Special Issue of the Bulletin of the Iranian Mathematical Society
Vol. 37 *No. 2 Part 2* (*2011*), *pp 187-234.*

CLUSTER ALGEBRAS AND CLUSTER CATEGORIES

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Communicated by Lidia Angeleri Hügel

Abstract. These are notes from introductory survey lectures given at the Institute for Studies in Theoretical Physics and Mathematics (IPM), Teheran, in 2008 and 2010. We present the definition and the fundamental properties of Fomin-Zelevinsky's cluster algebras. Then, we introduce quiver representations and show how they can be used to construct cluster variables, which are the canonical generators of cluster algebras. From quiver representations, we proceed to the cluster category, which yields a complete categorification of the cluster algebra and its combinatorial underpinnings.

1. Introduction

Cluster algebras, invented [37] by Sergey Fomin and Andrei Zelevinsky around the year 2000, are commutative algebras whose generators and relations are constructed in a recursive manner. Among these algebras, there are the algebras of homogeneous coordinates on the Grassmannians, on the flag varieties and on many other varieties which play an important role in geometry and representation theory. Fomin and Zelevinsky's main aim was to set up a combinatorial framework for the study of the so-called canonical bases which these algebras possess [69, 85] and which are closely related to the notion of total positivity

MSC(2010): Primary: 13F60; Secondary: 16G20, 18E30, 11Y50.
Keywords: Cluster algebra, quiver representation, cluster category, triangulated category.
Received: 9 February 2011, Accepted: 20 April 2011.

[86] in the associated varieties. It has rapidly turned out that the combinatorics of cluster algebras also appear in many other subjects, for example in

- Poisson geometry [53, 54, 55, 56, 11] ...;
- discrete dynamical systems [40, 71, 27, 64, 73, 78, 75] ...;
- higher Teichmüller spaces [31, 33, 32] ...;
- combinatorics and in particular the study of polyhedra like the Stasheff associahedra [23, 22, 63, 83, 35, 36, 88, 89, 100] ...;
- commutative and non commutative algebraic geometry and in particular the study of stability conditions in the sense of Bridgeland [13], Calabi-Yau algebras [66, 57], Donaldson-Thomas invariants [106, 68, 82, 81, 95, 47, 46, 21, 48] ...;
- and in the representation theory of quivers and finite-dimensional algebras; cf. for example the survey articles [6, 96, 98, 52, 78].

We refer to the introductory articles [39, 111, 112, 113, 114] and to the cluster algebras portal [34] for more information on cluster algebras and their links with other subjects in mathematics (and physics).

In these lectures, we give a concise introduction to cluster algebras and to their (additive) categorification using cluster categories. We start by recalling the most important facts on finite root systems (Section 2) because these play crucial roles in the classification of cluster algebras with finitely many cluster variables. Then, we prepare the ground for the formal definition of cluster algebras by giving an approximate description and the first examples in Section 3. In Section 4, we introduce the central construction of quiver mutation and define the cluster algebra associated with a quiver. We extend the definition to that of cluster algebras with coefficients and present some geometric examples in Section 5. Starting in Section 6, we turn to the (additive) categorification of cluster algebras. We start by recalling basic facts on representations of quivers. Then, we present the Caldero-Chapoton formula, which expresses the cluster variable associated with an indecomposable representation of a Dynkin quiver in terms of the geometry of the subrepresentations. For the proof of Caldero-Chapoton's formula and in order to prepare the grounds for the cluster category, we then recall the definition and the description of the derived category of a Dynkin quiver. Following Buan-Marsh-Reineke-Reiten-Todorov [7], we define the cluster category as an orbit category of the derived category. We then show how to categorify clusters via cluster-tilting objects. Finally, in Section 7, we extend the definitions given so far from the Dynkin quivers to acyclic quivers and

show on an example how, in certain cases, cluster categories are related to stable categories of the Cohen-Macaulay modules.

This introductory survey leaves out a number of important recent developments, notably monoidal categorification, as developed by Hernandez-Leclerc [61], and the theory of quivers with potentials [25] with its spectacular application to the solution [26] (cf. also [90, 93]) of a series of conjectures by Fomin-Zelevinsky [41]. We refer to [72] for a survey including these developments and to [74] for an introduction to the links between cluster theory and quantum dilogarithm identities.

2. **Reminder on root systems**

In the sequel, the classification of (finite, reduced) root systems will play an important role. For the convenience of the reader, we recall the main results from [103].

Let V be a finite-dimensional real vector space and v be a non zero vector in V. A *reflection at* v is a linear map $s : V \to V$ such that $s(v) = -v$ and s admits a hyperplane of fixed points in V.

Lemma 2.1. *Let G be a finite generating set of V and let s and s' be two reflections at v which leave G stable. Then, we have $s = s'$.*

Proof. Let us put $f = s \circ s'$. Then, we have $f(v) = v$ and f induces the identity in $V/\mathbb{R}v$. Hence, $f - \mathbf{1}_V$ is nilpotent. Since f leaves G stable and G generates V, the map f is of finite order and hence diagonalizable. It follows that f equals $\mathbf{1}_V$. □

A *root system* in V is a finite subset $\Phi \subset V$ such that

(a) Φ is non empty, does not contain the zero vector and generates V;
(b) for each $\alpha \in \Phi$, there is a reflection s_α at α which leaves Φ stable (notice that by the lemma, s_α is unique);
(c) for all $\alpha, \beta \in \Phi$, the vector $\beta - s_\alpha(\beta)$ is an integer multiple of α;
(d) for all $\alpha \in \Phi$ and all $\lambda \in \Phi$, if we have $\lambda\alpha \in \Phi$, then λ equals 1 or -1.

For two root systems (V, Φ) and (V', Φ'), an *isomorphism* is a linear map $f : V \xrightarrow{\sim} V'$ such that $f(\Phi) = \Phi'$. The *sum* of (V, Φ) and (V', Φ') is the space $V \oplus V'$ endowed with $\Phi \cup \Phi'$. The root system (V, Φ) is *irreducible* if it is not isomorphic to a sum of two root systems. The *rank* of a root system (V, Φ) is the dimension of V. It is not hard to

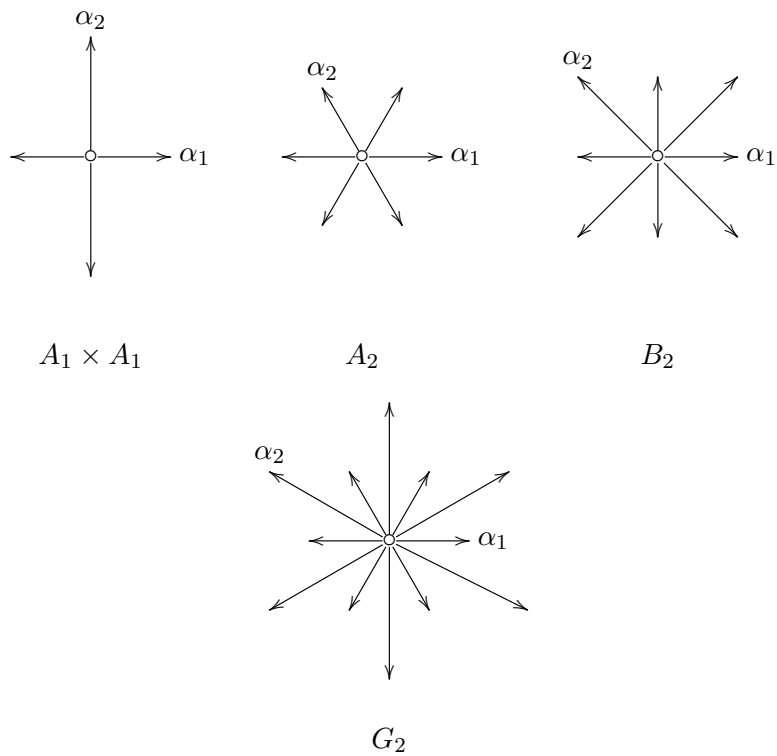


FIGURE 1. The root systems of rank 2

check that each root system of rank 1 is isomorphic to the following system, denoted by A_1,

$$A_1 \ : \ -\alpha_1 \longleftarrow\!\circ\!\longrightarrow \alpha_1$$

It is a less trivial exercise to check that each root system of rank 2 is isomorphic to one of the root systems depicted in Figure 1. Let (V, Φ) be a root system. A *root basis* is a subset $\alpha_1, \ldots, \alpha_n$ of Φ such that

(a) $\alpha_1, \ldots, \alpha_n$ is a basis of V and
(b) for each root α, either α or $-\alpha$ is a linear combination with non negative integer coefficients of $\alpha_1, \ldots, \alpha_n$.

In each of the examples in Figure 1, the roots α_1 and α_2 form a root basis. For a given root basis $\alpha_1, \ldots, \alpha_n$, the *root lattice* is the abelian subgroup of V formed by the integer linear combinations of the α_i and

the *positive roots* are the roots which are linear combinations with non negative coefficients of the α_i.

Lemma 2.2. *Let $f : V \to \mathbb{R}$ be a non zero linear form which does not vanish on any root. Let V^+ (respectively V^-) be the half-space formed by the vectors v such that $f(v) > 0$ (respectively $f(v) < 0$). Let $\Phi^\pm = V^\pm \cap \Phi$ and let S be the set of those roots $\alpha \in \Phi^+$ which are not sums of two roots in Φ^+. Then, S is a root basis and every root basis is obtained in this way.*

Once a root basis S is fixed, its elements are called *simple roots*. The *Weyl group* $W \subset GL(V)$ is the subgroup generated by the reflections s_α associated with all roots $\alpha \in \Phi$. Notice that the action of W leaves Φ stable. So, the group W is finite since Φ is a finite generating set for V.

Lemma 2.3. *The group W acts simply transitively on the set of root bases.*

Let $S \subset \Phi$ be a root basis. The *Cartan matrix* C has the entries $c_{\alpha,\beta}$, $\alpha, \beta \in S$, determined by

$$s_\alpha(\beta) = \beta - c_{\alpha,\beta}\alpha.$$

Thanks to Lemma 2.3, the Cartan matrix depends only on Φ and is unique up to conjugation with a permutation matrix. If α and β are distinct simple roots and $V_{\alpha,\beta}$ is the subspace generated by α and β, then the pair $(V_{\alpha,\beta}, V_{\alpha,\beta} \cap \Phi)$ is a root system of rank 2 with root basis α, β. Thus, the 2×2-submatrix of C given by α and β is one of the following:

$$A_1 \times A_1 : \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \; A_2 : \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \; B_2 : \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}, \; G_2 : \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

The *Coxeter-Dynkin diagram* of (Φ, V) is the labeled graph whose vertices are the simple roots $\alpha \in S$, where there is an edge

$$\alpha \text{ —— } \beta$$

if $c_{\alpha,\beta} \neq 0$, and this edge is labeled by the pair $(|c_{\alpha,\beta}|, |c_{\beta,\alpha}|)$. For example, the Coxeter-Dynkin diagrams of the root systems of rank 2 are:

$$A_1 \times A_1 : \circ \qquad \circ \;, \; A_2 : \circ \overset{(1,1)}{\text{———}} \circ \;, \; B_2 : \circ \overset{(2,1)}{\text{———}} \circ \;, \; G_2 : \circ \overset{(3,1)}{\text{———}} \circ \;.$$

By convention, in the sequel, we write

$$\circ \text{———} \circ \quad \text{instead of} \quad \circ \overset{(1,1)}{\text{———}} \circ \,.$$

If $\alpha_1, \ldots, \alpha_n$ is a root basis, the product $c = s_{\alpha_1} \ldots s_{\alpha_n}$ of the reflections at the simple roots is a *Coxeter element*. Up to conjugacy in W, the Coxeter element is independent of the choice of the root basis. For example, for A_2, the Coxeter element is the rotation by 120 degrees. The *Coxeter number* is the order of the Coxeter element.

