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ON THE USE OF KÜLSHAMMER TYPE INVARIANTS IN REPRESENTATION THEORY

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Abstract. Since 2005 a new powerful invariant of an algebra has emerged using the earlier work of Horváth, Héthelyi, Külshammer and Murray. The authors studied Morita invariance of a sequence of ideals of the center of a finite dimensional algebra over a field of finite characteristic. It was shown that the sequence of ideals is actually a derived invariant, and most recently a slightly modified version of it is an invariant under stable equivalences of Morita type. The invariant was used in various contexts to distinguish derived and stable equivalence classes of pairs of algebras in very subtle situations. Generalisations to non symmetric algebras and to higher Hochschild (co-)homology were given. This article surveys the results and gives some of the constructions in more details.

1. **Introduction**

Brauer studied representations of finite groups over fields of characteristic p dividing the order of the group. In 1956, he showed [12], amongst many other things, that if the field is algebraically closed, then the number of simple modules is equal to the number of conjugacy classes of elements of G of order prime to p. His method was rather general already and Külshammer used these ideas 25 years later to define very sophisticated invariants for general symmetric algebras. More

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precisely, Külshammer defined and studied in a series of four papers [30, 31, 32, 33] a sequence of ideals of the center of a symmetric algebra. In case of group rings over a finite group, Külshammer studied lower defect groups, and proved Brauer's main theorems and many other results known in modular representation theory of finite groups by these ideals and related invariants. Külshammer's approach was left untouched until Murray [38, 39] studied Külshammer's approach in connection with the question of the existence of real characters and of characters of defect 0. This was the starting point of the collaboration between Breuer, Héthelyi, Horváth, Külshammer and Murray (see [13] and [19]), where the authors study the existence of odd diagonal entries in the Cartan matrix of a group algebra. Moreover, in [19], the Morita invariance of the sequence of ideals was shown and the question of derived invariance was posed.

In [56], completely different methods were used to show that indeed at least for perfect base fields the sequence of ideals was invariant under derived equivalence. Still the assumption that the algebras were symmetric was needed.

After a lecture of the author in September 2005 in Oberwolfach, Holm became interested in the sequence of ideals and proposed several improvements and applications. First, in joint work of the author with Bessenrodt and Holm [4], using trivial extension algebras the derived invariance, and also the very definition of the Külshammer ideals was extended to not necessarily symmetric algebras. Further more, the Külshammer ideals were used in [23] to distinguish the derived equivalence class of two algebras of dihedral type and of two pairs of algebras of semidihedral type which were not seen to be not derived equivalent in the classification of Holm [20].

During the September 2005 Oberwolfach lecture Adem asked for a generalisation of the Külshammer ideal structure to higher Hochschild (co-)homology. The question is far from trivial and was solved in [57], where actually two approaches were taken, both of which were not exactly what was asked for. The first approach uses Hochschild homology instead of cohomology as in the Külshammer ideal structure, and the second approach uses the Stasheff approach to the Gerstenhaber structure to get a non linear analogue. The case of non symmetric algebras was answered in [58] again using trivial extension algebras and the Hochschild homology approach.

The derived equivalence classification of tame domestic weakly symmetric algebras was given by Bocian, et al. for domestic algebras [9, 10] and by Holm and Skowroński [21] using for the last remaining delicate questions, the Külshammer ideals. Similarly, also using Külshammer ideals in parts, Białkowski, et al. [6, 7] and Holm and Skowroński [22] gave a derived equivalence classification of tame algebras of polynomial growth up to some difficult problems concerning scalars in the relations of certain algebras, similar to the problem solved in [23].

Then, most recently Białkowski, Erdmann and Skowroński classified selfinjective algebras with the property that the third syzygy of every simple module S is again isomorphic to S. They obtain that these algebras are all certain deformations of preprojective algebras of a generalised Dynkin type. The deformations involve parameters in the relations of the algebra. As was seen in [21, 22, 23], Külshammer ideals are well suited for this kind of questions. Derived equivalence classes of one family called of type L, defined in detail in Example 3.19 and more generally in Section 6.2 below, were largely given in a joint work with Holm [24]. Here, we develop quite sophisticated methods to determine the Külshammer subspace structure at the beginning, which hold for a priori non symmetric algebras as well. We display this method, even though strictly speaking only small parts of it are really necessary in case of algebras of type L. Nevertheless, the method works in general, is potentially very useful and it seems reasonable to present it here.

During a lecture of the author in October 2007 at Beijing Normal University, the question of an invariance under stable equivalences was posed. In most recent results with Yuming Liu and Guodong Zhou, a generalisation of the Külshammer ideal theory was given for stable categories and an invariance was proved for stable equivalences of Morita type ([36] and [29]). Most interestingly, the result has strong links to the Auslander-Reiten conjecture [3, Conjecture 5, page 409], which says that a stable equivalence should preserve the number of simple non projective modules. The result [36, 29] was used in a joint work with Zhou [52] to give a classification of algebras of dihedral, of semidihedral and of quaternion type, as defined by Erdmann [16] up to stable equivalences of Morita type. Moreover, in [53], in a joint work with Zhou, we prove that the classification of weakly symmetric tame algebras of polynomial growth up to stable equivalence of Morita type coincides with the derived equivalence classification of Białkowski-Holm-Skowroński.

Here, we survey these results and give at certain points quite complete proofs for results which seem to be very useful also in further contexts. In Section 2, we trace some steps in the origins, starting from Brauer and Reynolds. Section 3 reviews properties of selfinjective algebras and develops tools to actually compute the Külshammer ideals for quite complicated algebras. These tools were developed during the past years for this purpose, but the origins are, of course, very classical. Section 4 presents the Külshammer ideals as they were developed originally by Külshammer and as they were generalised later to non-symmetric algebras. Section 5 displays the invariance of the various forms of the Külshammer ideals under Morita, derived and stable equivalences. Section 6 is devoted to various applications mentioned above including a detailed outline of the proof for the deformed preprojective algebras of type L. Section 7 gives the above mentioned two approaches to the Hochschild (co-)homology generalisations of the Külshammer invariants.

2. Historical facts and basic definitions: the origins by Brauer and Reynolds

Brauer developed in the 1950's the far reaching representation theory of groups over fields of finite characteristic. In 1956, he showed in particular the following result.

Theorem 2.1. *(Brauer [12, Statement 3B]) If K is an algebraically closed field of characteristic $p > 0$ and if G is a finite group, then the number of simple KG-modules up to isomorphism equals the number of conjugacy classes of G of elements of order prime to p.*

Of course, in the spirit of that time, Brauer did speak of irreducible characters rather than of modules, but the result translates into modern terms as it is shown above. The method of proof he used is somewhat indirect. He defines for any K-algebra A the space of commutators $[A, A]$, which is defined as the K-vector space generated as vector space by all possible expressions $ab - ba$ where $a, b \in A$.

Further, he defines

$$TA := \{a \in A \mid \exists n \in \mathbb{N} : a^{p^n} \in [A, A]\}.$$

The first lemma is a little more general than stated in [12], but with an identical proof. The generalisation comes from the fact that actually one can consider more generally

$$T_nA := \{a \in A \mid a^{p^n} \in [A, A]\},$$

for all $n \in \mathbb{N}$, a fact which is an observation due to Külshammer. We will need and study T_nA in more detail later.

Lemma 2.2. (*Brauer* [12, Statement 3A]) *Let K be a field of characteristic $p > 0$ and let A be a K-algebra. Then, T_nA is a K-subspace of A satisfying $TA = \bigcup_{n\in\mathbb{N}} T_nA$. If A is finite dimensional, and K is a splitting field for A, then the number of simple A-modules up to isomorphism equals the dimension of A/TA.*

The proof of this lemma is so simple that we may give it here in almost full detail.

Proof. (Brauer). Take $x, y \in T_nA$. Then develop $(x+y)^p$ and get a sum of all possible words in x and y with p factors, each occurring exactly once. If $1 < s < p$, then there are $n(s)$ of such words in which x occurs s times and y occurs $p - s$ times. Take $N(s)$ to be the set of these words. Then, the cyclic group of order p acts on this set by a cyclic permutation of the word: $c \cdot (a_1a_2 \dots a_{p-1}a_p) := (a_2a_3 \dots a_pa_1)$ for a generator c of the cyclic group and $a_i \in \{x, y\}$. Hence, $N(s)$ decomposes into orbits of length p, and the difference of two elements in the same orbit is clearly in $[A, A]$. Hence,

$$(x+y)^p - x^p - y^p \in [A, A].$$

Moreover,

$$\begin{aligned}(xy - yx)^p + [A, A] &= (xy)^p - (yx)^p + [A, A] \\ &= x\left((yx)^{p-1}y\right) - \left((yx)^{p-1}y\right)x + [A, A] = [A, A]\end{aligned}$$

and

$$(\lambda x)^p = \lambda^p x^p,$$

for all $x, y \in A$ and $\lambda \in K$, show that T_nA is a K-subspace of A.

Traces of commutators of matrices are 0. Therefore, if $A = Mat_n(K)$, then $[A, A] \subseteq \{M \in Mat_n(K) \mid trace(M) = 0\}$. On the other hand, using elementary matrices, one sees that the inclusion actually is an equality. The space of matrices with trace 0 is of codimension 1, and $Mat_n(K)$ has exactly one simple module up to isomorphism. We denote by $rad(A)$ the Jacobson radical of the algebra A. Hence, putting $\overline{A} := A/rad(A)$, one gets

$$\overline{A}/(T\overline{A}) = A/(TA)$$

as vector spaces since $rad(A)$ is nilpotent, and therefore $rad(A) \subseteq T(A)$. This shows the statement by Wedderburn's theorem. $\square$

Brauer's Theorem 2.1 follows from the fact that, for a group ring KG, a basis is formed by the elements of G, and any element $g \in G$ admits a unique (so-called the p-primary decomposition) $g = g_p \cdot g_{p'}$, where g_p is a p-element and $g_{p'}$ is of order prime to p commuting with g_p. Hence,

$$(g - g_{p'})^{p^n} + [KG, KG] = (g^{p^n} - g_{p'}^{p^n}) + [KG, KG] = [KG, KG]$$

for a certain large n. Then,

$$hgh^{-1} - g = hgh^{-1} - gh^{-1}h = [h, gh^{-1}] \;\; \forall g, h \in G$$

and the rest is straightforward.

