): we already observed that weakly \( G \)-periodic subcategories of a strongly simply connected cycle-finite category \( R \) are lines. For the converse, assume that (e) is satisfied. By theorem 3.3, every finite convex subcategory of \( R \) is of polynomial growth, that is, \( R \) is of polynomial growth. Assume, to get a contradiction, that \( B \) is a convex subcategory of \( R \) of type \((2, 2, \infty)\); in particular, there is a convex subcategory \( D \) of \( R \) tilted of type \( \mathbb{D}_s \) with \( s > n + 2 \) for \( n \), the number of vertices of the quiver \( Q_A \), given by a quiver with relations corresponding to one of the following frames of categories:
for some $x \in Q_D$ and some $g \in G$. Clearly, this yields a convex subcategory $D'$ of $R$ which is tame concealed of type $\tilde{D}$, and a convex line $x - x_1 - x_2 - \cdots - x_t - g(x)$ such that each $x_i$ has exactly two neighbors in the quiver of $C$. Applying Lemma 3.1, we get indecomposable $R$-modules $Y$ whose support contain at least 4 different tame concealed algebras. This contradicts the main result in [25].

$(c)$ implies $(f)$: Assume $(c)$ holds. Consider an indecomposable $A$-module of the first kind and a linear map $f : M \to M$ in $\text{rad}^\infty(\text{mod}_1 A)$. Suppose that $F_\lambda(X) = M$, for some indecomposable $R$-module $X$. Then, there are maps $f_g \in \text{Hom}_R(X, X^g)$, almost all $f_g = 0$, such that

$$\sum_{g \in G} F(f_g) = f.$$ 

Since $f \in \text{rad}^s(\text{mod}_1 A)$ then, $f^g \in \text{rad}^s(\text{mod} R)$, for any $s \geq 1$. Suppose $0 \neq f = f_1 \cdots f_r$, for some $f_i \in \text{rad}^\infty(M, M)$, there exist maps $f_{(i, g)} \in \text{rad}^\infty(X, X^g)$, for $1 \leq i \leq r$, with almost all $f_{(i, g)} = 0$, such that

$$\sum_{g \in G} F(f_{(i, g)}) = f_i.$$ 

We get

$$f_g = \sum_{g = g_1 \cdots g_1} f_{(r, g_r)}^{g_{r-1} \cdots g_1} \cdots f_{(2, g_2)}^{g_1} f_{(1, g_1)}.$$ 

Call $X_0 = X, X_1 = X^{g_1}, X_2 = X^{g_2 g_1}, \cdots, X_r = X^{g_r \cdots g_2 g_1}$ and consider a non-zero composition of maps $0 \neq f_r \cdots f_2 f_1$ with $f_i \in \text{rad}^\infty(X_{i-1}, X_i)$, $1 \leq i \leq r$. Since $R$ is cycle-finite and therefore $\text{rad}^\infty(Y, Y) = 0$ for any indecomposable $R$-module $Y$, then the modules $X_i, 0 \leq i \leq r,$ are pairwise non-isomorphic indecomposable $R$-modules with the same
dimension \(d = \dim_k M\). The Harada-Sai lemma yields a contradiction, in case \(r \geq 2^d\). This shows that \(\text{rad}^\infty(\text{mod}_1 A)\) is locally nilpotent. \(\Box\)

Given a Galois covering \(R \to R/G = A\) of a finite dimensional \(k\)-algebra \(A\), we observe that a component \(C'\) of \(\Gamma_A\) is either of the first kind, that is formed by the modules \(F_\lambda(X)\), for \(X \in C\) for a component \(C\) in \(\Gamma_R\), or of the second kind, that is formed by the modules \(\Phi^Y(V)\), for \(Y\) a fixed weakly \(G\)-periodic module and \(V\) an indecomposable \(k[T, T^{-1}]\)-module. The following consequence for the structure of components of the Auslander-Reiten quiver \(\Gamma_A\) is obtained.