Theorem 2.4.

(a) *The Coxeter-Dynkin diagram determines the root system up to isomorphism.*

(b) *The Coxeter-Dynkin diagrams of the irreducible root systems are those in the following table, where the index n denotes the number of vertices and h the Coxeter number:*

Name	Graph	n	h
A_n	$\circ$ —— $\circ$ —— $\ldots$ —— $\circ$	≥ 1	$n+1$
B_n	$\circ \overset{(2,1)}{\text{———}} \circ$ —— $\ldots$ —— $\circ$	≥ 2	$2n$
C_n	$\circ \overset{(1,2)}{\text{———}} \circ$ —— $\ldots$ —— $\circ$	≥ 2	$2n$
D_n	$\circ$ $\circ$ $\circ$ —— $\cdots$ —— $\circ$	≥ 4	$2n-2$
E_6	$\circ$ —— $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ $\circ$	6	12
E_7	$\circ$ —— $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ $\circ$	7	18
E_8	$\circ$ —— $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ —— $\circ$ $\circ$	8	30

The root systems whose Coxeter-Dynkin diagram has only labels $(1,1)$ are called *simply laced*. A root system is simply laced if and only if its Cartan matrix is symmetric if and only if it is a sum of root systems of the types A_n, D_n, E_6, E_7 and E_8.

3. Description and first examples of cluster algebras

3.1. **Description.** A *cluster algebra* is a commutative $\mathbb{Q}$-algebra endowed with a set of distinguished generators (the *cluster variables*) grouped into overlapping subsets (the *clusters*) of constant cardinality (the *rank*) which are constructed recursively via *mutation* from an *initial cluster*. The set of cluster variables can be finite or infinite.

Theorem 3.1. [38]. *The cluster algebras having only a finite number of cluster variables are parametrized by the finite root systems.*

Thus, the classification is analogous to the one of semi-simple complex Lie algebras. We will make the theorem more explicit (for simply laced root systems) in Section 4 below.

First example 3.2. In order to illustrate the description and the theorem, we present [114] the cluster algebra $\mathcal{A}_{A_2}$ associated with the root system A_2. By definition, it is generated as a $\mathbb{Q}$-algebra by the cluster variables x_m, $m \in \mathbb{Z}$, submitted to the *exchange relations*

$$x_{m-1}x_{m+1} = 1 + x_m \ , \ m \in \mathbb{Z}.$$

Its clusters are by definition the pairs of consecutive cluster variables $\{x_m, x_{m+1}\}$, $m \in \mathbb{Z}$. The initial cluster is $\{x_1, x_2\}$ and two clusters are linked by a mutation if and only if they share exactly one variable.

The exchange relations allow to express each cluster variable as a rational expression in the initial variables x_1, x_2 and thus to identify the algebra $\mathcal{A}_{A_2}$ with a subalgebra of the field $\mathbb{Q}(x_1, x_2)$. In order to make this subalgebra explicit, let us compute the cluster variables x_m for $m \geq 3$. We have

$$x_3 = \frac{1 + x_2}{x_1} \tag{3.1}$$

$$x_4 = \frac{1 + x_3}{x_2} = \frac{x_1 + 1 + x_2}{x_1 x_2} \tag{3.2}$$

$$x_5 = \frac{1 + x_4}{x_3} = \frac{x_1 x_2 + x_1 + 1 + x_2}{x_1 x_2} \ \div \ \frac{1 + x_2}{x_1} = \frac{1 + x_1}{x_2} \ . \tag{3.3}$$

Notice that, contrary to what one might expect, the denominator in (3.3) remains a monomial! In fact, each cluster variable in an arbitrary cluster algebra is a Laurent polynomial; cf. Theorem 4.1 below. Let us continue

the computation:

$$x_6 = \frac{1+x_5}{x_4} = \frac{x_2+1+x_1}{x_2} \div \frac{x_1+1+x_2}{x_1x_2} = x_1 \tag{3.4}$$

$$x_7 = (1+x_1) \div \frac{1+x_1}{x_2} = x_2. \tag{3.5}$$

It is now clear that the sequence of the x_m, $m \in \mathbb{Z}$, is 5-periodic and that the number of cluster variables is indeed finite and equal to 5. In addition to the two initial variables x_1 and x_2, we have three non initial variables x_3, x_4 and x_5. By examining their denominators, we see that they are in natural bijection with the positive roots α_1, $\alpha_1+\alpha_2$ and α_2 of the root system A_2. This generalizes to an arbitrary Dynkin diagram; cf. Theorem 4.1.

3.2. **Cluster algebras of rank 2.** To each pair of positive integers (b,c), there is associated a cluster algebra $\mathcal{A}_{(b,c)}$. It is defined in analogy with $\mathcal{A}_{A_2}$ by replacing the exchange relations with

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1 & \text{if } m \text{ is odd,} \\ x_m^c + 1 & \text{if } m \text{ is even.} \end{cases}$$

The algebra $\mathcal{A}_{(b,c)}$ has only a finite number of cluster variables if and only if we have $bc \leq 3$. In other words, if and only if the matrix

$$\begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}$$

is the Cartan matrix of a root system Φ of rank 2. The reader is invited to check that, in this case, the non initial cluster variables are still in natural bijection with the positive roots of Φ.

4. Cluster algebras associated with quivers

4.1. **Quiver mutation.** A *quiver* is an oriented graph, i.e., a quadruple $Q = (Q_0, Q_1, s, t)$ formed by a set of vertices Q_0, a set of arrows Q_1 and two maps s and t from Q_1 to Q_0 which send an arrow α respectively to its source $s(\alpha)$ and its target $t(\alpha)$. In practice, a quiver is given by a picture as in the following example:

$$\begin{tikzcd}
Q: & & 3 \arrow[dl, "\lambda"'] & \alpha \circlearrowright 5 \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & 6 \\
& 1 \arrow[rr, "\nu"'] & & 2 \arrow[ul, "\mu"'] \arrow[r, shift left, "\beta"] & 4. \arrow[l, shift left, "\gamma"]
\end{tikzcd}$$

An arrow α whose source and target coincide is a *loop*; a 2-*cycle* is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$. Similarly, one defines n-*cycles* for any positive integer n. A vertex i of a quiver is a *source* (respectively a *sink*) if there is no arrow with target i (respectively with source i).

By convention, in the sequel, by a quiver we always mean a finite quiver without loops nor 2-cycles whose set of vertices is the set of integers from 1 to n, for some $n \geq 1$. Up to an isomorphism fixing the vertices, such a quiver Q is given by the antisymmetric matrix $B = B_Q$ whose coefficient b_{ij} is the difference between the number of arrows from i to j and the number of arrows from j to i, for all $1 \leq i, j \leq n$. Conversely, each antisymmetric matrix B with integer coefficients comes from a quiver.

Let Q be a quiver and k be a vertex of Q. The *mutation* $\mu_k(Q)$ is the quiver obtained from Q as follows:

(1) for each subquiver $i \xrightarrow{\beta} k \xrightarrow{\alpha} j$, we add a new arrow $[\alpha\beta] : i \to j$;
(2) we reverse all arrows with source or target k;
(3) we remove the arrows in a maximal set of pairwise disjoint 2-cycles.

If B is the antisymmetric matrix associated with Q and B' is the one associated with $\mu_k(Q)$, then we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(0, b_{ik} b_{kj}) & \text{otherwise.} \end{cases}$$

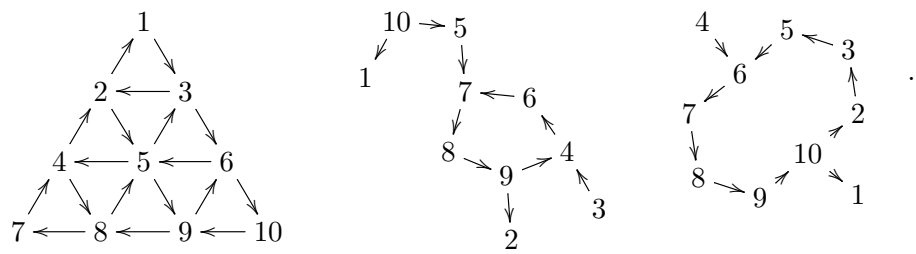
This is the *matrix mutation rule* for antisymmetric (more generally, antisymmetrizable) matrices introduced by Fomin-Zelevinsky in [37]; also cf [41].

One checks easily that μ_k is an involution. For example, the quivers

(4.1)
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd}$$

are linked by a mutation at the vertex 1. Notice that these quivers are drastically different: the first one is a cycle, the second one the Hasse diagram of a linearly ordered set.

Two quivers are *mutation equivalent* if they are linked by a finite sequence of mutations. It is easy to check, for example using the quiver mutation applet [76], that the following three quivers are mutation equivalent:
(4.2)



The common mutation class of these quivers contains 5739 quivers (up to isomorphism). The mutation class of 'most' quivers is infinite. The classification of the quivers having a finite mutation class is a difficult problem recently solved by Felikson-Shapiro-Tumarkin [30]: in addition to the quivers associated with triangulations of surfaces (with boundary and marked points), the list contains 11 exceptional quivers, the largest of which is in the mutation class of the quivers (4.2).

4.2. **Seed mutation, cluster algebras.** Let $n \geq 1$ be an integer and $\mathcal{F}$ be the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by n indeterminates $x_1, \ldots, x_n$. A *seed* (more precisely, X*-seed*) is a pair (R, u), where R is a quiver and u is a sequence $u_1, \ldots, u_n$ of elements of $\mathcal{F}$ which freely generates the field $\mathcal{F}$. If (R, u) is a seed and k is a vertex of R, then the *mutation* $\mu_k(R, u)$ is the seed (R', u'), where $R' = \mu_k(R)$ and u' is obtained from u by replacing the element u_k by the element u'_k defined with the *exchange relation*

$$u'_k u_k = \prod_{s(\alpha)=k} u_{t(\alpha)} + \prod_{t(\alpha)=k} u_{s(\alpha)}. \tag{4.3}$$

One checks that $\mu_k^2(R, u) = (R, u)$. For example, the mutations of the seed

$$(\ 1 \longrightarrow 2 \longrightarrow 3\ ,\ \{x_1, x_2, x_3\})$$

with respect to the vertices 1 and 2 are the seeds

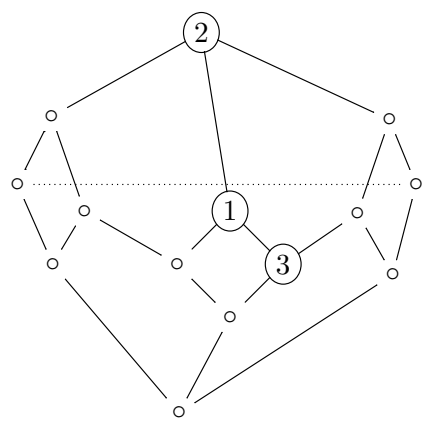
$$(4.4) \qquad \left(1 \longleftarrow 2 \longrightarrow 3 \ , \ \{\frac{1+x_2}{x_1}, x_2, x_3\}\right)$$

$$\text{and} \qquad \left(\begin{tikzcd} 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[ll, bend left] \end{tikzcd} \ , \ \{x_1, \frac{x_1+x_3}{x_2}, x_3\}\right).$$

Let us fix a quiver Q. The *initial seed* of Q is $(Q, \{x_1, \ldots, x_n\})$. A *cluster* associated with Q is a sequence u which appears in a seed (R, u) obtained from the initial seed by iterated mutation. The *cluster variables* are the elements of the clusters. The *cluster algebra* $\mathcal{A}_Q$ is the subalgebra of $\mathcal{F}$ generated by the cluster variables. Clearly, if (R, u) is a seed associated with Q, the natural isomorphism

$$\mathbb{Q}(u_1, \ldots, u_n) \xrightarrow{\sim} \mathbb{Q}(x_1, \ldots, x_n)$$

induces an isomorphism of $\mathcal{A}_R$ on $\mathcal{A}_Q$ which preserves the cluster variables and the clusters. Thus, the cluster algebra $\mathcal{A}_Q$ is an invariant of the mutation class of Q. It is useful to introduce a combinatorial object which encodes the recursive construction of the seeds: the *exchange graph*. By definition, its vertices are the isomorphism classes of seeds (isomorphisms of seeds renumber the vertices and the variables simultaneously) and its edges correspond to mutations. For example, the exchange graph obtained from the quiver $Q: \ 1 \longrightarrow 2 \longrightarrow 3$ is the 1-skeleton of the Stasheff associahedron [23, 105]:



The vertex 1 corresponds to the initial seed and the vertices 2 and 3 to correspond the seeds (4.4).