Remark 2.3. *Observe that actually much more is shown:* $TA = rad(A) + [A, A]$.

Another concept due to Reynolds [45] is closely linked. Let G be a finite group, and let K be an algebraically closed field of characteristic $p > 0$. For any $g \in G$ and $h \in G$, consider the p-primary decomposition $g = g_p \cdot g_{p'}$ and $h = h_p \cdot h_{p'}$. Then, let

$$S_h := \{g \in G \mid \exists x \in G : x \cdot g_{p'} \cdot x^{-1} = h_{p'}\}$$

be the set of elements in G, whose p'-part is conjugate to the p'-part of h. Put

$$C_h := \sum_{g \in S_h} g$$

to be the sum of all these elements in S_h. Recall that the center of KG has a basis consisting of all conjugacy class sums of elements of G. Therefore, $C_h \in Z(KG)$, for all $h \in G$.

Definition 2.4. (*Reynolds* [45, Theorem 1]) *The Reynolds ideal of KG is the ideal of $Z(KG)$ generated as a K-vector space by the elements C_h, for $h \in G$.*

We now get the following result.

Proposition 2.5. (*Reynolds* [45], *cf* [18, Theorem VI.4.6]) *The Reynolds ideal $R(KG)$ of KG is the annihilator of $rad(KG)$ in $Z(KG)$.*

3. Selfinjective and symmetric algebras revisited

In order to be able to explain more deeply the relations between Reynolds ideals, $T(KG)$ and related objects, we need to explain the structure of selfinjective and of symmetric algebras. The theory is classical and originates in Nakayama's work [40, 41] in the late 1930's.

Various approaches can be found in the literature, but as far as we know the approach using the Picard groups, which we will explain in Section 3.2, did not appear elsewhere, though Yamagata [51] gave some related thoughts.

Throughout this section, we suppose for simplicity that K is a field. However, many results stay true under weaker assumptions on K; sometimes, K being a commutative ring would be sufficient.

3.1. **Basic definitions and properties.** Recall that for a K-algebra A the space of linear forms $Hom_K(A, K)$ is an $A - A$-bimodule by

$$(afb)(x) := f(bxa) \ \ \forall a, b, x \in A \forall f \in Hom_K(A, K).$$

A group algebra is a symmetric algebra in the following sense.

Definition 3.1. *Let K be a field and let A be a K-algebra. Then, A is*

- *symmetric if $A \simeq Hom_K(A, K)$ as $A - A$-bimodules.*
- *selfinjective if $A \simeq Hom_K(A, K)$ as A left-modules.*

We shall derive some consequences.

Suppose A is selfinjective and let $\varphi : A \longrightarrow Hom_K(A, K)$ be an isomorphism of A left-modules. Then, we may define a K-bilinear form

$$\langle \ , \ \rangle : A \times A \longrightarrow K$$

by

$$\langle a, b\rangle := (\varphi(b))(a).$$

The fact that φ is an isomorphism of vector spaces is equivalent to the fact that $\langle \ , \ \rangle$ is non degenerate.

The fact that φ is A-linear is equivalent to

$$\langle a, bc\rangle = (\varphi(bc))(a) = (b\varphi(c))(a) = \varphi(c)(ab) = \langle ab, c\rangle,$$

for all a, b, c, where the linearity is used in the second equality. A bilinear form on an algebra A is called associative if

$$\langle a, bc\rangle = \langle ab, c\rangle, \quad \text{for all } a, b, c \in A.$$

Now, φ is an $A - A$-bimodule homomorphism if and only if $\langle \ , \ \rangle$ is associative (i.e., φ is left A-linear) and moreover,

$$\langle a, b\rangle = (\varphi(b))(a) = (\varphi(1)b)(a) = \varphi(1)(ba) = \langle ba, 1\rangle = \langle b, a\rangle,$$

and so A is symmetric if and only if the associative non degenerate form $\langle \ , \ \rangle$ may be chosen symmetric.

We summarise the statements in a (well known) proposition which gives an alternative definition of selfinjective and symmetric algebras.

Proposition 3.2. *Let K be a field and let A be a finite dimensional K-algebra. Then, we have the following statements.*

- *The algebra A is selfinjective if and only if there is a non degenerate associative bilinear form on A.*
- *The algebra A is symmetric if and only if there is a non degenerate associative and symmetric bilinear form on A.*

We should mention that the existence statement in Proposition 3.2 is constructive: The bilinear form is as explicit as is the isomorphism. That is, if one knows the explicit isomorphism of A to its dual by an explicit formula, then one knows the bilinear form by an explicit formula, and vice versa.

The non degenerate associative symmetric bilinear form is called symmetrising form for a symmetric algebra. In the remaining parts of Section 3, these ideas are developed further in particular with emphasis on the question how to actually determine the bilinear form and associated questions.

For the moment, we shall continue with Reynolds ideals and give the promised link.

In the following, we frequently use for a symmetric algebra A and subsets S of A the symbol $S^\perp$ to designate the orthogonal space with respect to the symmetrising form of the algebra.

Proposition 3.3. (*Külshammer* [34]*;* [30, Part I, Lemma A; Satz C; Satz D]) *Let K be a field and let A be a finite dimensional symmetric K-algebra. Then, $[A,A]^\perp = Z(A)$ and $soc(A) = rad(A)^\perp = Ann_A(rad(A))$. In particular, we have $R(KG) = Z(KG) \cap soc(KG)$ for a finite group G.*

Proof.

$\langle ab-ba, c\rangle = \langle ab, c\rangle - \langle ba, c\rangle = \langle a, bc\rangle - \langle c, ba\rangle = \langle bc, a\rangle - \langle cb, a\rangle = \langle bc-cb, a\rangle$,

and hence $c \in [A,A]^\perp$ if and only if $\langle ab-ba, c\rangle = 0$, for all a, b. Therefore, $c \in [A,A]^\perp$ if and only if $\langle bc-cb, a\rangle = 0$, for all a, b. In particular, $c \in [A,A]^\perp$ if and only if $bc - cb \in A^\perp$. But $A^\perp = 0$, since the form is non degenerate. Hence, $c \in [A,A]^\perp$ if and only if $bc = cb$, for all $b \in A$. This shows $[A,A]^\perp = Z(A)$. Also,

$$\langle I, rad(A)\rangle = \langle 1, I \cdot rad(A)\rangle$$

and hence

$$I \subseteq rad(A)^\perp \Leftrightarrow I \cdot rad(A) = 0 \Leftrightarrow I \subseteq soc(A),$$

which completes the proof. □

3.2. **The Nakayama automorphism.** Selfinjective algebras come along with an automorphism, called the Nakayama automorphism which will next be explained.

If A is a selfinjective K-algebra, then $A \simeq Hom_K(A,K)$ an A left-module. Hence, $Hom_K(A,K)$ is a free left A-module of rank 1. Moreover,

$$End_A({}_AHom_K(A,K)) \simeq End_A({}_AA) \simeq A,$$

and so $Hom_K(A,K)$ is a progenerator over A with endomorphism ring isomorphic to A, whence inducing a Morita self-equivalence of A. This implies that the isomorphism class of $Hom_K(A,K)$ is in the Picard group $Pic_K(A)$ (cf. e.g., [44, Section 37]). As is shown there, there is a group homomorphism,

$$\omega_0 : Aut_K(A) \longrightarrow Pic_K(A),$$

given by $\omega_0(\alpha) = {}_1A_\alpha$. Here, for any two automorphisms α and β of A, the $A-A$-bimodule ${}_\alpha A_\beta$ denotes A as a vector space, on which $a \in A$ acts by multiplication by $\beta(a)$ on the right and by $\alpha(a)$ on the left. To shorten the notation, we abbreviate in this context the identity on A by 1. One gets $ker(\omega_0) = Inn(A)$, the inner automorphisms of A, and hence

$$Out_K(A) := Aut_K(A)/Inn(A)$$

is a subgroup of $Pic_K(A)$. (Observe that the group of inner automorphisms does not depend on K.) The image of ω_0 consists of those isomorphism classes of invertible $A-A$-bimodules which are free on the left (cf. [44, (37.16) Theorem]). Observe that

$$\begin{array}{ccc} {}_{\alpha^{-1}}A_1 & \longrightarrow & {}_1A_\alpha \\ a & \mapsto & \alpha(a) \end{array}$$

is an $A-A$-bimodule homomorphism. Hence, ω_0 may also be defined by twisting the action on the left.

Now, as $Hom_K(A,K)$ is free of rank 1 as left-module, one gets that $Hom_K(A,K)$ is in the image of $Out_K(A)$ in $Pic_K(A)$ and therefore there is an automorphism $\nu \in Aut_K(A)$ so that

$$Hom_K(A,K) \simeq {}_1A_\nu,$$

as A-A-bimodules. The automorphism ν is unique up to an inner automorphism.

Definition 3.4. *Let A be a selfinjective K-algebra. Then, there is an automorphism ν of A so that $Hom_K(A,K) \simeq {}_1A_\nu$, as A-A-bimodules. This automorphism is unique up to inner automorphisms and is called the* Nakayama automorphism.

Remark 3.5. *In Nakayama's original approach, ${}_\nu A_1$ is used instead of ${}_1A_\nu$ and so the Nakayama automorphism in Nakayama's work corresponds to the inverse of what we define d here.*

In principle, the definition of selfinjectiveness uses an isomorphism of the A-left-module of linear forms on A with the regular A-module. One could use the right module structure as well. We get the well-known result that left selfinjective is equivalent to right selfinjective.

Corollary 3.6. ${}_AA \simeq {}_AHom_K(A,K) \Leftrightarrow A_A \simeq Hom_K(A,K)_A$.