**Proposition 4.2.** Let \(R\) be a cycle-finite strongly simply connected category and \(F : R \to A\) be a Galois covering functor of a finite dimensional algebra \(A\) defined by the action of a torsion free group \(G\). Let \(C\) be a component of the Auslander-Reiten quiver \(\Gamma_R\). The followings hold:

(a) the set of vertices \(a\) such that \(X(a) \neq 0\), for some indecomposable \(X \in C\), form a convex subcategory \(B(C)\) of \(R\);

(b) the stabilizer \(G' = G_C\) of \(C\) is a normal subgroup of \(G\);

(c) the category \(B(C)\) is strongly simply connected and cycle-finite, the induced functor \(F' : B(C) \to A'\) is a Galois covering defined by the action of a torsion free group \(G'\), and \(C\) is a component of \(\Gamma_{B(C)}\) with stabilizer \(G'_C = G'\);

(d) every component of the Auslander-Reiten quiver \(\Gamma(\text{mod}_1 A)\) is generalized standard.

**Proof.** (a) : Assume \(a_1 \to a_2 \to \cdots \to a_r\) is a path in the quiver \(Q_R\) such that \(X(a_1) \neq 0 \neq Y(a_r)\), for indecomposable modules \(X, Y \in C\) and \(Z(a_i) = 0\), for \(2 \leq i \leq r - 1\), and all \(Z \in C\). We shall construct a cycle in the componental quiver \(C(R)\). This contradicts \([42](4.1)\).

Indeed, consider the quotient \(R'\) of \(R\) obtained by adding relations \(a_1 \to a_2 \to b\) and \(c \to a_{r-1} \to a_r\), for all arrows \(a_2 \to b\) and \(c \to a_{r-1}\). Consider \(I_x\) to be the injective envelope and \(P_x\) to be the projective cover of the simple module \(S_x\) corresponding to a vertex \(x\) in the category \(\text{mod} R'\). We get a path of morphisms in \(\text{mod} R'\) to be

\[
Y \to I_{a_r} \to S_{a_{r-1}} \to F(a_{r-1}, a_{r-2}) \to S_{a_{r-2}} \to \cdots \to F(a_3, a_2) \to S_{a_2} \to P_{a_1} \to X
\]

where for any arrow \(y \to x\) in \(Q_R\), the \(R\)-module \(F(x, y)\) is the unique indecomposable whose composition factors are \(S_x\) and \(S_y\). Since \(S_{a_i}\) does not belong to \(C\), for \(2 \leq i \leq r - 1\), we get a cycle through \(C\) in the componental quiver \(C(R)\).
(b) and (c) are obvious.

(d): Let \( M \) and \( N \) be two modules in \( \mathcal{C}' \) and \( 0 \neq f \in \text{rad}^\infty_A(M, N) \). Assuming that \( \mathcal{C}' \) is of the first kind implies that there exists a component \( \mathcal{C} \) in \( \Gamma_R \) and indecomposable \( R \)-modules \( X, Y \in \mathcal{C} \) such that \( F_\lambda(X) = M \) and \( F_\lambda(Y) = N \). Lifting the morphism \( f \) provides morphisms \( f_g \in \text{rad}^\infty_R(X, Y^g) \), for \( g \in G' \), almost all zero, such that \( \sum_{g \in G'} F_\lambda(f_g) = f \).

We remark that, for the algebra \( A' \), we have \( \text{rad}^\infty_A(M, N) = 0 \). Indeed, a morphism \( f' \in \text{rad}^\infty_A(M, N) \) yields the existence of morphisms \( f'_g \in \text{rad}^\infty_R(X, Y^g) \), for \( g \in G' \), almost all zero, such that \( \sum_{g \in G'} F_\lambda(f'_g) = f' \). Since for \( g \in G' \) the module \( Y^g \in \mathcal{C} \), then \([42](4.1)\) implies that \( \text{rad}^\infty_R(X, Y^g) = 0 \). Hence \( f'_g = 0 \) and \( f' = 0 \).