Let Q be a connected quiver. If its underlying graph is a simply laced Dynkin diagram Δ, we say that Q is a *Dynkin quiver of type* Δ.

Theorem 4.1. [38].

(a) *Each cluster variable of $\mathcal{A}_Q$ is a Laurent polynomial with integer coefficients* [37].

(b) *The cluster algebra $\mathcal{A}_Q$ has only a finite number of cluster variables if and only if Q is mutation equivalent to a Dynkin quiver R. In this case, the underlying graph Δ of R is unique up to isomorphism and is called the* cluster type of Q.

(c) *If Q is a Dynkin quiver of type Δ, then the non initial cluster variables of $\mathcal{A}_Q$ are in bijection with the positive roots of the root system Φ of Δ; more precisely, if $\alpha_1, \ldots, \alpha_n$ are the simple roots, then for each positive root $\alpha = d_1\alpha_1 + \cdots + d_n\alpha_n$, there is a unique non initial cluster variable X_α whose denominator is $x_1^{d_1} \ldots x_n^{d_n}$.*

A *cluster monomial* is a product of non negative powers of cluster variables belonging to the same cluster. The construction of a 'canonical basis' of the cluster algebra $\mathcal{A}_Q$ is an important and largely open problem; cf., for example, [65, 29, 37, 104]. It is expected that such a basis should contain all cluster monomials. Whence, the following conjecture is at hand.

Conjecture 4.2. [37]. *The cluster monomials are linearly independent over the field $\mathbb{Q}$.*

If Q is a Dynkin quiver, then one knows [18] that the cluster monomials form a basis of $\mathcal{A}_Q$. If Q is *acyclic*, i.e., does not have any oriented cycles, the conjecture follows from a theorem by Geiss-Leclerc-Schröer [49], who show the existence of a 'generic basis' containing the cluster monomials. The conjecture has also been shown for classes of cluster algebras with coefficients (cf. Section 5); for example, see the papers [24, 42, 49].

Conjecture 4.3. [38]. *The cluster variables are Laurent polynomials with non negative integer coefficients in the variables of each cluster.*

The technique of monoidal categorification developed by Leclerc [84] and Hernandez-Leclerc [61] has recently allowed the proof of this conjecture first for the quivers of type A_n and D_4 (cf. [61]), and then for each bipartite quiver [91], i.e., a quiver where each vertex is a source or a sink. This has been shown in a combinatorial way, by Musiker-Schiffler-Williams [89], for all the quivers associated with triangulations

of surfaces (with boundary and marked points) and by Di Francesco-Kedem [27], for the quivers associated with the T-system of type A.

We refer to [39, 41] for numerous other conjectures on cluster algebras and to [26] for the solution of a large number of them using additive categorification.

5. Cluster algebras with coefficients

We will slightly generalize the definition in Section 4 in order to obtain the class of 'antisymmetric cluster algebras of geometric type'. This class contains many algebras of geometric origin which are equipped with 'dual semi-canonical bases'. The construction of a large part of such a basis in [50] is one of the most remarkable applications of cluster algebras so far.

We refer to [41] for the definition of the 'antisymmetrizable cluster algebras with coefficients in a semi-field', which constitutes so far the most general class considered.

Definition 5.1. *Let $1 \leq n \leq m$ be integers. Let $\widetilde{Q}$ be an* ice quiver of type (n, m)*, i.e., a quiver with m vertices and which does not have any arrows between vertices i and j which are both strictly greater than n. The* principal part *of $\widetilde{Q}$ is the full subquiver Q whose vertices are $1, \ldots, n$ (a subquiver is* full *if, with any two vertices, it contains all the arrows linking them). The vertices $n+1, \ldots, m$ are called the* frozen vertices*. The* cluster algebra associated with the ice quiver $\widetilde{Q}$*,*

$$\mathcal{A}_{\widetilde{Q}} \subset \mathbb{Q}(x_1, \ldots, x_m),$$

is defined in the same manner as the cluster algebra associated with a quiver (Section 4), but

- *only mutations with respect to non frozen vertices are allowed and no arrows between frozen vertices are added in the mutations;*
- *the variables $x_{n+1}, \ldots, x_m$, which belong to all clusters, are called* coefficients *rather than cluster variables;*
- *the* cluster type *of the ice quiver is that of its principal part (if it is defined).*

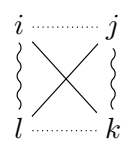
Often one considers localizations of $\mathcal{A}_{\widetilde{Q}}$ obtained by inverting some of the coefficients. If K is an extension field of $\mathbb{Q}$ and A is a K-algebra (associative with 1), a cluster structure of type $\widetilde{Q}$ on A *is given by an isomorphism φ from $\mathcal{A}_{\widetilde{Q}} \otimes_{\mathbb{Q}} K$ onto A. Such an isomorphism is*

determined by the images of the coefficients and of the initial cluster variables $\varphi(x_i)$, $1 \leq i \leq m$. We call the datum of $\widetilde{Q}$ and of the $\varphi(x_i)$ an initial seed for A.

Example 5.2. planes in a vector space Let $n \geq 1$ be an integer. Let A be the algebra of polynomial functions on the cone over the Grassmannian of planes in $\mathbb{C}^{n+3}$. This algebra is generated by the Plücker coordinates x_{ij}, $1 \leq i < j \leq n+3$, subject to the Plücker relations: for each quadruple of integers $i < j < k < l$ between 1 and $n+3$, we have

$$x_{ik}x_{jl} = x_{ij}x_{kl} + x_{jk}x_{il}. \tag{5.1}$$

Notice that the monomials in this relation are naturally associated with the diagonals and the sides of the square



The idea is to interpret this relation as an exchange relation in a cluster algebra with coefficients. In order to describe this algebra, let us consider, in the Euclidean plane, a regular polygon P whose vertices are numbered from 1 to $n+3$. Consider the variable x_{ij} as associated with the segment $[ij]$ which links the vertices i and j.

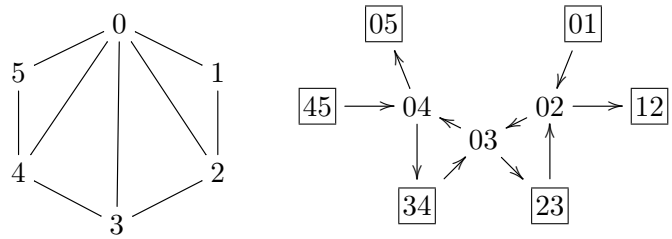
Proposition 5.2. [38, Example 12.6]. *The algebra A has a cluster algebra structure such that*

- *the coefficients are the variables x_{ij} associated with the sides of P;*
- *the cluster variables are the variables x_{ij} associated with the diagonals of P;*
- *the clusters are the n-tuples of cluster variables corresponding to diagonals which form a triangulation of P.*

Moreover, the exchange relations are exactly the Plücker relations and the cluster type is A_n.

A triangulation of P determines an initial seed for the cluster algebra and the exchange relations satisfied by the initial cluster variables determine the ice quiver $\widetilde{Q}$. For example, one can check that in the following picture, the triangulation and the ice quiver (whose frozen vertices are

in boxes) correspond to each other:



Many other (homogeneous) coordinate algebras of classical algebraic varieties admit cluster algebra structures (or 'upper cluster algebra structures'), and in particular the Grassmannians [102] and the double Bruhat cells [10]. Some of these algebras have only finitely many cluster variables and thus a well-defined cluster type. Here is a list of some examples extracted from [39], where N is a maximal unipotent subgroup:

$Gr_{2,n+3}$	$Gr_{3,6}$	$Gr_{3,7}$	$Gr_{3,8}$	SL_3/N	SL_4/N	SL_5/N	Sp_4/N	SL_2	SL_3
A_n	D_4	E_6	E_8	A_1	A_3	D_6	B_2	A_1	D_4

A theorem analogous to proposition 5.2 for 'reduced double Bruhat cells' is due to Yang and Zelevinsky [109]. They thus obtain a cluster algebra (with principal coefficients) with an explicit description of the cluster variables for each Dynkin diagram.

6. Categorification via cluster categories: the finite case

6.1. **Quiver representations and Gabriel's theorem.** We refer to the books [3, 4, 44, 99] and for a wealth of information on the representation theory of quivers and finite-dimensional algebras. Here, we will only need very basic notions.

Let Q be a finite quiver without oriented cycles. For example, Q can be an orientation of a simply laced Dynkin diagram or the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 3. \end{tikzcd}$$

Let k be an algebraically closed field. A *representation of* Q is a diagram of finite-dimensional vector spaces of the shape given by Q. More formally, a representation of Q is the datum V of

- a finite-dimensional vector space V_i for each vertex i of Q;
- a linear map $V_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$ of Q.

Thus, in the above example, a representation of Q is a (not necessarily commutative) diagram

$$\begin{tikzcd} & V_2 \arrow[dr, "V_\beta"] & \\ V_1 \arrow[ur, "V_\alpha"] \arrow[rr, "V_\gamma"'] & & V_3 \end{tikzcd}$$

formed by three finite-dimensional vector spaces and three linear maps. A *morphism of representations* is a morphism of diagrams. More formally, a morphism of representations $f : V \to W$ is the datum of a linear map $f_i : V_i \to W_i$ for each vertex i of Q such that the square

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"'] & W_j \end{tikzcd}$$

commutes for all arrows $\alpha : i \to j$ of Q. The *composition* of morphisms is defined in the natural way. We thus obtain the *category of representations* $\mathsf{rep}(Q)$. A morphism $f : V \to W$ of this category is an isomorphism if and only if its components f_i are invertible for all vertices i of Q_0.

For example, let Q be the quiver

$$1 \longrightarrow 2 \ ,$$

and

$$V : \ V_1 \xrightarrow{V_\alpha} V_2$$

be a representation of Q. By choosing basis in the spaces V_1 and V_2, we find an isomorphism of representations

$$\begin{tikzcd} V_1 \arrow[r, "V_\alpha"] & V_2 \\ k^n \arrow[u] \arrow[r, "A"'] & k^p \arrow[u] \end{tikzcd} ,$$

where, by abuse of notation, we denote by A the multiplication by a $p \times n$ matrix A. We know that we have

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for invertible matrices P and Q, where r is the rank of A. Let us denote the right hand side by $I_r \oplus 0$. Then, we have an isomorphism of representations

$$\begin{tikzcd} k^n \arrow[r, "A"'] & k^p \\ k^n \arrow[u, "Q"] \arrow[r, "I_r \oplus 0"'] & k^p \arrow[u, "P^{-1}"'] \end{tikzcd} .$$

We thus obtain a normal form for the representations of this quiver.