To prove the corollary, one just needs to see that the isomorphism as left-modules implies the following bimodule isomorphisms:

$$Hom_K(A,K) \simeq {}_1A_\nu \simeq {}_{\nu^{-1}}A_1$$

and so $Hom_K(A,K) \simeq A$, as A right-modules.

Now, for a selfinjective K-algebra A, given a simple A-module S, then ${}_1A_\nu \otimes_A S \simeq Hom_K(A,K) \otimes_A S$ is again a simple A-module.

Definition 3.7. *Let K be a field and let A be a finite dimensional K-algebra. Then, A is* weakly symmetric *if A is selfinjective and $Hom_K(A, K) \otimes_A S \simeq S$, for all simple A-modules S.*

This definition will be important in Section 6.3.

3.3. **The Nakayama twisted center.** Let K be a field and let A be a finite dimensional K-algebra. For an explicitly given algebra, say as quiver with relations, it is not very hard to write down many commutators. This gives an upper bound for the dimension of $A/[A,A]$. However, to prove that the commutators found really generate $[A,A]$ is quite difficult in general. The method is to interpret $A/[A,A]$ as a different space, in which it is easier to find many linearly independent elements. This then gives a lower bound for the dimension of $A/[A,A]$. If the lower and the upper bounds coincide, then one has proved that the commutators found actually generate the whole space $[A,A]$.

Let A be a selfinjective K-algebra. Given a ring R and a right R-module M and a left R-module N, the very definition of the tensor product $M \otimes_R N$, as free abelian on symbols $m \otimes n$ with relations $m \otimes$

$rn = mr \otimes n$ and additivity in each variable, gives an isomorphism $A/[A,A] \simeq A \otimes_{A\otimes_K A^{op}} A$. An alternative way to see this is by the bar resolution of the Hochschild homology.

Now,

$$\begin{array}{rcl} Hom_K(A/[A,A],K) & \simeq & Hom_K(A \otimes_{A\otimes_K A^{op}} A, K) \\ & \simeq & Hom_{A\otimes_K A^{op}}(A, Hom_K(A,K)) \\ & \simeq & Hom_{A\otimes_K A^{op}}(A, \ {}_1A_\nu), \end{array}$$

which gives

$$Hom_K(A/[A,A],K) \quad \simeq \quad \{a \in A \mid b \cdot a = a \cdot \nu(b) \ \forall b \in A\},$$

where the isomorphism is given by sending a homomorphism to the image of $1 \in A$, which will satisfy the equation by the property of the homomorphism being $A \otimes_K A^{op}$-linear.

Definition 3.8. (*Holm and Zimmermann* [24, Definition 2.2]) *Let K be a field and let A be a selfinjective K-algebra with the Nakayama automorphism ν. Then, the* Nakayama twisted center *is defined to be*

$$Z_\nu(A) := \{a \in A \mid b \cdot a = a \cdot \nu(b) \ \forall b \in A\}.$$

Remark 3.9. *The definition works for K a commutative ring as well. The automorphism ν is unique only up to an inner automorphism. If ν differs from ν' by an inner automorphism, let $\nu(b) = u \cdot \nu'(b) \cdot u^{-1}$, for all $b \in A$ and some unit u of A. Then,*

$$\begin{array}{rcl} Z_\nu(A) & = & \{a \in A \mid b \cdot a = a \cdot \nu(b) \ \forall b \in A\} \\ & = & \{a \in A \mid b \cdot a = a \cdot u \cdot \nu'(b) \cdot u^{-1} \ \forall b \in A\} \\ & = & \{a \in A \mid b \cdot (a \cdot u) = (a \cdot u) \cdot \nu'(b) \ \forall b \in A\} \\ & = & \{a \in A \mid a \cdot u \in Z_{\nu'}(A)\} \\ & = & Z_{\nu'}(A) \cdot u^{-1}. \end{array}$$

Remark 3.10. *In general, the Nakayama twisted center will not be a ring: if $a, b \in Z_\nu(A)$, then*

$$b(a_1a_2) = (ba_1)a_2 = (a_1\nu(b))a_2 = a_1(\nu(b)a_2) = a_1a_2\nu^2(b)$$

and $\nu^2 = \nu$ is equivalent to $\nu = id$. Nevertheless, if $z \in Z(A)$ and $a \in Z_\nu(A)$, then

$$b \cdot za = zba = za \cdot \nu(b)$$

and $za \in Z_\nu(A)$. Hence, $Z_\nu(A)$ is a $Z(A)$-submodule of A. The module structure does not depend on the chosen Nakayama automorphism, up to isomorphism of $Z(A)$-modules.

Remark 3.11. *In case one follows Nakayama's original definition of a Nakayama automorphism we need to replace ν by ν^{-1} and hence there the Nakayama twisted center would consist of elements a satisfying $\nu(b)\cdot a = a \cdot b$.*

We summarize our results in the following proposition.

Proposition 3.12. *(Holm and Zimmermann* [24, Lemma 2.4]*) If A is a selfinjective K-algebra over a field K with the Nakayama automorphism ν, then $Hom_K(A/[A,A],K) \simeq Z_\nu(A)$, as $Z(A)$-modules.*

Again, the proposition holds as well for K being a commutative ring.

3.4. **How to get the Nakayama automorphism explicitly.** Let K be a field and let A be a finite dimensional selfinjective K-algebra. In order to compute the Nakayama automorphism ν we need to find an explicit isomorphism $A \longrightarrow Hom_K(A,K)$, as A-modules.

Proposition 3.13. *(Holm and Zimmermann* [24, Lemma 2.7]*) Let K be a field and let A be a finite dimensional selfinjective K-algebra with associated bilinear form $\langle\ ,\ \rangle$. Then, the Nakayama automorphism ν of A satisfies $\langle a,b\rangle = \langle b,\nu(a)\rangle$, for all $a,b \in A$, and any automorphism satisfying this formula is a Nakayama automorphism.*

Remark 3.14. *If one would use Nakayama's original definition, then the form would satisfy $\langle \nu(a),b\rangle = \langle b,a\rangle$, for all $a,b \in A$.*

Proof of Proposition 3.13. There is a non-degenerate associative bilinear form on A, which induces an isomorphism between A and the linear forms on A, as A-modules, by Proposition 3.2. The isomorphism gives an isomorphism of A-A-bimodules of ${}_1A_\nu$ and $Hom_K(A,K)$ by

$$\begin{array}{ccc} {}_1A_\nu & \xrightarrow{\varphi} & Hom_K(A,K) \\ a & \mapsto & \langle -,a\rangle = \varphi(a). \end{array}$$

Therefore, $\varphi(a) = \varphi(1\cdot\nu^{-1}(a)) = \varphi(1)\cdot\nu^{-1}(a)$ and $\varphi(a) = \varphi(a\cdot 1) = a\cdot\varphi(1)$. Since for $f \in Hom_K(A,K)$ one has $(fa)(b) = f(ab)$ and $(af)(b) = f(ba)$, for all $a,b \in A$, one gets

$$\begin{array}{rcl} \langle b,a\rangle & = & (\varphi(a))(b) = (\varphi(1)\cdot\nu^{-1}(a))(b) = \varphi(1)(\nu^{-1}(a)b) \\ & = & (b\cdot\varphi(1))(\nu^{-1}(a)) = \varphi(b)(\nu^{-1}(a)) = \langle \nu^{-1}(a),b\rangle. \end{array}$$

Now, putting $a := \nu(a')$, one gets

$$\langle b,\nu(a')\rangle = \langle a',b\rangle,$$

for all $a', b \in A$. Hence, the Nakayama automorphism has the above property. Conversely, if an automorphism ν satisfies $\langle a, b\rangle = \langle b, \nu(a)\rangle$, for all $a, b \in A$, then the mapping $A \longrightarrow Hom_K(A, K)$, given by $a \mapsto \langle -, a\rangle$, gives an isomorphism of A and $Hom_K(A, K)$, as A-modules, inducing the element ${}_1A_\nu$ in the Picard group of A.

3.5. **Practical questions for algebras given by quivers and relations.** In [23], the following very useful result was proven for weakly symmetric algebras. However, the statement holds in a more general form.

Proposition 3.15. *Let K be a field and let $A = KQ/I$ be a selfinjective algebra given by the quiver Q and ideal of relations I, and fix a k-basis $\mathcal{B}$ of A consisting of pairwise distinct non-zero paths of the quiver Q. Assume that $\mathcal{B}$ contains a basis of the socle $soc(A)$ of A. Define a K-linear mapping ψ on the basis elements by*

$$\psi(b) = \begin{cases} 1 & \text{if } b \in soc(A) \\ 0 & \text{otherwise,} \end{cases}$$

for $b \in \mathcal{B}$. Then, an associative non-degenerate K-bilinear form $\langle -, -\rangle$ for A is given by $\langle x, y\rangle := \psi(xy)$.

Remark 3.16. *In case A is weakly symmetric, Proposition 3.15 was proven in [23]. The assumption that A is weakly symmetric was used in [23] only to prove the non degeneracy of the form. For the reader's convenience, we include a complete proof.*

Proof. By definition, since A is an associative algebra, ψ is associative on basis elements, and hence is associative on all of A.

Let ν be a Nakayama automorphism of A. We observe now that $\psi(x \cdot \nu(e)) = \psi(e \cdot x)$, for all $x \in A$, and all primitive idempotents $e \in A$. Indeed, since ψ is linear, we need to show this only on the elements in $\mathcal{B}$. Let $b \in \mathcal{B}$. If b is a path not in the socle of A, then $b\nu(e)$ and eb are either zero or not contained in the socle either, and hence $0 = \psi(b) = \psi(b\nu(e)) = \psi(eb)$. If $b \in \mathcal{B}$ is in the socle of A, then $b = e_b b = b\nu(e_b)$, for exactly one primitive idempotent e_b, and $e'b = b\nu(e') = 0$ for each primitive idempotent $e' \neq e_b$. Therefore, $\psi(e'b) = \psi(b\nu(e')) = 0$ and $\psi(e_b b) = \psi(b) = \psi(b\nu(e_b))$.