Since \( \text{rad}^\infty_A(M, N) = 0 \) then, for every \( g \in G \) such that \( f_g \neq 0 \), we have \( \text{rad}^\infty_B(X, Y^g) = 0 \) and there is a chain of irreducible maps connecting \( X \) and \( Y^g \), that is, \( Y^g \in \mathcal{C} \) and \( g \in G_\mathcal{C} \). Up to a change of orientation, we may assume that there is an indecomposable projective \( R \)-module \( P_a \notin \mathcal{C} \) such that \( \text{rad} P_a = L \) and the one-point extension category \( B' = B(\mathcal{C})[L] \) is convex in \( R \). Moreover, \( f_g \in \text{rad}^\infty_B(X, Y^g) \) factorizes through a module \( Z \in \text{mod}\, B' \) satisfying \( Z(a) \neq 0 \). Therefore, there is a direct summand \( Z' \) of \( Z \) satisfying \( Z' \notin \mathcal{C} \) and there is a cycle in the componental quiver \( C(R) \) of the form \( \mathcal{C} = [X] \to [Z] \to [Y^g] = \mathcal{C} \), where \([Z]\) denotes the component in \( \Gamma_R \) containing \( Z' \). \( \square \)

5. Criteria for polynomial growth

The aim of this section is to establish a criterion for an algebra with a cycle-finite Galois covering to be of polynomial growth (respectively, domestic type). We start by recalling a criterion in [30].

**Theorem 5.1.** Let \( R \) be a connected cycle-finite locally bounded \( k \)-category, \( G \) be a torsion-free admissible group of \( k \)-linear automorphisms of \( R \), and \( A = R/G \). Then the followings hold.

(i) \( A \) is of polynomial growth if and only if the number of \( G \)-orbits of isoclasses of weakly \( G \)-periodic \( R \)-modules is finite.

(ii) \( A \) is domestic if and only if \( R \) does not contain a convex subcategory which is tubular and the number of \( G \)-orbits of isoclasses of weakly \( G \)-periodic \( R \)-modules is finite.

Part of the following result is explicit in [30].
Theorem 5.2. Let $R$ be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional algebra $A$ defined by the action of a torsion free group $G$. Let $S_0$ be a set of representatives of the $G$-orbits in a separating family $S$ of convex subcategories of $R$ with respect to $G$. Then the followings hold.

1. The category $\text{mod}_1 A$ of modules of the first type is of polynomial growth.
2. The category $\text{mod}_2 A$ of modules of the second kind is of polynomial growth if and only if $S_0$ is a finite set.
3. The algebra $A$ is of polynomial growth if and only if the cardinality of $S_0$ is bounded by the number $n$ of vertices in $Q_A$.

Proof. (1): Since every convex subcategory of $R$ is of polynomial growth, by [16], Lemma 3, the category of modules of the first kind $\text{mod}_1 A$ is of polynomial type.

(2): This results from Theorem 4.1 in [30].

(3): Assume there are different lines $L_1, \ldots, L_s \in S_0$ for any $s > n$. Obviously, not all sets of vertices $F(L_i)$ are disjoint. We may suppose $x$ is a vertex in $L_1 \cap L_2$. Let $1 \neq g \in G_{L_1}$ and observe that $g(x) \notin L_2$, since otherwise, by Proposition 2.3, we would have $L_1 = L_2$. Consider the line $L'_s$, for $s \in \mathbb{N}$ formed by the vertices

$$\cdots - y_2 - y_1 - x - x_1 - \cdots x_t = g(x) - \cdots - g^2(x) - \cdots - g^s(x) -$$

$$- g(y_1) - g(y_2) - \cdots$$

where $x - x_1 - \cdots x_t = g(x) - \cdots - x_{2t} = g^2(x) - \cdots - x_{st} = g^s(x)$ is the convex segment of $L_1$ connecting $x$ and $g^s(x)$, and $\cdots - y_2 - y_1 - x - y_1 - y_2 - \cdots$ is the line $L_2$. We may assume that $y_1$ and $g(y_1)$ are not in the line $L_1$. We claim that the lines $L'_s$ determine pairwise different elements in $S_0$. Indeed, assume that $h(L'_p) = L'_q$, for some $p \leq q$, and $h \in G$. Then $h$ sends infinite segments of $L_2$ to $L_2$, and hence $h \in G_{L_2}$. Moreover, $L'_p$ contains exactly $tp$ vertices of $L_1$, which yields $p = q$. □

The structure of $G$ is sometimes a source of information on the families of second kind modules, and hence on the representation type of $R/G$. Namely, we show the following proposition.

Proposition 5.3. Let $R$ be a cycle-finite strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional
algebra $A$ defined by the action of a torsion-free group $G$. If $G$ is cyclic, then $A$ is of polynomial growth.