Now, the category $\mathsf{rep}_k(Q)$ is in fact an *abelian category*: its direct sums, kernels and cokernels are computed componentwise. Thus, if V and W are two representations, then the *direct sum* $V \oplus W$ is the representation given by

$$(V \oplus W)_i = V_i \oplus W_i \text{ and } (V \oplus W)_\alpha = V_\alpha \oplus W_\alpha \,,$$

for all vertices i and all arrows α of Q. For example, the above representation in normal form is isomorphic to the direct sum

$$(\ k \xrightarrow{1} k\ )^r \oplus (\ k \longrightarrow 0\ )^{n-r} \oplus (\ 0 \longrightarrow k\ )^{p-r}.$$

The *kernel* of a morphism of representations $f : V \to W$ is given by

$$\mathsf{ker}(f)_i = \mathsf{ker}(f_i : V_i \to W_i)$$

endowed with the maps induced by the V_α and similarly for the cokernel. A *subrepresentation* V' of a representation V is given by a family of subspaces $V'_i \subset V_i$, $i \in Q_0$, such that the image of V'_i under V_α is contained in V'_j, for each arrow $\alpha : i \to j$ of Q. A sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

of representations is a *short exact sequence* if the sequence

$$0 \longrightarrow U_i \longrightarrow V_i \longrightarrow W_i \longrightarrow 0$$

is exact for each vertex i of Q.

A representation V is *simple* if it is non zero and if for each subrepresentation V' of V, we have $V' = 0$ or $V/V' = 0$. Equivalently, a representation is simple if it has exactly two subrepresentations. A representation V is *indecomposable* if it is non zero and in each decomposition $V = V' \oplus V''$, we have $V' = 0$ or $V'' = 0$. Equivalently, a representation is indecomposable if it has exactly two direct factors.

In the above example, the representations

$$k \longrightarrow 0 \ \text{ and } \ 0 \longrightarrow k$$

are simple. The representation

$$V = (\ k \xrightarrow{1} k\ )$$

is not simple: it has the non trivial subrepresentation $0 \longrightarrow k$. However, it is indecomposable. Indeed, each endomorphism $f : V \to V$ is given by two equal components $f_1 = f_2$ so that the endomorphism algebra of V is one-dimensional. If V was a direct sum $V' \oplus V''$ for two non zero subspaces, the endomorphism algebra of V would contain the product of the endomorphism algebras of V' and V'' and thus would have to be at least of dimension 2. Since V is indecomposable, the exact sequence

$$0 \to (\ 0 \longrightarrow k\ ) \to (\ k \xrightarrow{1} k\ ) \to (\ k \longrightarrow 0\ ) \to 0$$

is not a split exact sequence.

If Q is an arbitrary quiver, then for each vertex i, we define the representation S_i by

$$(S_i)_j = \begin{cases} k & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then, clearly the representations S_i are simple and pairwise non isomorphic. As an exercise, the reader may show that if Q does not have oriented cycles, then each representation admits a finite filtration whose subquotients are among the S_i. Thus, in this case, each simple representation is isomorphic to one of the representations S_i.

Recall that a (possibly non commutative) ring is *local* if its non invertible elements form an ideal.

Theorem 6.1. *decomposition [Azumaya-Fitting-Krull-Remak-Schmidt]*

(a) *A representation is indecomposable iff and only if its endomorphism algebra is local.*
(b) *Each representation decomposes into a finite sum of indecomposable representations, unique up to isomorphism and permutation.*

As seen above, for quivers without oriented cycles, the classification of the simple representations is trivial. On the other hand, the problem of classifying the indecomposable representations is non trivial. Let us examine this problem in a few examples: for the quiver $1 \to 2$, we have checked the existence in part (b) directly. The uniqueness in (b) then

implies that each indecomposable representation is isomorphic to exactly one of the representations S_1, S_2 and

$$k \xrightarrow{1} k\ .$$

Similarly, using elementary linear algebra, it is not hard to check that each indecomposable representation of the quiver

$$\vec{A}_n:\ 1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n$$

is isomorphic to a representation $I[p,q]$, $1 \le p < q \le n$, which takes the vertices i in the interval $[p,q]$ to k, the arrows linking them to the identity and all other vertices to zero. In particular, the number of isomorphism classes of indecomposable representations of $\vec{A}_n$ is $n(n+1)/2$.

The representations of the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

are the pairs (V_1, V_α) consisting of a finite-dimensional vector space and an endomorphism and the morphisms of representations are the 'intertwining operators'. It follows from the existence and uniqueness of the Jordan normal form that a system of representatives of the isomorphism classes of indecomposable representations is formed by the representations $(k^n, J_{n,\lambda})$, where $n \ge 1$ is an integer, λ is a scalar and $J_{n,\lambda}$ is the Jordan block of size n with eigenvalue λ.

The *Kronecker quiver*

$$1 \rightrightarrows 2$$

admits the following infinite family of pairwise non isomorphic representations:

$$k \underset{\mu}{\overset{\lambda}{\rightrightarrows}} k\ ,$$

where $(\lambda : \mu)$ runs through the projective line.

Question 6.2. *For which quivers are there only finitely many isomorphism classes of indecomposable representations?*

To answer this question, we define the *dimension vector* of a representation V to be the sequence $\underline{\dim}\, V$ of the dimensions $\dim V_i$, $i \in Q_0$. For example, the dimension vectors of the indecomposable representations of $\vec{A}_2$ are the pairs

$$\underline{\dim}\, S_1 = [10]\ ,\ \underline{\dim}\, S_2 = [01]\ ,\ \underline{\dim}\,(k \to k) = [11].$$

We define the *Tits form*

$$q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

by

$$q_Q(v) = \sum_{i \in Q_0} v_i^2 - \sum_{\alpha \in Q_1} v_{s(\alpha)} v_{t(\alpha)}.$$

Notice that the Tits form does not depend on the orientation of the arrows of Q but only on its underlying graph. We say that the quiver Q is *representation-finite* if, up to isomorphism, it has only finitely many indecomposable representations. We say that a vector $v \in \mathbb{Z}^{Q_0}$ is a *root* of q_Q if $q_Q(v) = 1$ and that it is *positive* if its components are ≥ 0.

Theorem 6.3. *Gabriel* [43]. *Let Q be a connected quiver and assume that k is algebraically closed. The followings are equivalent.*

(i) *Q is representation-finite.*
(ii) *q_Q is positive definite.*
(iii) *The underlying graph of Q is a simply laced Dynkin diagram Δ.*

Moreover, in this case, the map taking a representation to its dimension vector yields a bijection from the set of isomorphism classes of indecomposable representations to the set of positive roots of the Tits form q_Q.

It is not hard to check that if the conditions hold, then the positive roots of q_Q are in turn in bijection with the positive roots of the root system Φ associated with Δ, via the map taking a positive root v of q_Q to the element

$$\sum_{i \in Q_0} v_i \alpha_i$$

of the root lattice of Φ.

Let us consider the example of the quiver $Q = \vec{A}_2$. In this case, the Tits form is given by

$$q_Q(v) = v_1^2 + v_2^2 - v_1 v_2.$$

It is positive definite and its positive roots are indeed precisely the dimension vectors

$$[01]\,,\ [10]\,,\ [11]$$

of the indecomposable representations.

Gabriel's theorem has been generalized to non algebraically closed ground fields by Dlab and Ringel [28]. Let us illustrate the main idea

on one simple example: consider the category of diagrams

$$V: \ V_1 \xrightarrow{f} V_2$$

where V_1 is a finite-dimensional real vector space, V_2 is a finite-dimensional complex vector space and f is an $\mathbb{R}$-linear map. Morphisms are given in the natural way. Then, we have the following complete list of representatives of the isomorphism classes of indecomposables:

$$\mathbb{R} \to 0\,,\ \mathbb{R}^2 \to \mathbb{C}\,,\ \mathbb{R} \to \mathbb{C}\,,\ 0 \to \mathbb{C}.$$

The corresponding dimension vectors are

$$[10]\,,\ [21]\,,\ [11]\,,\ [01].$$

They correspond bijectively to the positive roots of the root system B_2.

6.2. **The Caldero-Chapoton formula.** Let Δ be a simply laced Dynkin diagram and Q be a quiver with underlying graph Δ. Suppose that the set of vertices of Δ and Q is the set of the natural numbers 1, 2, ..., n. We already know from part (c) of Theorem 4.1 that for each positive root

$$\alpha = \sum_{i=1}^{n} d_i \alpha_i$$

of the corresponding root system, there is a unique non initial cluster variable X_α with denominator

$$x_1^{d_1} \dots x_n^{d_n}.$$

By combining this with Gabriel's theorem, we get the following result.

Corollary 6.4. *The map taking an indecomposable representation V with dimension vector $[d_1 \cdots d_n]$ of Q to the unique non initial cluster variable X_V, whose denominator is $x_1^{d_1} \dots x_n^{d_n}$, induces a bijection from the set of ismorphism classes of indecomposable representations to the set of non initial cluster variables.*

Let us consider this bijection for $Q = \vec{A}_2$:

$$S_2 = (0 \to k) \qquad P_1 = (k \to k) \qquad S_1 = (k \to 0)$$

$$X_{S_2} = \frac{1 + x_1}{x_2} \qquad X_{P_1} = \frac{x_1 + 1 + x_2}{x_1 x_2} \qquad S_{S_1} = \frac{1 + x_2}{x_1}.$$

We observe that for the two simple representations, the numerator contains exactly two terms: the number of subrepresentations of a simple

representation! Moreover, the representation P_1 has exactly three subrepresentations and the numerator of X_{P_1} contains three terms. In fact, it turns out that this phenomenon is general in type A. But, now let us consider the following quiver, with the underlying graph D_4,

$$\begin{tikzcd} 3 \arrow[dr] & \\ 2 \arrow[r] & 4 \\ 1 \arrow[ur] & \end{tikzcd}$$

and the dimension vector $d = [1112]$. The unique (up to isomorphism) indecomposable representation V with dimension vector d consists of a plane V_4 together with three lines in general position $V_i \subset V_4$, $i = 1, 2, 3$. The corresponding cluster variable is

$$X_4 = \frac{1}{x_1x_2x_3x_4^2}\,(1 + 3x_4 + 3x_4^2 + x_4^3 + 2x_1x_2x_3 + 3x_1x_2x_3x_4 + x_1^2x_2^2x_3^2).$$

Its numerator contains a total of 14 monomials. On the other hand, it is easy to see that V_4 has only 13 types of submodules: twelve submodules are determined by their dimension vectors but for the dimension vector $e = (0, 0, 0, 1)$, we have a family of submodules, and each submodule of this dimension vector corresponds to the choice of a line in V_4. Thus for this dimension vector e, the family of submodules is parametrized by a projective line. Notice that the Euler characteristic of the projective line is 2 (since it is a sphere: the Riemann sphere). So, if we attribute weight 1 to the submodules determined by their dimension vector and weight 2 to this $\mathbb{P}^1$-family, then we find a 'total submodule weight' equal to the number of monomials in the numerator. These considerations led Caldero-Chapoton [15] to the following definition, whose ingredients we describe next. Let Q be a finite quiver with vertices 1, ..., n, and V be a finite-dimensional representation of Q. Let $d = [d_1 \cdots d_n]$ be the dimension vector of V. Define

$$CC(V) = \frac{1}{x_1^{d_1}x_2^{d_2}\dots x_n^{d_n}}\,(\sum_{0\le e\le d} \chi(\mathrm{Gr}_e(V)) \prod_{i=1}^{n} x_i^{\sum_{j\to i} e_j + \sum_{i\to j}(d_j - e_j)}).$$

Here, the sum is taken over all vectors $e \in \mathbb{N}^n$ such that $0 \le e_i \le d_i$, for all i. For each such vector e, the *quiver Grassmannian*, $\mathrm{Gr}_e(V)$, is the variety of n-tuples of subspaces $U_i \subset V_i$ such that $\dim U_i = e_i$ and the U_i form a subrepresentation of V. By taking such a subrepresentation

to the family of the U_i, we obtain a map

$$\mathrm{Gr}_e(V) \to \prod_{i=1}^{n} \mathrm{Gr}_{e_i}(V_i)\ ,$$

where $\mathrm{Gr}_{e_i}(V_i)$ denotes the ordinary Grassmannian of e_i-dimensional subspaces of V_i. Recall that the Grassmannian carries a canonical structure of projective variety. It is not hard to see that for a family of subspaces (U_i) the condition of being a subrepresentation is a closed condition so that the quiver Grassmannian identifies with a projective subvariety of the product of ordinary Grassmannians. If k is the field of complex numbers, then the Euler characteristic χ is taken with respect to singular cohomology with coefficients in $\mathbb{Q}$ (or any other field). If k is an arbitrary algebraically closed field, we use étale cohomology to define χ. The most important properties of χ are (*e.g.* , *cf.* Section 7.4 in [51]):

(1) χ is additive with respect to disjoint unions;
(2) if $p : E \to X$ is a morphism of algebraic varieties such that the Euler characteristic of the fiber over a point $x \in X$ does not depend on x, then $\chi(E)$ is the product of $\chi(X)$ by the Euler characteristic of the fiber over any point $x \in X$.