It remains to show that the map $(x, y) \mapsto \psi(xy)$ is non-degenerate. Suppose we had $x \in A \setminus \{0\}$ so that $\psi(xy) = 0$, for all $y \in A$. In particular, for each primitive idempotent e_i of A we get $\psi(e_i xy) = \psi(xy\nu(e_i)) =$

0, for all $y \in A$. Hence, we may suppose that $x \in e_i A$, for some primitive idempotent $e_i \in A$.

Now, xA is a right A-module. Choose a simple submodule S of xA and $s \in S \setminus \{0\}$. Then, since $s \in S \leq xA$, there is a $y \in A$ so that $s = xy$. Since $S \leq xA \leq A$, and since S is simple, $s \in soc(A) \setminus \{0\}$. Moreover, since $x \in e_i A$, also $s = e_i s$, i.e., s is in the (1-dimensional) socle of the projective indecomposable module $e_i A$. So, up to a non zero scalar factor, s is a path contained in the basis $\mathcal{B}$ (recall that by assumption, $\mathcal{B}$ contains a basis of the socle). This implies that

$$\psi(xy) = \psi(s) = \psi(e_i s) \neq 0,$$

contradicting the choice of x, and hence proving non-degeneracy. □

Remark 3.17. *It should be noted that the form depends on the chosen basis of the algebra. Indeed, take $A = K[X]/X^2$. The socle is one-dimensional, and take a basis $\{X\}$. Then, one may complete with an element $1+\mu X$, for $\mu \in K$, to a basis of A. Hence, $1+X = (1+\mu X) + (1-\mu)X$ and we get $\langle 1+X, 1\rangle = 1-\mu$, which depends heavily on μ.*

Example 3.18. *This example was communicated to me by Guodong Zhou in January 2010 during a visit in Paderborn. Let K be a field and let $A_q = K\langle X, Y\rangle/(X^2, Y^2, XY - qYX)$, for $q \neq 0$. Then, A_q is always selfinjective and A_q is symmetric if and only if $q = 1$. Now, $\nu(X) = qX$ and $\nu(Y) = q^{-1}Y$ defines a Nakayama automorphism. Indeed, if we use the K-basis $\{1, X, Y, XY\}$, for A_q we get*

$$1 = \langle X, Y\rangle = \langle Y, \nu(X)\rangle = \langle Y, qX\rangle = 1$$
$$1 = \langle Y, X\rangle = \langle X, \nu(Y)\rangle = \langle X, q^{-1}Y\rangle = 1$$

and likewise for the other basis elements.

Now, if $q \neq 1$, we get

$$Z(A_q) = K \cdot 1 + K \cdot XY,$$

whereas,

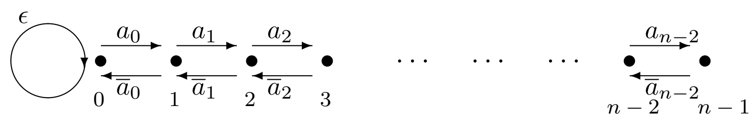
$$Z_\nu(A_q) = K \cdot X + K \cdot Y + K \cdot XY.$$

Hence, the $Z(A_q)$-module $Z_\nu(A_q)$ is isomorphic to $(Z(A_q)/rad(Z(A_q)))^3$.

If A is a symmetric algebra, and ν is chosen to be inner, then $Z_\nu(A)$ is a rank 1 free $Z(A)$-module.

Observe that the order of the above automorphism is the multiplicative order of q in K. Hence, for big fields K it is possible to create algebras with the Nakayama automorphisms of any given order, even infinite order.

Example 3.19. *Proposition 3.15 is stated in [21] and in [22] with an additional conclusion. Namely, it is stated there that the form $\langle\ ,\ \rangle$ is symmetric in case the algebra is symmetric. This is not true in general. A counterexample was given during the collaboration on [24]. Namely, the deformed preprojective algebra in the sense of Białkowski, Erdmann and Skowroński [5] of type L_n for $n \geq 3$ gives an example (see Remark 6.6 below). This algebra is defined by the following quiver*



subject to the following relations

$$a_i\overline{a}_i + \overline{a}_{i-1}a_{i-1} = 0 \text{ for all } i \in \{1, \dots, n-2\}\ ,$$

$$\overline{a}_{n-2}a_{n-2} = 0\ ,\ \epsilon^{2n} = 0\ ,\ \epsilon^2 + a_0\overline{a}_0 + \epsilon^3 p(\epsilon) = 0,$$

for a polynomial $p(X) \in K[X]$. These algebras are the deformed preprojective algebras of type L_n, in the sense of Białkowski, Erdmann and Skowroński [5].

For the special case $p(X) = X^{2j}$, for $j \in \mathbb{N}$, and for abbreviation we call this algebra by L_n^j and assume that K is of characteristic 2. Here, we just give an example where the bilinear form Proposition 3.15 does not yield a symmetric bilinear form. We will deal with the general case later in Section 6.2.

In order to be able to apply Proposition 3.15, we need to fix a basis of the socle of L_n^j. The fact that the elements below is indeed a basis of the algebra is shown in [24] and the basis displayed in Proposition 6.9 can easily be transformed into the basis below. Most recently, in a completely independent approach, Andreu [1] shows that the basis below is indeed a basis.

For our purpose, it seems to be most natural to take as K-basis of the socle the set

$$\{\epsilon^{2n-1}, \overline{a}_{i-1}\overline{a}_{i-2}\dots\overline{a}_0\epsilon^{2n-3-2i}a_0a_1\dots a_{i-1} \mid i \in \{1, 2\dots, n-2\}\}.$$

Complete the elements

$$\overline{a}_{i-1}\overline{a}_{i-2}\dots\overline{a}_0\epsilon^{2n-3-2i}a_0a_1\dots a_{i-1}$$

of $e_iL_n^je_i$, for $i \geq 1$, to a basis of $e_iL_n^je_i$ by the elements

$$\overline{a}_{i-1}\overline{a}_{i-2}\dots\overline{a}_0\epsilon^{\ell}a_0a_1\dots a_{i-1},$$

for $\ell \leq 2n-4-3i$ *and*

$$a_i a_{i+1} a_{i+2} \dots a_j \overline{a}_j \overline{a}_{j-1} \dots \overline{a}_{i+2} \overline{a}_{i+1} \overline{a}_i,$$

for $i+1 \leq j \leq n-2$. *A basis of* $e_0 L_n^j e_0$ *is given by* ϵ^ℓ, *for* $0 \leq \ell \leq 2n-1$.

Now, we verify

$$\langle \overline{a}_0 \epsilon^m, a_0 \rangle = \begin{cases} 1 & \text{if } m = 2n-3 \\ 0 & \text{if } m \neq 2n-3 \end{cases}$$

$$\langle a_0, \overline{a}_0 \epsilon^m \rangle = \begin{cases} 1 & \text{if } m = 2n-3 \text{ or } m = 2n-4-2j \\ 0 & \text{else.} \end{cases}$$

Hence, the bilinear form from Proposition 3.15 is not symmetric. However, the algebras L_n^j *are symmetric (cf. Proposition 6.10 below).*

4. Külshammer, the new idea in the 1980s

We come back to Brauer's proof displayed in Section 2, Reynolds' discoveries and present the original approach which was introduced by Külshammer to improve and unify these earlier approaches. Moreover, we explain the generalisation to non symmetric algebras.

4.1. Külshammer's original construction for symmetric algebras.

Recall that for a symmetric K-algebra A, one defines $T_n(A) := \{a \in A|\ a^{p^n} \in [A,A]\}$ and $T_n^\perp(A)$ is the orthogonal space with respect to the symmetrizing form.

Definition 4.1. *(Külshammer [33]) Let* A *be a symmetric* K*-algebra. Then, the ideal* $T_n(A)^\perp$ *of* $Z(A)$ *is the* n *th Külshammer ideal.*

Remark 4.2.

- *Remark 2.3 gives that* $T(A) = [A,A] + rad(A)$. *Hence,*
$$T(A)^\perp = rad(A) \cap soc(A) =: R(A).$$
- *Külshammer calls the ideals* $T_n(A)^\perp$ *the generalized Reynolds' ideals.*
- *Since* $T_n(A) \subseteq T_{n+1}(A)$ *we get* $T_n(A)^\perp \supseteq T_{n+1}(A)^\perp$ *and the set of Külshammer ideals is a decreasing sequence of ideals of the center with first term being the center and last term being the Reynolds ideal* $R(A)$.

Külshammer obtained in [30, 31, 32, 33] many properties of a group in terms of properties of the sequence of ideals $T_n(KG)^{\perp}$. Already in the first discussions in [30] the setup was completely general and the definitions were given for a symmetric algebra over a field of finite characteristic in general. Nevertheless, the applications in focus in Külshammer's discussions mainly have been group algebras and representations of finite groups.

The main tool for technical proofs is still the symmetrizing form. Since we have by Proposition 3.3 that $[A, A]^{\perp} = Z(A)$, the restriction of the symmetrizing form $\langle\ ,\ \rangle$ to $Z(A)$ on the left argument induces a non degenerate form, also denoted by $\langle\ ,\ \rangle$,

$$\langle\ ,\ \rangle : Z(A) \times A/[A, A] \longrightarrow K.$$

As seen in Lemma 2.2, the mapping

$$\begin{array}{ccc} A/[A, A] & \stackrel{\mu}{\longrightarrow} & A/[A, A] \\ a + [A, A] & \mapsto & a^p + [A, A] \end{array}$$

is additive and semilinear, i.e., linear if one applies in addition a twist with the Frobenius automorphism of the field. If V and W are finite dimensional vector spaces over a field K and if $\langle ., . \rangle$ is a non degenerate bilinear pairing $V \times W \longrightarrow K$, then any endomorphism φ of W has a unique left adjoint $\varphi^* \in End_K(V)$ satisfying $\langle v, \varphi(w)\rangle = \langle \varphi^*(v), w\rangle$, for all $v \in V$ and $w \in W$.