Proof. Assume that $S_0$ is not empty and assume that $G$ is cyclic. Take lines $L_1, \ldots, L_s \in S_0$, for any $s > n$, where $n$ is the number of vertices in $Q_A$. Obviously, not all the sets of vertices $F(L_i)$ are disjoint. We may suppose $x$ to be a vertex in $L_1 \cap L_2$. Since $G_{L_1}$ and $G_{L_2}$ are non-trivial cyclic subgroups of $G$, then $G/(G_{L_1} \cap G_{L_2})$ is a finite group. Let $1 \neq g \in G_{L_1} \cap G_{L_2}$ and observe that $x$ and $g(x)$ belong to $L_1 \cap L_2$ which, according to Proposition 2.3, is formed by a unique connected segment. This yields $L_1 = L_2$, and the cardinality of $S_0$ is at most $n$. □

6. Examples

Here we illustrate some results of our work in four parts.

(1) We start by giving an example (see [12]) of the relation between structural properties of the category $R$ and the group $G$ defining the Galois covering.

Theorem 6.1. Let $R$ be a strongly simply connected category and $F : R \to A$ be a Galois covering functor of a finite dimensional triangular algebra $A$ defined by the action of a torsion free group $G$. Then, $G$ is a free (non-abelian) group.

Sketch of proof: (i) Assume $A = B[M]$ to be a one-point extension of an algebra $B$ by a module $M$. Let $a$ be a source vertex in $Q_A$ such that $\text{rad} P_a = M$ and $x$ be a vertex in $Q_R$ such that $F(x) = a$. Consider $R'$ the convex subcategory formed by those vertices at the preimage $F^{-1}(B)$ and choose a connected component $R^B$ of $R'$. The stabilizer $G^B$ of $R^B$ is a normal subgroup of $G$. Consider $F^B : R^B \to B$ to be the functor obtained as the restriction of $F$. We get that $R^B$ is a strongly simply connected category and $F^B : R^B \to B$ is a Galois covering functor of a finite dimensional triangular algebra $B$ defined by the action of a torsion free group $G^B$.

By induction on the dim$_k A$, we may assume that $G^B$ is a free group.

(ii) We show that $F$ is a covering of the first kind, that is, if $\text{rad} P_a = M_1 \oplus \cdots \oplus M_t$ is an indecomposable decomposition in mod $B$, then there is an indecomposable decomposition $\text{rad} P_a = Y_1 \oplus \cdots \oplus Y_s$ in mod $R'$ such that $s = t$ and a permutation $\sigma$ satisfying $F\lambda(Y_i) = M_{\sigma(i)}$, for $1 \leq i \leq t$. 
Indeed, since $R$ is strongly simply connected, then the source $x$ separates $R$, that is, there are connected components $R_1, \cdots, R_s$ of $R'$ such that the support of $Y_i$ is contained in $R_i$, for $1 \leq i \leq s$. Therefore, \[ \text{rad} P_a = \text{rad} F_\lambda(Px) = \bigoplus_{i=1}^{s} F_\lambda(Y_i) \] is an indecomposable decomposition and the claim follows.

(iii) The group $G/G^B$ is a free group $F(t-1)$ of rank $t-1$.

(iv) $G$ is isomorphic to the free product $G^B * F(t-1)$ and it is therefore a free group.

To illustrate the idea of the proof, assume that $t = 3$. Construct the category $\bar{B}$ as a model for a covering $F' : \bar{B} \to A$ defined by the action of $F(t-1) = F(2)$. Substitute each $o$ in the diagram by the category $R_B$ in such a way that, for every vertex $a$, the radical $P_a = Y_1 \oplus Y_2 \oplus Y_3$ (in the representations $Y_i$ the arrows stand for identity maps; observe that the vertical arrows in $\bar{B}$ contribute $Y_1$ to the radical of $P_a$). The functor $F : R \to A$ factorizes as $F = \bar{F}F'$ by a Galois covering functor $\bar{F} : R \to \bar{B}$ defined by the action of $G^B$ (in the example $G^B = \mathbb{Z}$):