Theorem 6.5. *Caldero-Chapoton* [15]. *Let Q be a Dynkin quiver and V be an indecomposable representation. Then, we have $CC(V) = X_V$, the cluster variable obtained from V by composing Fomin-Zelevinsky's bijection with Gabriel's.*

Caldero-Chapoton's proof of the theorem was by induction. One of the aims of the following sections is to explain 'on what' they did in the induction.

6.3. **The derived category.** Let k be an algebraically closed field and Q be a (possibly infinite) quiver without oriented cycles (we will impose more restrictive conditions on Q later). For example, Q could be the quiver

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\gamma"'] & 3 \arrow[d, "\alpha"] \\ 2 \arrow[r] \arrow[ur, "\beta"] & 4 \end{tikzcd}$$

A *path* of Q is a formal composition of arrows. For example, the sequence $(4|\alpha|\beta|\gamma|1)$ is a path of length 3 in the above example (notice that, for the moment, we include the source and target vertices of the path in

the notation; later, we will simply write $\alpha\beta\gamma$ for this composition). For each vertex i of Q, we have the *lazy path* $e_i = (i|i)$, the unique path of length 0 which starts at i and stops at i and does nothing in between. The *path category* has a set of objects Q_0 (the set of vertices of Q) and, for any vertices i, j, the morphism space from i to j is the vector space whose basis consists of all paths from i to j. Composition is induced by composition of paths and the unit morphisms are the lazy paths. If Q is finite, then we define the *path algebra* to be the matrix algebra

$$kQ = \bigoplus_{i,j \in Q_0} \mathsf{Hom}(i,j)$$

where multiplication is matrix multiplication. Equivalently, the path algebra has as a basis all paths and its product is given by concatenating composable paths and equating the product of non composable paths to zero. The path algebra has the sum of the lazy paths as its unit element

$$1 = \sum_{i \in Q_0} e_i.$$

The idempotent e_i yields the *projective right module*

$$P_i = e_i kQ.$$

The modules P_i generate the *category of k-finite-dimensional right modules* $\mathsf{mod}\, kQ$. Each arrow α from i to j yields a map $P_i \to P_j$ given by left multiplication by α. (If we were to consider - heaven forbid - left modules, the analogous map would be given by right multiplication by α and it would go in the direction opposite to that of α. Whence, our preference for right modules is justified.)

Notice that we have an equivalence of categories

$$\mathsf{rep}_k(Q^{op}) \to \mathsf{mod}\, kQ$$

sending a representation V of the opposite quiver Q^{op} to the sum

$$\bigoplus_{i \in Q_0} V_i$$

endowed with the natural right action of the path algebra. Conversely, a kQ-module M gives rise to the representation V with $V_i = Me_i$, for each vertex i of Q, and V_α given by right multiplication by α, for each arrow α of Q. The category $\mathsf{mod}\, kQ$ is *abelian*, *i.e.* , it is additive, has kernels and cokernels and for each morphism f, the cokernel of its kernel is canonically isomorphic to the kernel of its cokernel.

The category $\mathsf{mod}\, kQ$ is *hereditary*. Recall from [20] that this means that submodules of projective modules are projective; equivalently, that all extension groups in degrees $i \geq 2$ vanish:

$$\mathsf{Ext}^i_{kQ}(L, M) = 0$$

for all L and M; equivalently, that kQ is of global dimension ≤ 1. Thus, in the spirit of noncommutative algebraic geometry approached via abelian categories, we should think of $\mathsf{mod}\, kQ$ as a 'non commutative curve'.

We define $\mathcal{D}_Q$ to be the *bounded derived category* $\mathcal{D}^b(\mathsf{mod}\, kQ)$ of the abelian category $\mathsf{mod}\, kQ$. Thus, the objects of $\mathcal{D}_Q$ are the bounded complexes of (right) kQ-modules

$$\ldots \to 0 \to \ldots \to M^p \xrightarrow{d^p} M^{p+1} \to \ldots \to 0 \to \ldots.$$

Its morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms. We refer to [70, 107] for in depth treatments of the fundamentals of this construction. Below, we will give a complete and elementary description of the category $\mathcal{D}_Q$ if Q is a Dynkin quiver. We have the following general facts. The functor

$$\mathsf{mod}\, kQ \to \mathcal{D}_Q$$

taking a module M to the complex concentrated in degree 0,

$$\ldots \to 0 \to M \to 0 \to \ldots,$$

is a fully faithful embedding. From now on, we will identify modules with complexes concentrated in degree 0. If L and M are two modules, then we have a canonical isomorphism

$$\mathsf{Ext}^i_{kQ}(L, M) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}_Q}(L, M[i]),$$

for all $i \in \mathbb{Z}$, where $M[i]$ denotes the complex M shifted by i degrees to the left, $M[i]^p = M^{p+i}$, $p \in \mathbb{Z}$, and endowed with the differential $d_{M[i]} = (-1)^i d_M$. The category $\mathcal{D}_Q$ has all finite direct sums (and they are given by direct sums of complexes) and Theorem 6.1 holds. Moreover, each object is isomorphic to a direct sum of shifted copies of modules (this holds more generally in the derived category of any hereditary abelian category; for example, the derived category of coherent sheaves on an algebraic curve). The category $\mathcal{D}_Q$ is abelian if and only if the quiver Q does not have any arrows. However, it is always *triangulated*. This means that it is k-linear (it is additive, and the morphism sets

are endowed with k-vector space structures so that the composition is bilinear) and endowed with the following extra structure:

(a) a *suspension (or shift) functor* $\Sigma : \mathcal{D}_Q \to \mathcal{D}_Q$, namely the functor taking a complex M to $M[1]$;

(b) a class of *triangles* (sometimes called 'distinguished triangles'), namely the sequences

$$L \to M \to N \to \Sigma L$$

which are 'induced' by short exact sequences of complexes. We sometimes abbreviate the notation for such a triangle to (L, M, N).

The class of triangles satisfies certain axioms; *e.g.* , *cf.* [107]. The most important consequence of these axioms is that the triangles induce long exact sequences in the functors $\mathsf{Hom}(X, ?)$ and $\mathsf{Hom}(?, X)$, *i.e.* for each object X of $\mathcal{D}_Q$, the sequences

$$\ldots (X, \Sigma^{-1}N) \to (X, L) \to (X, M) \to (X, N) \to (X, \Sigma L) \to \ldots$$

and

$$\ldots (\Sigma^{-1}N, X) \leftarrow (L, X) \leftarrow (M, X) \leftarrow (N, X) \leftarrow (\Sigma L, X) \leftarrow \ldots$$

are exact, where we abbreviate $\mathsf{Hom}(U, V)$ to (U, V). For the set of morphisms from an object U to an object V in a category $\mathcal{C}$, we will also use the notation $\mathcal{C}(U, V)$.

6.4. **Presentation of the derived category of a Dynkin quiver.** From now on, we assume that Q is a Dynkin quiver. Let $\mathbb{Z}Q$ be its *repetition* [97]: the vertices of $\mathbb{Z}Q$ are the pairs (p, i), where p is an integer and i a vertex of Q and the arrows of $\mathbb{Z}Q$ are obtained as follows: each arrow $\alpha : i \to j$ of Q yields the arrows

$$(p, \alpha) : (p, i) \to (p, j)\ ,\ p \in \mathbb{Z}\ ,$$

and the arrows

$$\sigma(p, \alpha) : (p - 1, j) \to (p, i)\ ,\ p \in \mathbb{Z}.$$

We extend σ to a map (sometimes called the *polarization*) defined on all arrows of $\mathbb{Z}Q$ by defining

$$\sigma(\sigma(p, \alpha)) = (p - 1, \alpha).$$

Notice that σ is only defined on the arrows, not on the vertices of $\mathbb{Z}Q$. We endow $\mathbb{Z}Q$ with the map σ and with the automorphism $\tau : \mathbb{Z}Q \to \mathbb{Z}Q$ taking (p, i) to $(p - 1, i)$ and (p, α) to $(p - 1, \alpha)$, for all vertices i of Q, all arrows α of Q and all integers p.

For a vertex v of $\mathbb{Z}Q$, the *mesh ending at* v is the full subquiver

(6.1)
$$\begin{tikzcd}
 & u_1 \arrow[dr, "\alpha"] & \\
 & u_2 \arrow[dr] & \\
\tau v \arrow[uur, "\sigma(\alpha)"] \arrow[ur] \arrow[dr] & \vdots & v \\
 & u_s \arrow[ur] &
\end{tikzcd}$$

formed by v, $\tau(v)$ and all sources u of arrows $\alpha : u \to v$ of $\mathbb{Z}Q$ ending in v. We define the *mesh ideal* $\mathcal{M}$ to be the (two-sided) ideal of the path category of $\mathbb{Z}Q$ (cf. Section 6.3) which is generated by all *mesh relators*

$$r_v = \sum_{\text{arrows } \alpha : u \to v} \alpha\sigma(\alpha)\,,$$

where v runs through the vertices of $\mathbb{Z}Q$. Thus, for two vertices u and w of $\mathbb{Z}Q$, the space $\mathcal{M}(u, w)$ is formed by all morphisms of the path category which are linear combinations of compositions of the form pr_vq, where v is a vertex of $\mathbb{Z}Q$, p is a path from v to w and q is a path from u to τv. The *mesh category* is the quotient of the path category of $\mathbb{Z}Q$ by the mesh ideal. Thus, its objects are the vertices of $\mathbb{Z}Q$ and, for two vertices u and w, the space of morphisms in the mesh category is the quotient of the space of morphisms in the path category by its subspace $\mathcal{M}(u, w)$.

Theorem 6.6. *Happel* [58].