Now, this fact holds for semi-linear maps as well as for linear maps, and the map μ has a left adjoint,

$$\zeta : Z(A) \longrightarrow Z(A).$$

Now, $a \in T_n(A) \Leftrightarrow a \in ker(\mu^n)$ gives the following lemma.

Lemma 4.3. $z \in T_n(A)^{\perp} \Leftrightarrow z \in im(\zeta^n)$.

This characterization will be the main tool for most of abstract statements later.

4.2. **Extending to general algebras: trivial extension algebras and Külshammer's theory.** Up to now, in order to establish a Külshammer's ideal theory it was necessary already for the very definition to work over symmetric algebras. There is a method to circumvent this difficulty.

Let A be any finite dimensional K-algebra. Then, as already mentioned, $Hom_K(A,K)$ is an $A-A$-bimodule by

$$(afb)(c) := f(bca) \ \ \forall a,b,c \in A \text{ and } f \in Hom_K(A,K).$$

Recall the construction of the trivial extension algebra, which is well-known and very useful in the representation theory of associative algebras. We may form $\mathbb{T}A := Hom_K(A,K) \times A$, which is naturally a K-vector space. We may define an algebra structure on this space:

$$(f,a)\cdot(g,b) := (ag+fb, ab) \ \ \forall\, a,b \in A, f,g \in Hom_K(A,K)\ .$$

It is a tedious but straightforward computation to verify that $\mathbb{T}A$ is a K-algebra by this multiplication. Moreover, the projection to the second component is an algebra homomorphism $\mathbb{T}A \longrightarrow A$ with kernel $Hom_K(A,K)$ being an ideal with square 0.

Definition 4.4. *For any finite dimensional K-algebra A, the algebra $\mathbb{T}A$ is called the trivial extension algebra.*

The property that is most interesting for our purposes is that $\mathbb{T}A$ is a symmetric algebra, whatever may be the structure of the algebra A. Indeed,

$$\langle (f,a),(g,b)\rangle := g(a) + f(b) \ \ \forall (f,a),(g,b) \in \mathbb{T}A$$

is a symmetric associative non degenerate bilinear form on $\mathbb{T}A$. This fact can be found in [4, Section 3], for example. Proposition 3.2 shows then the statement.

In [4], Bessenrodt, Holm and the author compute the Külshammer ideals of $\mathbb{T}A$. We denote

$$Ann_{Hom_K(A,K)}(I) := \{f \in Hom_K(A,K) \mid f(I) = \{0\}\},$$

for any subset $I \subseteq A$. With this notation we showed the following.

Proposition 4.5. (*Bessenrodt, Holm, Zimmermann* [4, Theorem 4.1]) *Let A be a finite-dimensional algebra over a field of characteristic $p>0$, and let $\mathbb{T}A$ be its trivial extension.*

(1) *We have $T_0(\mathbb{T}A)^\perp = Z(\mathbb{T}A) = Ann_{Hom_K(A,K)}([A,A]) \times Z(A)$.*
(2) *For all $n \geq 1$, we have $T_n(\mathbb{T}A)^\perp = Ann_{Hom_K(A,K)}(T_nA) \times 0$.*

This result, though not difficult to prove, is most remarkable, since for symmetric algebras we may use the symmetrizing form to transport $T_n(A)$ via orthogonality to an ideal $T_n(A)^\perp$ of the center of A. The ideal structure allows to consider many invariants from commutative algebra attached to this ideal $T_n(A)^\perp$. If A is not symmetric, this is not easily possible. Hence, it is surprising that enlarging A to $\mathbb{T}A$, the space $T_n(\mathbb{T}A)^\perp$ keeps the trace of $T_n(A)^\perp$ faithfully in the sense that $T_n(A)^\perp$ can be fully recovered by $T_n(\mathbb{T}A)^\perp$.

If A is already symmetric, then the isomorphism $\mathbb{T}A \longrightarrow Hom_K(\mathbb{T}A, K)$ takes $T_n(A)^\perp$ to $Ann_{Hom_K(A,K)}(T_n(A))$. This fact is not hard to see. Indeed, if A is symmetric, then $Hom_K(A,K)$ is the space of all $\langle a, -\rangle$, for $a \in A$. Now, a linear form being in $Ann_{Hom_K(A,K)}(T_n(A))$ is equivalent to $\langle a, T_n(A)\rangle = 0$, which in turn is equivalent to $a \in T_n(A)^\perp$.

5. Morita's derived and stable invariance

5.1. **The Morita invariance.** In [19], Héthélyi, Horváth, Külshammer and Murray studied amongst other questions the invariance of Külshammer's ideals $T_n(A)^\perp$ under the Morita equivalence. Recall that if

$$M \otimes_A - : A-mod \longrightarrow B-mod$$

is an equivalence, then for any $z \in Z(A)$ there is a unique $\varphi_M(z) \in Z(B)$ so that $m \cdot z = \varphi_M(z) \cdot m$, for all $m \in M$. Then, $\varphi_M : Z(A) \longrightarrow Z(B)$ is an isomorphism of algebras.

Proposition 5.1. *(Héthelyi, Horváth, Külshammer and Murray* [19, Corollary 5.3]*) Let K be a perfect field of characteristic $p > 0$ and let A and B be finite dimensional K-algebras. If*

$$M \otimes_A - : A-mod \longrightarrow B-mod$$

is a Morita equivalence, then $\varphi_M(T_n(A)^\perp) = T_n(B)^\perp$, for all $n \in \mathbb{N}$.

The authors show that the mapping ζ of Lemma 4.3 behaves well with respect to multiplication by idempotents. Using this statement it is possible to reduce to basic algebras, and to use then that two Morita equivalent basic algebras are isomorphic.

The existence of basic algebras, i.e., an up to isomorphism unique minimal algebra which is Morita equivalent to the given algebra, is very specific for Morita equivalences. Such a concept does not exist for weaker equivalences such as derived equivalences or stable equivalences of Morita type.

5.2. **Derived invariance.** [19, Question 5.4] asked if Külshammer's ideals are also invariant under derived equivalences. The method used for the Morita invariance does not apply since as said before a concept of "derived basic" algebras do not exist. Nevertheless, an equivalence between derived categories of finite dimensional algebras imply the existence of an isomorphism of the centers.

More precisely, denote by $D^b(A)$ the derived category of bounded complexes of finitely generated A-modules with non zero homology in only finitely many degrees.

The main tool is the following result.

Theorem 5.2. (*Rickard* [47, Theorem 3.3]*, Keller* [26, Section 8]*; cf. also, e.g.* [28, Theorem 6.2.8]) *Let K be a field, let A and B be finite dimensional K-algebras and suppose that $D^b(A) \simeq D^b(B)$ is an equivalence of triangulated categories. Then, there is a complex $X \in D^b(B \otimes_K A^{op})$ which is formed by modules which are projective as B-modules and projective as A-modules, so that*

$$X \otimes_A^{\mathbb{L}} - : D^b(A) \longrightarrow D^b(B)$$

is an equivalence of triangulated categories.

A complex X as in the theorem is called two-sided tilting complex. It is unknown if every equivalence between derived categories is of the form

$$X \otimes_A^{\mathbb{L}} - : D^b(A) \longrightarrow D^b(B).$$

This is shown to hold on the level of objects, but it is not known if there may be an exotic equivalence behaving differently on morphisms (cf. Rickard [47, Corollary 3.5]).

The result has many consequences. In particular,

$$X \otimes_A^{\mathbb{L}} (- \otimes_A^{\mathbb{L}} Hom_A(X, A)) : D^b(A \otimes_K A^{op}) \longrightarrow D^b(B \otimes_K B^{op})$$

is an equivalence. Therefore, if $D^b(A) \simeq D^b(B)$, then there exists a two-sided tilting complex realizing an equivalence. The given equivalence one started with may be different. Then, this two-sided tilting complex induces the equivalence of the derived categories of bimodules, and this then has the property that

$$End_{D^b(A \otimes A^{op})}(A) \xrightarrow{\simeq} End_{D^b(B \otimes B^{op})}(B)$$

is an equivalence.

It is known that the for any algebra C the module category is a full subcategory of the derived category by identifying a module with the complex concentrated in a single degree 0 (cf. e.g. Verdier [49, Chapitre III Section 1.2.9]). Therefore, for a C-module M one gets $End_C(M) \simeq End_{D^b(C)}(M)$. Hence,

$End_{D^b(A\otimes A^{op})}(A) \simeq End_{A\otimes A^{op}}(A) \simeq Z(A)$ and $End_{D^b(B\otimes B^{op})}(B) \simeq End_{B\otimes B^{op}}(B) \simeq Z(B)$.

Corollary 5.3. (*Rickard* [47, Proposition 2.5]) *Let K be a field and let A and B be finite dimensional K-algebras. If $D^b(A) \simeq D^b(B)$, then a choice of a two-sided tilting complex X realizing this equivalence as tensor product induces an isomorphism φ_X as algebras between the center of A and the center of B.*

We see that still an equivalence between the derived categories of finite dimensional algebras yield an isomorphism between the centers of the algebras. The isomorphism however is far less explicit and somewhat complicated.

Despite these difficulties, we obtain the following result.

Theorem 5.4. [56, Theorem 1] *Let A and B be finite dimensional symmetric K-algebras over a perfect field K of characteristic $p > 0$. Suppose that $D^b(A) \simeq D^b(B)$, as triangulated categories. Then, any choice of a two-sided tilting complex X yields an isomorphism $\varphi_X : Z(A) \longrightarrow Z(B)$ satisfying $\varphi_X(T_n(A)^\perp) = T_n(B)^\perp$, for all $n \in \mathbb{N}$.*

It is worth writing that we use here the mapping that is induced by a functor on the morphisms. We recall that possibly non standard derived equivalences exist. Non standard derived equivalences are not standard only on morphisms, but we only use morphisms here. Hence, non standard derived equivalences would possibly induce an isomorphism between the centers which does not preserve the Külshammer's ideal structure.