(2) As another series of examples, consider the categories $R_{\alpha, \beta}$ given by the quiver with relations
and \((\alpha, \beta) \neq (1, 1)\), to be locally support finite; it is simply connected but not strongly simply connected. Moreover, the group \(\mathbb{Z}\) generated by the action \((a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1})\) acts freely on \(R_{\alpha, \beta}\) and on \(\text{ind} R_{\alpha, \beta}/\cong\). Hence, the Galois covering \(F: R_{\alpha, \beta} \to A_{\alpha, \beta}/\mathbb{Z}\) yields a bijection \(F: \text{ind} R_{\alpha, \beta}/\cong \to \text{ind} A_{\alpha, \beta}/\cong\). The algebra \(A_{\alpha, \beta}\) is given by the quiver with relations

\[
\begin{align*}
\sigma_{i+1}\sigma_i &= \alpha\nu_{i+1}\gamma_i \\
\nu_{i+1}\rho_i &= \beta\sigma_{i+1}\nu_i \\
\gamma_{i+1}\nu_i &= \rho_{i+1}\rho_i \\
\rho_{i+1}\gamma_i &= \gamma_{i+1}\sigma_i
\end{align*}
\]

Since \(R_{\alpha, \beta}\) is tame (respectively polynomial growth for \(\alpha\beta \neq 1\)), so is \(A_{\alpha, \beta}\).

(3) Consider the Galois covering

\[
F: A = A_{1,1}^{(2)} \to \bar{A} = A_{1,1}^{(2)}/\mathbb{Z}_2
\]

and assume that the characteristic of \(k\) is 2. As a tubular algebra, we know that \(A\) is tame. We show that \(\bar{A}\) is a wild algebra.
Set \( x_0 = \alpha_0 + \beta_0, \ y_0 = \beta_0, \ x_1 = \alpha_1 + \beta_1, \ y_1 = \beta_1. \) Then, \( A \) is isomorphic to the algebra \( A' \) given by the quiver with relations

\[
A': \begin{matrix}
\bullet & \xrightarrow{x_0} & \bullet & \xrightarrow{x_1} & \bullet \\
\downarrow & & \downarrow & & \downarrow \\
y_0 & & y_1 & & \\
\end{matrix}
\]

\[
x_1x_0 = 0 \\
y_1x_0 = x_1y_0
\]

Observe that \( A' \) accepts a Galois covering \( R \to R/Z = A' \), given by the category

which is strongly simply connected. Therefore, \( R \) is tame if and only if the Tits form \( q_R \) is weakly non-negative. Observe that the vector \( y \) marked on the vertices of \( R \) determines a convex subcategory \( B \) of \( R \) whose Tits form takes value \( q_B(y) = -1 \). Therefore, \( R \) is a wild category and \( A' \) a wild algebra.

(4) Our last example is similar to an example given in [30]. Let \( A \) be the bound quiver algebra \( kQ/I \) given by the quiver
and the ideal $I$ of the path algebra $kQ$ of $Q$ be generated by the elements $\alpha \varphi, \alpha \kappa, \sigma \varphi, \sigma \kappa, \gamma \sigma, \gamma \rho, \nu \rho, \nu \sigma, \eta \gamma, \eta \beta, \xi \beta, \xi \gamma, \varphi \delta, \delta \psi, \psi \eta, \psi \chi, \kappa \chi, \kappa \eta, \alpha \nu, b a, d c a, e b, g b - g d c, \pi \alpha, \lambda \pi, \mu \lambda \mu, \varepsilon^2 - \lambda \mu, \mu \lambda - \mu \varepsilon \lambda$.

For $k$ of characteristic 2, the convex subcategory $B$ of $A$ given by the objects 14 and 15 is a penny-farthing, and hence is a non-standard representation-finite algebra. Hence, for $k$ of characteristic 2, the algebra $A$ does not admit a simply connected (even triangular) Galois covering. For characteristic different from 2, $A$ is isomorphic to $kQ/I'$, where $I'$ is obtained by substituting the last given relation by $\mu \lambda$. For this presentation, the algebra $A$ accepts a covering $R \to R/G = A$, where $R$ is strongly simply connected and $G$ is a torsion-free group.

The convex subcategory $C$ given by the vertices 9, 10, 11, 12, 13 determines a convex subcategory of $R$ which contains subcategories of type $(2, 2, \infty)$; that is, $R$ is not cycle-finite.
References


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