(a) *There is a canonical bijection $v \mapsto M_v$ from the set of vertices of $\mathbb{Z}Q$ to the set of isomorphism classes of indecomposables of $\mathcal{D}_Q$ which takes the vertex $(1, i)$ to the indecomposable projective P_i.*

(b) *Let $\mathsf{ind}\,\mathcal{D}_Q$ be the full subcategory of indecomposables of $\mathcal{D}_Q$. The bijection of (a) lifts to an equivalence of categories from the mesh category of $\mathbb{Z}Q$ to the category $\mathsf{ind}\,\mathcal{D}_Q$.*

In Figure 2, we see the repetition for $Q = \vec{A}_5$ and the map taking its vertices to the indecomposable objects of the derived category. The vertices marked by $\bullet$ belonging to the left triangle are mapped to indecomposable modules. The vertex $(1, i)$ corresponds to the indecomposable projective P_i. The arrow $(1, i) \to (1, i + 1)$, $1 \le i \le 4$, is mapped to the left multiplication by the arrow $i \to i + 1$. The functor of Theorem 6.6, part (b), takes a mesh (6.1) to a triangle

(6.2)
$$M_{\tau v} \longrightarrow \bigoplus_{i=1}^{s} M_{u_i} \longrightarrow M_v \longrightarrow \Sigma M_{\tau v}$$

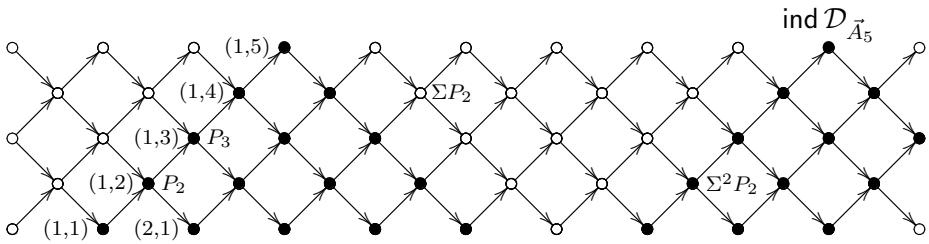


FIGURE 2. The repetition of type A_n

called an *Auslander-Reiten triangle* or *almost split triangle*; *cf.* [59]. If M_v and $M_{\tau v}$ are modules, then so is the middle term and the triangle comes from an exact sequence of modules

$$0 \longrightarrow M_{\tau v} \longrightarrow \bigoplus_{i=1}^{s} M_{u_i} \longrightarrow M_v \longrightarrow 0$$

called an *Auslander-Reiten sequence* or *almost split sequence*; *cf.* [4]. These almost split triangles (respectively sequences) can be characterized intrinsically in $\mathcal{D}_Q$ (respectively $\mathsf{mod}\, kQ$).

Recall that the Grothendieck group $K_0(\mathcal{T})$ of a triangulated category is the quotient of the free abelian group on the isomorphism classes $[X]$ of objects X of $\mathcal{T}$ by the subgroup generated by all elements

$$[X] - [Y] + [Z]$$

arising from triangles (X, Y, Z) of $\mathcal{T}$. In the case of $\mathcal{D}_Q$, the natural map

$$K_0(\mathsf{mod}\, kQ) \to K_0(\mathcal{D}_Q)$$

is an isomorphism (its inverse sends a complex to the alternating sum of the classes of its homologies). Since $K_0(\mathsf{mod}\, kQ)$ is free on the classes $[S_i]$ associated with the simple modules, the same holds for $K_0(\mathcal{D}_Q)$ so that its elements are given by n-tuples of integers. We write $\underline{\dim}\, M$ for the image in $K_0(\mathcal{D}_Q)$ of an object M of $\mathcal{D}_Q$ and call $\underline{\dim}\, M$ the *dimension vector of* M. Then, each triangle (6.2) yields an equality

$$\underline{\dim}\, M_v = \sum_{i=1}^{s} \underline{\dim}\, M_{u_i} - \underline{\dim}\, M_{\tau v}.$$

Using these equalities, we can easily determine $\underline{\dim}\, M$ for each indecomposable M starting from the known dimension vectors $\underline{\dim}\, P_i$, $1 \leq i \leq n$. In the above example, we find the dimension vectors listed in Figure 3.

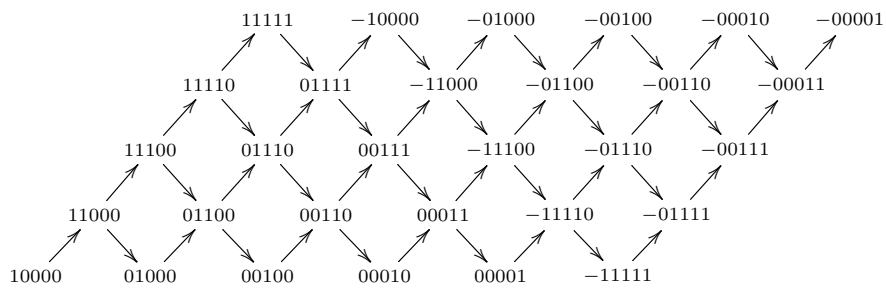


FIGURE 3. Some dimension vectors of indecomposables in $\mathcal{D}_{\vec{A}_5}$

Thanks to the theorem, the automorphism τ of the repetition yields a k-linear automorphism, still denoted by τ, of the derived category $\mathcal{D}_Q$. This automorphism has several intrinsic descriptions:

(1) As shown in [45], it is the right derived functor of the left exact Coxeter functor $\mathsf{rep}(Q^{op}) \to \mathsf{rep}(Q^{op})$ introduced by Bernstein-Gelfand-Ponomarev [12] in their proof of Gabriel's theorem. If we identify $K_0(\mathcal{D}_Q)$ with the root lattice via Gabriel's theorem, then the automorphism induced by τ^{-1} equals the the Coxeter transformation c. As shown by Gabriel [45], the identity $c^h = \mathbf{1}$, where h is Coxeter number, lifts to an isomorphism of functors

$$\tau^{-h} \xrightarrow{\sim} \Sigma^2. \tag{6.3}$$

(2) It can be expressed in terms of the Serre functor of $\mathcal{D}_Q$: recall that for a k-linear triangulated category $\mathcal{T}$ with finite-dimensional morphism spaces, a Serre functor is an autoequivalence $S : \mathcal{T} \to \mathcal{T}$ such that the Serre duality formula holds. We have bifunctorial isomorphisms

$$D\,\mathsf{Hom}(X, Y) \xrightarrow{\sim} \mathsf{Hom}(Y, SX)\ ,\ X, Y \in \mathcal{T}\ ,$$

where D is the duality $\mathsf{Hom}_k(?, k)$ over the ground field. Notice that this determines the functor S uniquely up to isomorphism. In the case of $\mathcal{D}_Q = \mathcal{D}^b(\mathsf{mod}\, kQ)$, it is not hard to prove that a Serre functor exists; it is given by the left derived functor of the tensor product by the bimodule $D(kQ)$. Now, the autoequivalence τ, the suspension functor Σ and the Serre functor S are linked by the fundamental isomorphism

$$\tau\Sigma \xrightarrow{\sim} S. \tag{6.4}$$

6.5. **Caldero-Chapoton's proof.** The above description of the derived category yields in particular a description of the module category, which is a full subcategory of the derived category. This description was used by Caldero-Chapoton [15] to prove their formula. Let us sketch the main steps in their proof. Recall that we have defined a surjective map $v \mapsto X_v$ from the set of vertices of the repetition to the set of cluster variables such that

(a) we have $X_{(0,i)} = x_i$ for $1 \leq i \leq n$ and
(b) we have
$$X_{\tau v} X_v = 1 + \prod_{\text{arrows } w \to v} X_w$$
for all vertices v of the repetition.

We wish to show that we have
$$X_v = CC(M_v)$$
for all vertices v such that M_v is an indecomposable module. This is done by induction on the distance of v from the vertices $(1, i)$ in the quiver $\mathbb{Z}Q$. More precisely, one shows the followings.

(a) We have $CC(P_i) = X_{(1,i)}$ for each indecomposable projective P_i. Here, we use the fact that submodules of projectives are projective in order to explicitly compute $CC(P_i)$.
(b1) For each split exact sequence
$$0 \to L \to E \to M \to 0\,,$$
we have
$$CC(L)CC(M) = CC(E).$$
Thus, if $E = E_1 \oplus \ldots \oplus E_s$ is a decomposition into indecomposables, then
$$CC(E) = \prod_{i=1}^{s} CC(E_i).$$
(b2) If
$$0 \to L \to E \to M \to 0$$
is an almost split exact sequence, then we have
$$CC(E) + 1 = CC(L)CC(M).$$

It is now clear how to prove the equality $X_v = CC(M_v)$, by induction, by proceeding from the projective indecomposables to the right.

6.6. **The cluster category.** Recall that Q denotes a Dynkin quiver. The *cluster category*

$$\mathcal{C}_Q = \mathcal{D}_Q/(\tau^{-1}\Sigma)^{\mathbb{Z}} = \mathcal{D}_Q/(S^{-1}\Sigma^2)^{\mathbb{Z}}$$

is the orbit category of the derived category under the action of the cyclic group generated by the autoequivalence $\tau^{-1}\Sigma = S^{-1}\Sigma^2$ (notice that Σ commutes with S and τ since both are triangle functors and that we have $\tau\Sigma = S$ by (6.4)). This means that the objects of $\mathcal{C}_Q$ are the same as those of the derived category $\mathcal{D}_Q$ and that for two objects X and Y, the morphism space from X to Y in $\mathcal{C}_Q$ is

$$\mathcal{C}_Q(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathcal{D}_Q(X,(S^{-1}\Sigma^2)^p Y).$$

Morphisms are composed in the natural way. This definition is due to Buan-Marsh-Reineke-Reiten-Todorov [7], who tried to obtain a better understanding of the 'decorated quiver representations' introduced by Reineke-Marsh-Zelevinsky [87]. For quivers of type A, an equivalent category was defined independently by Caldero-Chapoton-Schiffler [16] using an entirely different description. Clearly, the category $\mathcal{C}_Q$ is k-linear. It is not hard to check that its morphism spaces are finite-dimensional.

One can show [77] that $\mathcal{C}_Q$ admits a canonical structure of triangulated category such that the projection functor $\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor (in general, orbit categories of triangulated categories are no longer triangulated). The Serre functor S of $\mathcal{D}_Q$ clearly induces a Serre functor in $\mathcal{C}_Q$, which we still denote by S. Now, by the definition of $\mathcal{C}_Q$ (and its triangulated structure), we have an isomorphism of triangle functors

$$S \xrightarrow{\sim} \Sigma^2.$$

This means that $\mathcal{C}_Q$ is 2-Calabi-Yau. Indeed, for an integer $d \in \mathbb{Z}$, a triangulated category $\mathcal{T}$ with finite-dimensional morphism spaces is d-Calabi-Yau if it admits a Serre functor isomorphic as a triangle functor to the dth power of its suspension functor.

6.7. **From cluster categories to cluster algebras.** We keep the notations and hypotheses of the previous section. If R is a quiver and G is a group of automorphisms of R, then the *orbit quiver* R/G has as vertices the set of orbits R_0/G of G on the set of vertices R_0 and the set of arrows between the orbit of a vertex u and that of a vertex v is the set of orbits of G in the disjoint union of the sets of arrows from gu

to hv, where g and h range through G. We have a canonical morphism $\pi : R \to R/G$ satisfying $\pi \circ g = \pi$ for all g in G and universal for this property.

The suspension functor Σ and the Serre functor S of the derived category induce automorphisms of the repetition $\mathbb{Z}Q$ which we still denote by Σ and S, respectively. The orbit quiver $\mathbb{Z}Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$ inherits the automorphism τ and the map σ (defined on arrows only) and thus has a well-defined mesh category. Recall that we write $\mathsf{Ext}^1(X, Y)$ for $\mathsf{Hom}(X, \Sigma Y)$ in any triangulated category.

Theorem 6.7. [7, 8].