The proof is much more involved than the proof for the Morita invariance in the sense that one needs to reformulate the construction of Külshammer's ideals in a "derived category readable form". Instead of explicit constructions of particular sets, one needs to argue via homological properties of morphism spaces.

One should mention that being symmetric is an invariant under derived equivalences.

Proposition 5.5. (*Rickard* [47, Corollary 5.3] *for fields R,* [55] *for more general rings*) *Let R be a Dedekind domain and let A and B be R-algebras of finite rank over R so that $D^b(A) \simeq D^b(B)$. Then, if A is symmetric, B is symmetric as well.*

Applications of Theorem 5.4 will be given in Section 6. Actually, the invariance of Külshammer's ideals proved to be a rather powerful tool in particular in order to distinguish algebras given by quivers and relations where the relations depend on certain parameters. The structure of the quotient $Z(A)/T_n(A)^{\perp}$ tends to depend on the parameters in several cases.

5.3. **Stable invariance.** As usual, the stable category $A-\underline{mod}$ of a module category $A-mod$ is the category with objects being A-modules and morphisms between two A-modules in the stable category are equivalence classes of morphisms between these A-modules modulo those which factor through a projective A-module. We denote by $\underline{Hom}_A(M,N)$ the morphisms in $A-\underline{mod}$ from M to N.

Equivalences between stable categories can behave badly in general. An example was given by Auslander and Reiten in 1973 [2, Example 3.5].

Example 5.6. *Let K be a field. Then, the algebras*

$$A := \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \times \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$$

and

$$B := \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the quotient of the upper triangular matrix ring by the ideal generated by the upper right component matrices have equivalent stable categories. Indeed, A has six indecomposable modules: four simple modules, two of which are projective, and the projective cover of the non projective simple. Hence, the stable module category is equivalent to two copies of $K-mod$. The algebra B has five indecomposable modules: three simple modules, one of which is projective and the projective covers of the other two simple modules. Hence, the stable module category is equivalent to two copies of $K-mod$ as well.

The algebras A and B are hence stably equivalent and B is indecomposable, whereas A is not. Neither A nor B has any simple direct factor.

Given two self-injective algebras A and B, suppose that X is a complex of $A-B$-bimodules inducing a standard equivalence $D^b(B) \longrightarrow D^b(A)$. The quasi-inverse is again a standard equivalence, given by a complex Y of $B-A$-bimodules. Then, a somewhat technical construction on X and on Y produces an $A-B$-bimodule M, projective on either side, and a $B-A$-bimodule N, projective on either side, so that the $A-A$-bimodule $M\otimes_B N$ is isomorphic to $A\oplus P$, for some projective $A-A$-bimodule P, and so that one has an isomorphism of $B-B$-bimodules $N\otimes_A M\simeq B\oplus Q$, for some projective $B-B$-bimodule Q. This motivated Broué to define a class of stable equivalences with nicer properties.

Definition 5.7. (*Broué* [14, Section 5]) *Let K be a commutative ring, A and B be two K-algebras, $M\in A\otimes_K B^{op}-mod$, and let $N\in B\otimes_K A^{op}-mod$. Then, (M,N) is said to induce a stable equivalence of Morita type if*

- *M as well as N are projective as A-modules and as B-modules.*
- *$M\otimes_B N\simeq A\oplus P$, as $A-A$-bimodules, for a projective $A-A$-bimodule P.*
- *$N\otimes_A M\simeq B\oplus Q$, as $B-B$-bimodules, for a projective $B-B$-bimodule Q.*

Remark 5.8. *Liu showed in* [35, Theorem 2.2] *that a stable equivalence of Morita type between two finite dimensional algebras with no separable summands restricts to a stable equivalence between their summands. Therefore, the algebras A and B in Example 5.6 are not stably equivalent of Morita type.*

In the meantime, several properties have been shown to be invariant under stable equivalence of Morita type, whereas the general stable equivalences are still rather poorly understood.

In particular, the following definition will be of importance in our discussion. Let A be a K-algebra. Then,

$$Z(A)=Hom_{A\otimes_K A^{op}}(A,A)$$

and define the stable center,

$$Z^{st}(A)=\underline{Hom}_{A\otimes_K A^{op}}(A,A).$$

The natural homomorphism

$$Hom_{A\otimes_K A^{op}}(A,A)\longrightarrow \underline{Hom}_{A\otimes_K A^{op}}(A,A)$$

has a kernel denoted by $Z^{pr}(A)$, the projective center.

Proposition 5.9. (*Broué* [14, Proposition 5.4]) *Let A and B be finite dimensional K-algebras and let (M,N) be bimodules inducing a stable equivalence of Morita type. Then, $Z^{st}(A)\simeq Z^{st}(B)$, as algebras.*

How can we determine $Z^{pr}(A)$? This is a result due to Liu, Zhou and the author. The Cartan matrix of the algebra A is denoted by C_A. Recall that the Cartan matrix is square of size n, where n is the number of simple A-modules up to isomorphism. If we label the rows and the columns by the isomorphism classes $[S]$ of simple modules, then we have that the coefficient in position $([S],[T])$ is $Hom_A(P_S,P_T)$, where P_S denotes the projective cover of S. Hence, C_A has integer coefficients and can therefore be interpreted as a linear endomorphism of the K-vector space K^n.

Proposition 5.10. (*Liu, Zhou, Zimmermann* [36, Proposition 2.4, Corollary 2.9, Lemma 7.8, Proposition 7.10]) *Let K be an algebraically closed field and let A be a finite dimensional symmetric K-algebra. Then, $Z^{pr}(A)\subseteq Z(A)\cap soc(A)=R(A)$ and $dim_K(Z^{pr}(A))=rank_K C_A$.*

In order to adapt the Külshammer ideal theory for stable equivalences of Morita type we need to replace the center by the stable center, since we know that a stable equivalence gives an isomorphism of the stable centers and we do not have enough information about the center.

Further more, we need to find a replacement of $A/[A,A]$. For this purpose, we recall the Hattori-Stallings trace which was generalised by Bouc to a trace function on the whole Hochschild homology.

Definition 5.11. (*Bouc* [11] *for higher dimensional Hochschild homology, Hattori-Stallings in degree* 0) *Let K be a field and let A and B be two finite dimensional K-algebras. Given an $A-B$-bimodule M, which is projective as B-module, there are elements $m_i\in M,\varphi_i\in Hom_B(M,B)$, for $i=1,\dots,n$, so that the identity on M in $End_B(M)\simeq M\otimes_B Hom_B(M,B)$ is mapped to $\sum_{i=1}^n m_i\otimes\varphi_i$. The fact that M is an $A-B$-bimodule gives a mapping*

$$\begin{array}{ccc} A & \xrightarrow{\alpha_M} & End_B(M)\simeq M\otimes_B Hom_B(M,B) \\ a & \mapsto & \sum_{i=1}^n (am_i)\otimes\varphi_i. \end{array}$$

We produce

$$eval : M\otimes_B Hom_B(M,B)\longrightarrow B/[B,B]$$

by $eval(m\otimes\psi):=\psi(m)+[B,B]$, for $\psi\in Hom_B(M,B)$ and $m\in M$. The composition $eval\circ\alpha$ factorizes through $A/[A,A]$ and the resulting mapping

$$A/[A,A]\longrightarrow B/[B,B]$$

is called the trace of M, denoted by tr_M. Similar statements hold if M is projective on the left.

Using the Hattori-Stallings trace we give the following definition.

Definition 5.12. [36, Defintion 4.1] *Let A be a finite dimensional K-algebra*

$$HH_0^{st}(A) := \bigcap_{P \text{ projective indecomposable } A-mod} ker(tr_P)$$

observing that any projective A-module P is a $K-A$-bimodule as required by Definition 5.11.

With this preparation we obtain that the dimension of $HH_0^{st}(A)$ is an invariant under stable equivalence of Morita type. Denote by $\ell(A)$ the number of simple A-modules up to isomorphism.

Theorem 5.13. *Let K be an algebraically closed field and let A and B be finite dimensional K-algebras without any semisimple direct factor and suppose that A and B are stably equivalent of Morita type.*

- (*Liu, Zhou, Zimmermann* [36, Theorem 6.1]) *Then,*

  $$dim_K(A/[A,A]) = dim_K(B/[B,B]) \Leftrightarrow \ell(A) = \ell(B).$$

  Moreover, $\dim(T_n(A)/K(A)) = \dim(T_n(B)/K(B))$.
- (*Liu, Zhou, Zimmermann* [36, Corollary 6.2]) *If in addition A is symmetric, then*

  $$\dim_K(Z(A)) = \dim_K(Z(B)) \Leftrightarrow \ell(A) = \ell(B).$$

  and (*König, Liu, Zhou* [29, Proposition 5.8])

  $$Z(A)/T_n(A)^\perp \simeq Z(B)/T_n(B)^\perp,$$

  for all $n \geq 1$.

Proof. The proof of the first part uses first that $HH_0^{st}(A)$ is an invariant under stable equivalences of Morita type. Then, one shows [36, Theorem 4.4] that

$$dim(HH_0^{st}(A)) + rank_K(C_A) = dim(A/[A,A])$$

Further more, it is shown in [36, Section 5] that $rank_K(C_A) - dim_K(K \otimes_{\mathbb{Z}} K_0(A))$ equals the dimension of the so-called stable Grothendieck group, which is known to be an invariant under stable equivalences of Morita type by the work of Xi [50, Section 5].

We should mention the long standing Auslander-Reiten conjecture.

Conjecture. (*Auslander-Reiten* [3, page 409, Conjecture 5]) *Let A and B be finite dimensional K-algebras. If A and B are stably equivalent, then the number of simple non projective A-module up to isomorphism equals the number of simple non projective B-modules up to isomorphism.*

A priori I feel that there is no obvious reason why the invariance of the number of simple A-modules has anything to do with the commutator quotient. This fact appears somewhat surprisingly in this connection.