(a) *The decomposition theorem holds for the cluster category and the mesh category of $\mathbb{Z}Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$ is canonically equivalent to the full subcategory $\mathsf{ind}\,\mathcal{C}_Q$ of the indecomposables of $\mathcal{C}_Q$. Thus, we have an induced bijection $L \mapsto X_L$ from the set of isomorphism classes of indecomposables of $\mathcal{C}_Q$ to the set of all cluster variables of $\mathcal{A}_Q$, which takes the shifted projective ΣP_i to the initial variable x_i, $1 \leq i \leq n$.*

(b) *Under this bijection, the clusters correspond to the* cluster-tilting sets, i.e. *, the sets of pairwise non isomorphic indecomposables $T_1, \ldots, T_n$ such that we have*

$$\mathsf{Ext}^1(T_i, T_j) = 0$$

for all i, j.

c) *If $T_1, \ldots, T_n$ is cluster-tilting, then the quiver (*cf. *below) of the endomorphism algebra of the sum $T = \bigoplus_{i=1}^n T_i$ does not have loops nor 2-cycles and the associated antisymmetric matrix is the exchange matrix of the unique seed containing the cluster $X_{T_1}, \ldots, X_{T_n}$.*

In part (b), the condition implies in particular that $\mathsf{Ext}^1(T_i, T_i)$ vanishes. However, for a Dynkin quiver Q, we have $\mathsf{Ext}^1(L, L) = 0$, for each indecomposable L of $\mathcal{C}_Q$. A *cluster-tilting object of $\mathcal{C}_Q$* is the direct sum of the objects $T_1, \ldots, T_n$ of a cluster-tilting set. Since these are pairwise non isomorphic indecomposables, the datum of T is equivalent to that of the T_i. A *cluster-tilted algebra of type Q* is the endomorphism algebra of a cluster-tilting object of $\mathcal{C}_Q$. In part (c), the most subtle point is that the quiver does not have loops or 2-cycles [8]. Let us recall what one means by the *quiver of a finite-dimensional algebra over an algebraically closed field.*

Proposition-Definition 6.8 (Gabriel)**.** *Let B be a finite-dimensional algebra over the algebraically closed ground field k.*

(a) *There exists a quiver Q_B , unique up to isomorphism, such that B is Morita equivalent to the algebra kQ_B/I, where I is an ideal of kQ_B contained in the square of the ideal generated by the arrows of Q_B.*

(b) *The ideal I is not unique, in general, but we have $I = 0$ if and only if B is hereditary.*

(c) *There is a bijection $i \mapsto S_i$ between the vertices of Q_B and the isomorphism classes of simple B-modules. The number of arrows from a vertex i to a vertex j equals the dimension of $\mathsf{Ext}^1_B(S_j, S_i)$.*

In our case, the algebra B is the endomorphism algebra of the sum T of the cluster-tilting set $T_1, \ldots, T_n$ in $\mathcal{C}_Q$. In this case, the Morita equivalence of (a) even becomes an isomorphism (because the T_i are pairwise non isomorphic). For a suitable choice of this isomorphism, the idempotent e_i associated with the vertex i is sent to the identity of T_i and the images of the arrows from i to j yield a basis of the space of *irreducible morphisms*

$$\mathsf{irr}_T(T_i, T_j) = \mathsf{rad}_T(T_i, T_j)/\,\mathsf{rad}^2_T(T_i, T_j)\ ,$$

where $\mathsf{rad}_T(T_i, T_j)$ denotes the vector space of non isomorphisms from T_i to T_j (thanks to the locality of the endomorphism rings, this set is indeed closed under addition) and rad^2_T is the subspace of non isomorphisms admitting a non trivial factorization:

$$\mathsf{rad}^2_T(T_i, T_j) = \sum_{r=1}^{n} \mathsf{rad}_T(T_r, T_j)\,\mathsf{rad}_T(T_i, T_r).$$

As an illustration of Theorem 6.7, we consider the cluster-tilting set $T_1, \ldots, T_5$ in $\mathcal{C}_{\vec{A}_5}$, as depicted in Figure 4. Here, the vertices labeled 0, 1, …, 4 have to be identified with the vertices labeled 20, 21, …, 24 (in this order) to obtain the orbit quiver $\mathbb{Z}Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$. In the orbit category $\mathcal{C}_Q$, we have $\tau \xrightarrow{\sim} \Sigma$ so that ΣT_1 is the indecomposable associated to vertex 0, for example. Using this and the description of the morphisms in the mesh category, it is easy to check that we do have

$$\mathsf{Ext}^1(T_i, T_j) = 0$$

for all i, j. It is also easy to determine the spaces of morphisms

$$\mathsf{Hom}_{\mathcal{C}_Q}(T_i, T_j)$$

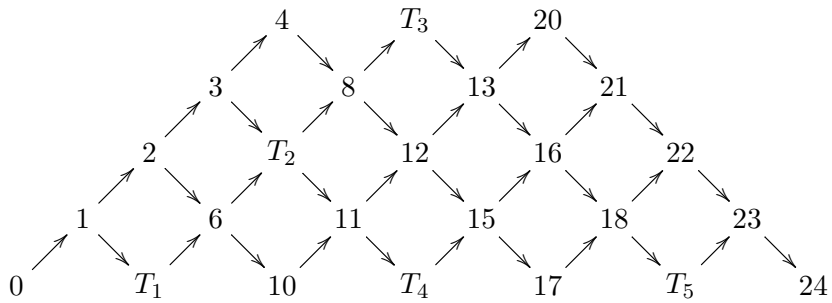


FIGURE 4. A cluster-tilting set in A_5

and the compositions of morphisms. Determining these is equivalent to determining the endomorphism algebra

$$\mathsf{End}(T) = \mathsf{Hom}(T,T) = \bigoplus_{i,j} \mathsf{Hom}(T_i, T_j).$$

This algebra is easily seen to be *isomorphic* to the algebra given by the following quiver Q':

$$\begin{tikzcd}
5 \arrow[dr, "\beta_1"'] & & 3 \arrow[ll, "\gamma_1"'] \\
& 2 \arrow[ur, "\alpha_1"'] \arrow[dr, "\gamma_2"] & \\
1 \arrow[ur, "\alpha_2"] & & 4 \arrow[ll, "\beta_2"]
\end{tikzcd}$$

with the relatiors

$$\alpha_1\beta_1\ ,\ \beta_1\gamma_1\ ,\ \gamma_1\alpha_1\ ,\ \alpha_2\beta_2\ ,\ \beta_2\gamma_2\ ,\ \gamma_2\alpha_2.$$

Thus, the quiver of $\mathsf{End}(T)$ is Q'. It encodes the exchange matrix of the associated cluster

$$X_{T_1} = \frac{1+x_2}{x_1}$$

$$X_{T_2} = \frac{x_1x_2 + x_1x_4 + x_3x_4 + x_2x_3x_4}{x_1x_2x_3}$$

$$X_{T_3} = \frac{x_1x_2x_3 + x_1x_2x_3x_4 + x_1x_2x_5 + x_1x_4x_5 + x_3x_4x_5 + x_2x_3x_4x_5}{x_1x_2x_3x_4x_5}$$

$$X_{T_4} = \frac{x_2+x_4}{x_3}$$

$$X_{T_5} = \frac{1 + x_4}{x_5} ,$$

where the variables $x_1, \ldots, x_5$ form the cluster corresponding to the vertices 0, 1, 2, 3, 4 of Figure 4.

6.8. **A K-theoretic interpretation of the exchange matrix.** Keep the notations and hypotheses of the preceding section. Let $T_1, \ldots, T_n$ be a cluster-tilting set, T the sum of the T_i and B its endomorphism algebra. For two finite-dimensional right B-modules L and M put

$$\langle L, M\rangle_a = \dim \mathsf{Hom}(L, M) - \dim \mathsf{Ext}^1(L, M) - \dim \mathsf{Hom}(M, L) + \dim \mathsf{Ext}^1(M, L).$$

This is the antisymmetrization of a truncated Euler form. A priori it is defined on the split Grothendieck group of the category $\mathsf{mod}\, B$ (*i.e.* the quotient of the free abelian group on the isomorphism classes divided by the subgroup generated by all relations obtained from direct sums in $\mathsf{mod}\, B$).

Proposition 6.9. *Palu* [92]. *The form $\langle , \rangle_a$ descends to an antisymmetric form on $K_0(\mathsf{mod}\, B)$. Its matrix in the basis of the simples is the exchange matrix associated with the cluster corresponding to $T_1, \ldots, T_n$.*

6.9. **Mutation of cluster-tilting sets.** Let us recall two axioms of triangulated categories:

TR1: For each morphism $u : X \to Y$, there exists a triangle

$$X \xrightarrow{u} Y \to Z \to \Sigma X.$$

TR2: A sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is a triangle if and only if the sequence

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-u} \Sigma Y$$

is a triangle.

One can show that in TR1, the triangle is unique up to (non unique) isomorphism. In particular, up to isomorphism, the object Z is uniquely determined by u. Notice the sign in TR2. It follows from TR1 and TR2 that a given morphism also occurs as the second (respectively third) morphism in a triangle.

Now, with the notations and hypotheses of the preceding section, suppose that $T_1, \ldots, T_n$ is a cluster-tilting set and Q' is the quiver of

the endomorphism algebra B of the sum of the T_i. As explained after Proposition-Definition 6.8, we have a surjective algebra morphism

$$kQ' \to \bigoplus_{i,j} \mathsf{Hom}(T_i, T_j)$$

which takes the idempotent e_i to the identity of T_i and the arrows $i \to j$ to irreducible morphisms $T_i \to T_j$, for all vertices i, j of Q' (*cf.* the above example, computation of B and $Q' = Q_B$).

Now, let k be a vertex of Q' (the mutating vertex). We choose triangles

$$T_k \xrightarrow{u} \bigoplus_{\substack{\text{arrows}\\ k\to i}} T_i \to T_k^* \to \Sigma T_k$$

and

$$^*T_k \to \bigoplus_{\substack{\text{arrows}\\ j\to k}} T_j \xrightarrow{v} T_k \to \Sigma {}^*T_k$$

where the component of u (respectively v) corresponding to an arrow $\alpha : k \to i$ (respectively $j \to k$) is the corresponding morphism $T_k \to T_i$ (respectively $T_j \to T_k$). These triangles are unique up to isomorphism and called the *exchange triangles* associated with k and $T_1, \ldots, T_n$.

Theorem 6.10. [7].

(a) *The objects T_k^* and $^*T_k$ are isomorphic.*
(b) *The set obtained from $T_1, \ldots, T_n$ by replacing T_k with T_k^* is cluster-tilting and its associated cluster is the mutation at k of the cluster associated with $T_1, \ldots, T_n$.*
(c) *Two indecomposables L and M appear as the the pair (T_k, T_k^*) associated with an exchange if and only if the space $\mathsf{Ext}^1(L, M)$ is one-dimensional. In this case, the exchange triangles are the unique (up to isomorphism) non split triangles*

$$L \to E \to M \to \Sigma L \text{ and } M \to E' \to L \to \Sigma M.$$

Let us extend the map $L \mapsto X_L$ from indecomposable to decomposable objects of $\mathcal{C}_Q$ by requiring that we have

$$X_N = X_{N_1} X_{N_2}$$

whenever $N = N_1 \oplus N_2$ (this is compatible with the muliplicativity of the Caldero-Chapoton map). We know that if $u_1, \ldots, u_n$ is a cluster

and $B = (b_{ij})$ is the associated exchange matrix, then the mutation at k yields the variable u'_k such that

$$u_k u'_k = \prod_{\substack{\text{arrows} \\ k \to i}} u_i + \prod_{\substack{\text{arrows} \\ j \to k}} u_j.$$

By combining this with the exchange triangles, we see that in the situation of (c), we have

$$X_L X_M = X_E + X_{E'}.$$

We would like to generalize this identity to the case where the space $\mathsf{Ext}^1(L, M)$ is of higher dimension. For three objects L, M and N of $\mathcal{C}_Q$, let $Ext^1(L, M)_N$ be the subset of $\mathsf{Ext}^1(L, M)$ formed by those morphisms $\varepsilon : L \to \Sigma M$ such that in the triangle

$$M \to E \to L \xrightarrow{\varepsilon} \Sigma M\ ,$$

the object E is isomorphic to N (we do not fix an isomorphism). Notice that this subset is a cone (*i.e.* , stable under multiplication by non zero scalars) in the vector space $\mathsf{Ext}^1(L, M)$.