The conjecture has been verified for quite a few classes of algebras. Few general positive results are known so far.

6. Applications

In the last five years, Külshammer's ideals were successfully employed to distinguish algebras up to derived and up to stable equivalences of Morita type for various classes of algebras which were extremely difficult to deal with previously. In particular, if two algebras are defined by the same quiver Q and a set of relations $I(c)$ subject to some parameter c in the base field, then the technique of computing Külshammer's ideals and the quotient of the center by the ideal proves to be fruitful in various cases.

6.1. **Algebras of dihedral, semidihedral and quaternion type.** Many authors during the past decades proved ring theoretic properties for group algebras, and still the area is an active field of research. In particular, many properties are shown to hold for the Cartan matrices and the occurrence of certain components in the stable Auslander-Reiten quiver for blocks of group algebras with dihedral, semidihedral or quaternion defect groups. Moreover, it was shown at that time that a block of a group algebra is of tame representation type (cf. Section 6.3 below for the precise definition) if and only if the defect group is a dihedral, a semidihedral or a quaternion group.

Erdmann showed in [16] that these properties determine the Morita equivalence classes of these algebras as belonging to a finite number of families, given by quivers with relations, subject to certain parameters in the relations. Up to these parameters in the relations, the algebras are classified in a finite number of classes up to the Morita equivalences.

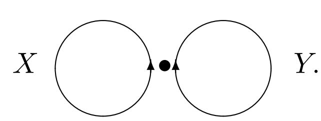
Holm [20] classified further these Morita equivalence classes up to derived equivalences. Many classes merge to a common derived equivalence class. However, Holm could not determine for a certain number of parameters if two algebras within one class but with different parameters are derived equivalent or not.

Külshammer ideals manage to distinguish derived equivalence classes in some of these cases.

We display Thorsten Holm's list [20] of algebras of dihedral, semidihedral and quaternion type up to derived equivalences. Each of these types form a family. Each family is subdivided into three subclasses: algebras with one simple module, algebras with two simple modules and algebras with three simple modules. Each subfamily contains algebras defined by quivers and relations, depending on parameters.

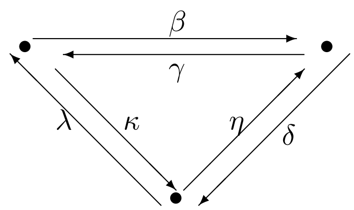
	dihedral	semidihedral	quaternion
1 simple	$K[X,Y]/(XY, X^m - Y^n)$, $m \geq n \geq 2, m+n > 4$; $D(1\mathcal{A})_1^1 = K[X,Y]/(X^2, Y^2)$; $(char K = 2)$ $K[X,Y]/(X^2, YX - Y^2)$; $D(1\mathcal{A})_1^k, k \geq 2$; $(char K = 2)$ $D(1\mathcal{A})_2^k(d)$, $k \geq 2, d = 0$ *or* 1;	$SD(1\mathcal{A})_1^k, k \geq 2$; $(char K = 2)$ $SD(1\mathcal{A})_2^k(c,d)$ $k \geq 2, (c,d) \neq (0,0)$;	$Q(1\mathcal{A})_1^k, k \geq 2$; $(char K = 2)$ $Q(1\mathcal{A})_2^k(c,d)$, $k \geq 2, (c,d) \neq (0,0)$;
2 simples	$D(2\mathcal{B})^{k,s}(c)$, $k \geq s \geq 1, c \in \{0,\ 1\}$	$SD(2\mathcal{B})_1^{k,t}(c)$ $k \geq 1, t \geq 2, c \in \{0,\ 1\}$; $SD(2\mathcal{B})_2^{k,t}(c)$ $k \geq 1, t \geq 2$, $k+t \geq 4, c \in \{0,\ 1\}$;	$Q(2\mathcal{B})_1^{k,s}(a,c)$ $k \geq 1, s \geq 3, a \neq 0$;
3 simples	$D(3\mathcal{K})^{a,b,c}$, $a \geq b \geq c \geq 1$; $D(3\mathcal{R})^{k,s,t,u}$, $s \geq t \geq u \geq k \geq 1, t \geq 2$	$SD(3\mathcal{K})^{a,b,c}$ $a \geq b \geq c \geq 1, a \geq 2$;	$Q(3\mathcal{K})^{a,b,c}$ $a \geq b \geq c \geq 1, b \geq 2$, $(a,b,c) \neq (2,2,1)$; $Q(3\mathcal{A})_1^{2,2}(d)$ $d \notin \{0,1\}$

All algebras with one simple module in the above list have the quiver of type $1\mathcal{A}$

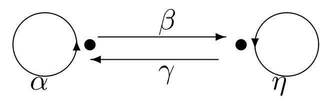


The quivers of the algebras of type $2\mathcal{B}$, $3\mathcal{K}$, $3\mathcal{A}$ and $3\mathcal{R}$ are respectively:

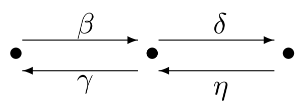
type $3\mathcal{K}$



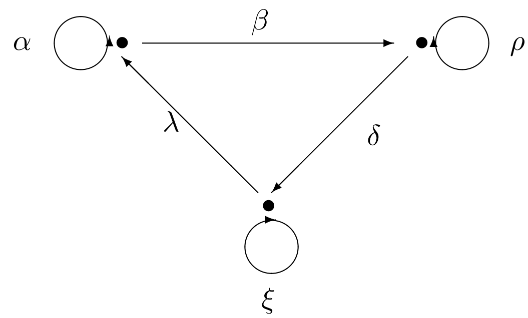
type $2\mathcal{B}$



type $3\mathcal{A}$



type $3\mathcal{R}$



The relations are respectively:

$$\begin{array}{rcl}
D(1\mathcal{A})_1^k & : & X^2, Y^2, (XY)^k - (YX)^k; \\
D(1\mathcal{A})_2^k(d) & : & X^2 - (XY)^k, Y^2 - d\cdot(XY)^k, (XY)^k - (YX)^k, \\
 & & \qquad (XY)^k X, (YX)^k Y; \\
SD(1\mathcal{A})_1^k & : & (XY)^k - (YX)^k, (XY)^k X, Y^2, X^2 - (YX)^{k-1}Y; \\
SD(1\mathcal{A})_2^k(c,d) & : & (XY)^k - (YX)^k, (XY)^k X, Y^2 - d(XY)^k, \\
 & & \qquad X^2 - (YX)^{k-1}Y + c(XY)^k; \\
Q(1\mathcal{A})_1^k & : & (XY)^k - (YX)^k, (XY)^k X, Y^2 - (XY)^{k-1}X, \\
 & & \qquad X^2 - (YX)^{k-1}Y; \\
Q(1\mathcal{A})_2^k(c,d) & : & X^2 - (YX)^{k-1}Y - c(XY)^k, \\
 & & \qquad Y^2 - (XY)^{k-1}X - d(XY)^k, \\
 & & \qquad (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y,
\end{array}$$

as well as,

$$\begin{array}{rcl}
D(2\mathcal{B})^{k,s}(c) & : & \beta\eta, \eta\gamma, \gamma\beta, \alpha^2 - c(\alpha\beta\gamma)^k, (\alpha\beta\gamma)^k - (\beta\gamma\alpha)^k, \\
& & \eta^s - (\gamma\alpha\beta)^k; \\
SD(2\mathcal{B})_1^{k,t}(c) & : & \gamma\beta, \eta\gamma, \beta\eta, \alpha^2 - (\beta\gamma\alpha)^{k-1}\beta\gamma - c(\alpha\beta\gamma)^k, \eta^t - (\gamma\alpha\beta)^k, \\
& & (\alpha\beta\gamma)^k - (\beta\gamma\alpha)^k; \\
SD(2\mathcal{B})_2^{k,t}(c) & : & \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha, \gamma\beta - \eta^{t-1}, \\
& & \alpha^2 - c(\alpha\beta\gamma)^k, \beta\eta^2, \eta^2\gamma; \\
Q(2\mathcal{B})_1^{k,s}(a,c) & : & \gamma\beta - \eta^{s-1}, \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha, \\
& & \alpha^2 - a(\beta\gamma\alpha)^{k-1}\beta\gamma - c(\beta\gamma\alpha)^k, \alpha^2\beta, \gamma\alpha^2;
\end{array}$$

$$\begin{array}{rcl}
D(3\mathcal{K})^{a,b,c} & : & \beta\delta, \delta\lambda, \lambda\beta, \gamma\kappa, \kappa\eta, \eta\gamma, (\beta\gamma)^a - (\kappa\lambda)^b, (\lambda\kappa)^b - (\eta\delta)^c, \\
& & (\delta\eta)^c - (\gamma\beta)^a; \\
D(3\mathcal{R})^{k,s,t,u} & : & \alpha\beta, \beta\rho, \rho\delta, \delta\xi, \xi\lambda, \lambda\alpha, \alpha^s - (\beta\delta\lambda)^k, \rho^t - (\delta\gamma\beta)^k, \\
& & \xi^u - (\lambda\beta\delta)^k; \\
SD(3\mathcal{K})^{a,b,c} & : & \kappa\eta, \eta\gamma, \gamma\kappa, \delta\gamma - (\gamma\alpha)^{a-1}\gamma, \beta\delta - (\kappa\lambda)^{b-1}\kappa, \\
& & \lambda\beta - (\eta\delta)^{c-1}\eta; \\
Q(3\mathcal{K})^{a,b,c} & : & \beta\delta - (\kappa\lambda)^{a-1}\kappa, \eta\gamma - (\lambda\kappa)^{a-1}\lambda, \delta\lambda - (\gamma\beta)^{b-1}\gamma, \\
& & \kappa\eta - (\beta\gamma)^{b-1}\beta, \lambda\beta - (\eta\delta)^{c-1}\eta, \\
& & \gamma\kappa - (\delta\eta)^{c-1}\delta, \gamma\beta\delta, \delta\eta\gamma, \lambda\kappa\eta; \\
Q(3\mathcal{A})_1^{2,2}(d) & : & \beta\delta\eta - \beta\gamma\beta, \delta\eta\gamma - \gamma\beta\gamma, \eta\gamma\beta - d\eta\delta\eta, \gamma\beta\delta - d\delta\eta\delta, \beta\delta\eta\delta, \\
& & \eta\gamma\beta\gamma.
\end{array}$$

For the dihedral type algebras with two simple modules a result due to Kauer and Roggenkamp [25, Corollary 5.3] shows that the parameters $c = 0$ and $c = 1$ yield different derived equivalence classes of algebras. The method employed there is rather involved. The authors define graph algebras and show that being a graph algebra is invariant under derived equivalences. Further more, for one of the scalars, the algebra is a graph algebra, for the other it is not. Holm and the author gave a much simpler proof in [23] avoiding graph algebras.