Proposition 6.11. [18]. *The subset $\mathsf{Ext}^1(L, M)_N$ is constructible in $\mathsf{Ext}^1(L, M)$. In particular, it is a union of algebraic subvarieties. It is empty for all but finitely isomorphism classes of objects N.*

If k is the field of complex numbers, then we denote by χ the Euler characteristic with respect to singular cohomology with coefficients in a field. If k is an arbitrary algebraically closed field, then we denote by χ the Euler characteristic with respect to étale cohomology with proper support.

Theorem 6.12. [18]. *Suppose that L and M are objects of $\mathcal{C}_Q$ such that $\mathsf{Ext}^1(L, M) \neq 0$. Then, we have*

$$X_L X_M = \sum_N \frac{\chi(\mathbb{P}\,\mathsf{Ext}^1(L, M)_N) + \chi(\mathbb{P}\,\mathsf{Ext}^1(M, L)_N)}{\chi(\mathbb{P}\,\mathsf{Ext}^1(L, M))}\, X_N$$

where the sum is taken over all isomorphism classes of objects N of $\mathcal{C}_Q$.

Notice that in the theorem, the objects L and M may be decomposable so that X_L and X_M will not be cluster variables, in general, and the X_N do not form a linearly independent set in the cluster algebra. Thus, the formula should be considered as a relation rather than as an alternative definition for the multiplication of the cluster algebra. Notice that it nevertheless bears a close resemblance to the product formula

in a dual Hall algebra: for two objects L and M in a finitary abelian category of finite global dimension, we have

$$[L] * [M] = \sum_{[N]} \frac{|\operatorname{Ext}^1(L,M)_N|}{|\operatorname{Ext}^1(L,M)|}\,[N]$$

where the brackets denote isomorphism classes and the vertical bars the cardinalities of the underlying sets; *cf.* Proposition 1.5 of [101].

7. Categorification via cluster categories: the acyclic case

7.1. **Categorification.** Let Q be a connected finite quiver without oriented cycles with vertex set $\{1,\dots,n\}$. Let k be an algebraically closed field. We have seen in Section 6.3 how to define the bounded derived category $\mathcal{D}_Q$. We still have a fully faithful functor from the mesh category of $\mathbb{Z}Q$ to the category of indecomposables of $\mathcal{D}_Q$, but this functor is very far from being essentially surjective. In fact, its image does not even contain the injective indecomposable kQ-modules. The methods of the preceding section therefore do not generalize, but most of the results continue to hold. The derived category $\mathcal{D}_Q$ still has a Serre functor (the total left derived functor of the tensor product functor $? \otimes_B D(kQ)$). We can form the cluster category

$$\mathcal{C}_Q = \mathcal{D}_Q/(S^{-1}\Sigma^2)^{\mathbb{Z}}$$

as before and it is still a triangulated category in a canonical way such that the projection $\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor [77]. Moreover, the decomposition Theorem 6.1 holds for $\mathcal{C}_Q$ and each object L of $\mathcal{C}_Q$ decomposes into a direct sum

$$L = \pi(M) \oplus \bigoplus_{i=1}^{n} \pi(\Sigma P_i)^{m_i}$$

for some module M and certain multiplicities m_i, $1 \le i \le n$; *cf.* [7]. We put

$$X_L = CC(M) \prod_{i=1}^{n} x_i^{m_i}$$

where $CC(M)$ is defined as in Section 6.2. Notice that, in general, X_L can only be expected to be an element of the fraction field $\mathbb{Q}(x_1,\dots,x_n)$, not of the cluster algebra $\mathcal{A}_Q$ inside this field.

Theorem 7.1. *Let Q be a finite quiver without oriented cycles with vertex set $\{1,\dots,n\}$.*

(a) *The map* $L \mapsto X_L$ *induces a bijection from the set of isomorphism classes of rigid indecomposables of the cluster category* $\mathcal{C}_Q$ *onto the set of cluster variables of the cluster algebra* $\mathcal{A}_Q$.

(b) *Under this bijection, the clusters correspond exactly to the cluster-tilting sets, i.e., the sets* $T_1, \ldots, T_n$ *of rigid indecomposables such that*

$$Ext^1(T_i, T_j) = 0,$$

for all i, j.

(c) *For a cluster-tilting set* $T_1, \ldots, T_n$, *the quiver of the endomorphism algebra of the sum of the* T_i *does not have loops nor* 2*-cycles and encodes the exchange matrix of the seed containing the corresponding cluster.*

(d) *If* L *and* M *are rigid indecomposables such that the space* $\mathsf{Ext}^1(L, M)$ *is one-dimensional, then we have the relation*

$$X_L X_M = X_B + X_{B'} \tag{7.1}$$

where B *and* B' *are the middle terms of 'the' non split triangles*

$$L \longrightarrow B \longrightarrow M \longrightarrow \Sigma L \text{ and } M \longrightarrow B' \longrightarrow L \longrightarrow \Sigma M.$$

Concerning part (c), let us point out that the uniqueness of a seed containing a given cluster is proved in full generality in [55]. The relation in part (d) generalizes the exchange relations as they appear in the definition of a cluster algebra. Parts (a), (b) and (d) of the theorem are proved in [17] and part (c) in [8]. The proofs build on the work by many authors notably Buan-Marsh-Reiten-Todorov [9], Buan-Marsh-Reiten [8], Buan-Marsh-Reineke-Reiten-Todorov [7], Marsh-Reineke-Zelevinsky [87], and especially on Caldero-Chapoton's explicit formula for X_L proved in [15] for orientations of simply laced Dynkin diagrams. Another crucial ingredient of the proof is the Calabi-Yau property of the cluster category. An alternative proof of part (c) was given by Hubery [62], for quivers whose underlying graph is an extended simply laced Dynkin diagram.

We describe the main steps of the proof of (a). The mutation of cluster-tilting sets is defined using the construction of Section 6.9.

(1) If T is a cluster-tilting object, then the quiver Q_T of its endomorphism algebra does not have loops or 2-cycles. If T' is obtained from T by mutation at the summand T_1, then the quiver $Q_{T'}$ of the endomorphism algebra of T' is the mutation at the vertex 1 of the quiver Q_T; *cf.* [8].

(2) Each rigid indecomposable is contained in a cluster-tilting set. Any two cluster-tilting sets are linked by a finite sequence of mutations. This is deduced in [7] from the work of Happel-Unger [60].

(3) If (T_1, T_1^*) is an exchange pair and

$$T_1^* \to E \to T_1 \to \Sigma T_1^* \text{ and } T_1 \to E' \to T_1^* \to \Sigma T_1$$

are the exchange triangles, then we have

$$X_{T_1} X_{T_1^*} = X_E + X_{E'}.$$

This is shown in [17].

It follows from (1)-(3) that the map $L \to X_L$ does take rigid indecomposables to cluster variables and that each cluster variable is obtained in this way. It remains to be shown that a rigid indecomposable L is determined up to isomorphism by X_L. This follows from

(4) If M is a rigid indecomposable module, then the denominator of X_M is $x_1^{d_1} \dots x_n^{d_n}$; *cf.* [17].

Indeed, a rigid indecomposable module M is determined, up to isomorphism, by its dimension vector.

We sum up the relations between the cluster algebra and the cluster category in the following table:

cluster algebra	cluster category
multiplication	direct sum
addition	?
cluster variables	rigid indecomposables
clusters	cluster-tilting sets
mutation	mutation
exchange relation	exchange triangles
$xx^* = m + m'$	$T_k \to M \to T_k^* \to \Sigma T_k$
	$T_k^* \to M' \to T_k \to \Sigma T_k^*$

7.2. **Two applications.** Theorem 7.1 does shed new light on cluster algebras. In particular, thanks to the theorem, Caldero and Reineke [19] have made significant progress towards the following.

Theorem 7.2. *Qin* [94], *Nakajima* [91]. *Suppose that Q does not have oriented cycles. Then, all cluster variables of $\mathcal{A}_Q$ belong to $\mathbb{N}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.*

This theorem is a consequence of a general conjecture of Fomin-Zelevinsky [37], which here is specialized to the case of cluster algebras

associated with acyclic quivers, for cluster expansions in the initial cluster. Notice that in [19], the above is also stated as a theorem. However, a gap in the proof was found by Nakajima [91]: the authors incorrectly identify their parameter q with Lusztig's parameter v, whereas the correct identification is $v = -\sqrt{q}$. Fomin-Zelevinsky's general positivity conjecture [37] has now been proved for quivers of type A_n and D_4 by Hernandez-Leclerc [61] and for bipartite acyclic quivers by Nakajima [91]. Both proofs rely on the method of 'monoidal categorification' developed by Leclerc and Hernandez-Leclerc.

Here are two applications to the exchange graph of the cluster algebra associated with an acyclic quiver Q.

Corollary 7.3. [17].

(a) *For any cluster variable x, the set of seeds, whose clusters contain x, form a connected subgraph of the exchange graph.*
(b) *The set of seeds, whose quiver does not have oriented cycles, form a connected subgraph (possibly empty) of the exchange graph.*

For acyclic cluster algebras, parts (a) and (b) confirm conjecture 4.14, parts (3) and (4), given by Fomin-Zelevinsky in [39]. By (b), the cluster algebra associated with a quiver without oriented cycles has a well-defined cluster-type.

7.3. **Cluster categories and singularities.** The construction of cluster categories may seem a bit artificial. Nevertheless, cluster categories do occur 'in nature'. In particular, certain triangulated categories associated with singularities are equivalent to cluster categories. We illustrate this on the following example. Let the cyclic group G of order 3 act on a three-dimensional complex vector space V by scalar multiplication with a primitive third root of unity. Let S be the completion at the origin of the coordinate algebra of V and let $R = S^G$, the fixed point algebra corresponding to the completion of the singularity at the origin of the quotient $V//G$. The algebra R is a Gorenstein ring, (*e.g.* , *cf.*) [108], and an isolated singularity of dimension 3 (*e.g.* , *cf.*) Corollary 8.2 of [67]. The category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay modules is an exact Frobenius category and its stable category $\underline{\mathrm{CM}}(R)$ is a triangulated category. By Auslander's results [5] (*cf.* Lemma 3.10 of [110]), it is 2-Calabi Yau. One can show that it is equivalent to the cluster category $\mathcal{C}_Q$ for the quiver

$$Q: \ 1 \rightrightarrows 2$$

by an equivalence which takes the cluster-tilting object $T = kQ$ to S considered as an R-module. This example can be found in [79], where it is deduced from an abstract characterization of cluster categories. A number of similar examples can be found in [14, 80]. A far-reaching link between singularities and 'generalized cluster categories' [1] is established in [2].

Acknowledgments

The author extends his warmest thanks to the IPM, and in particular to Javad Asadollahi, for the exquisite hospitality he was offered during his visits devoted to these lectures. He is grateful to an anonymous referee for his meticulous reading of the manuscript and many helpful remarks.

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