The semidihedral type case can be dealt with at least partially. Again, we consider the derived equivalence classes of semidihedral type algebras with two simple modules, and again the question that if the parameters $c = 0$ and $c = 1$ yield different derived equivalence classes is still open.

Theorem 6.1. (*Holm and Zimmermann* [23, Theorem 1.1, Theorem 1.2 and Theorem 1.3]) *Let K be an algebraically closed field of characteristic 2.*

- *For any given integers k and $s \geq 1$, consider the algebras of dihedral type $D(2A)^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that $k \geq 2$. Suppose that if $k = 2$, then $s \geq 3$ is odd, and if $s = 2$, then $k \geq 3$ is odd.*

  Put $A_0^{k,s} := D(2A)^{k,s}(0)$ and $A_1^{k,s} := D(2A)^{k,s}(1)$.
  Then, the factor rings $Z(A_0^{k,s})/T_1(A_0^{k,s})^{\perp}$ and $Z(A_1^{k,s})/T_1(A_1^{k,s})^{\perp}$ are not isomorphic as rings.

  In particular, the algebras $D(2A)^{k,s}(0)$ and $D(2A)^{k,s}(1)$ are not derived equivalent and are not stably equivalent of the Morita type.
- *For any given integers $k \geq 1$ and $s \geq 1$, consider the algebras of semidihedral type $SD(2B)_1^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that $k \geq 2$. Suppose that if $k = 2$, then $s \geq 3$ is odd, and if $s = 2$, then $k \geq 3$ is odd.*

  Put $B_0^{k,s} := SD(2B)_1^{k,s}(0)$ and $B_1^{k,s} := SD(2B)_1^{k,s}(1)$.
  Then, the factor rings $Z(B_0^{k,s})/T_1(B_0^{k,s})^{\perp}$ and $Z(B_1^{k,s})/T_1(B_1^{k,s})^{\perp}$ are not isomorphic as rings.

  In particular, the algebras $SD(2B)_1^{k,s}(0)$ and $SD(2B)_1^{k,s}(1)$ are not derived equivalent and are not stably equivalent of the Morita type in these cases.
- *For any given integers $k \geq 1$ and $s \geq 1$, consider the algebras of semidihedral type $SD(2B)_2^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that $k \geq 2$. Put $C_0^{k,s} := SD(2B)_2^{k,s}(0)$ and $C_1^{k,s} := SD(2B)_2^{k,s}(1)$.*

  If the parameters k and s are both odd, then the factor rings $Z(C_0^{k,s})/T_1(C_0^{k,s})^{\perp}$ and $Z(C_1^{k,s})/T_1(C_1^{k,s})^{\perp}$ are not isomorphic as rings.

  In particular, the algebras $SD(2B)_2^{k,s}(0)$ and $SD(2B)_2^{k,s}(1)$ are not derived equivalent and are not stably equivalent of the Morita type in these cases.

Remark 6.2.

(1) *We should mention that actually the dimension of the quotients of the centers modulo the Külshammer ideals do not depend on the scalar c. The algebraic structure of the quotient is needed.*

(2) *It is worth noticing that these algebras are all symmetric and so* [56] *applies directly. Moreover, the dimension of the center of the algebra equals the dimension of the quotient of the algebra by the commutator subspace. This immediate consequence of the fact that the algebras are symmetric is not clear in case the algebra is selfinjective only. Example 3.18 gives an easy example for how complicated the situation might become already for very small selfinjective algebras. For general selfinjective algebras, a rather sophisticated theory needs to be developed in order to compute the commutator subspace.*

Remark 6.3. *I would like to mention that in* [23, Theorem 1.1] *the condition that $k \geq 2$ in case of algebras of dihedral type in Theorem 6.1 is unfortunately missing. The condition is necessary. A recent result of Bleher* [8, Theorem 2] *determines the parameter c in the relations for a specific group by completely different methods. If $k = 1$ would be allowed, then the parameter would be different than determined by Bleher. This observation is due to Zhou.*

In a recent work, Zhou and the author studied if the derived equivalence classification of the Holm of dihedral, semidihedral and quaternion type algebras also give a classification up to stable equivalence. Some partial statements are already given in Theorem 6.1.

One has to deal with several additional problems for stable equivalences of the Morita type.

The first problem is that the Auslander-Reiten conjecture is open, that is, we might a priori have a stable equivalence of the Morita type between two algebras of dihedral, semidihedral or quaternion type with different numbers of simple modules. This does not happen for derived equivalences, since there the rank of the Grothendieck group is an invariant.

The second problem is that a derived equivalence between an algebra and a local algebra is in fact a Morita equivalence. This was shown by Roggenkamp and the author [54, Section 5]. The statement is false for stable equivalences of the Morita type. Given a finite group G and a field K of characteristic p dividing the order of G, a KG-module M is endotrivial if the KG-module $End_K(M)$ has the property $End_K(M) \simeq K \oplus P$ for K being the trivial K-module and P a projective KG-module. Every endotrivial module over a p-group gives a stable self-equivalence of the Morita type for the group ring over this p-group. The set of endotrivial modules over a fixed p-group up to some equivalence relation form a group, whose structure was completely determined by Carlson and Thévenaz, and which is non trivial free abelian in most cases [15].

Since the statement of the result might be technical for the non specialist reader, we illustrate the result in a coarser form.

Recall that we have the following rough classification of algebras up to derived equivalences.

	dihedral	semidihedral	quaternion
1 simple	five types of algebras depending on parameters	two types of algebras depending on parameters	two types of algebras depending on parameters
2 simples	one type of algebras depending on parameters	two types of algebras depending on parameters	one type of algebras depending on parameters
3 simples	two types of algebras depending on parameters	one type of algebras depending on parameters	two types of algebras depending on parameters

Theorem 6.4 below states mainly that the columns are preserved under stable equivalences of the Morita type and that the rows are preserved under stable equivalences of the Morita type.

The actual statement is finer than this, but this scheme gives a relatively good approximation of what is proven in Theorem 6.4.

The details we obtain are given in the following result.

Theorem 6.4. (*Zhou and Zimmermann* [52, Theorem 7.1]) *Let K be an algebraically closed field.*

Suppose that A and B are indecomposable algebras which are stably equivalent of the Morita type.

- *If A is an algebra of dihedral type, then B is of dihedral type. If A is of semidihedral type, then B is of semidihedral type. If A is of quaternion type, then B is of quaternion type.*
- *If A and B are of dihedral, semidihedral or quaternion type, then A and B have the same number of simple modules.*
- *Let A be an algebra of dihedral type.*
  (1) *If A is local, then A is stably equivalent of the Morita type to one and exactly one algebra in the following list:*
    - $A_1(n,m)$ *with* $m \geq n \geq 2$ *and* $m+n > 4$*;*
    - C_1*;*
    - $D(1\mathcal{A})_1^k$ *with* $k \geq 2$*;*
    - *if* $p = 2$, B_1 *and* $D(1\mathcal{A})_2^k(d)$ *with* $k \geq 2$ *and* $d \in \{0,1\}$, *except that we do not know whether* $D(1\mathcal{A})^k(0)$ *and* $D(1\mathcal{A})^k(1)$ *are stably equivalent of the Morita type or not.*

(2) *If A has two simple modules, then A is stably equivalent of the Morita type to one and exactly one of the following algebras: $D(2\mathcal{B})^{k,s}(0)$ with $k \geq s \geq 1$ or if $p = 2$, $D(2\mathcal{B})^{k,s}(1)$ with $k \geq s \geq 1$.*

(3) *If A has three simple modules, then A is stably equivalent of the Morita type to one and exactly one of the following algebras: $D(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$ or $D(3\mathcal{R})^{k,s,t,u}$ with $s \geq t \geq u \geq k \geq 1$ and $t \geq 2$.*

- *Let A be an algebra of semidihedral type.*
  (1) *If A has one simple module, then A is stably equivalent of the Morita type to one of the following algebras: $SD(1\mathcal{A})_1^k$, for $k \geq 2$ or $SD(1\mathcal{A})_2^k(c,d)$, for $k \geq 2$ and $(c,d) \neq (0,0)$ if the characteristic of K is 2. Different parameters k yield algebras in different stable equivalence classes of the Morita type.*
  (2) *If A has two simple modules, then A is stably equivalent of the Morita type to $SD(2\mathcal{B})_1^{k,s}(c)$, for $k \geq 1, s \geq 2, c \in \{0,1\}$ or to $SD(2\mathcal{B})_2^{k,s}(c)$, for $k \geq 1, s \geq 2, c \in \{0,1\}, k+s \geq 4$.*
  (3) *If A has three simple modules, then A is stably equivalent of the Morita type to one and only one algebra of the type $SD(3\mathcal{K})^{a,b,c}$, for $a \geq b \geq c \geq 1$.*
- *Let A be an algebra of quaternion type.*
  (1) *If A has one simple module, then A is stably equivalent of the Morita type to one of the algebras $Q(1\mathcal{A})_1^k$, for $k \geq 2$ or $Q(1\mathcal{A})_2^k(c,d)$, for $k \geq 2, (c,d) \neq (0,0)$ if characteristic of the K is 2. Different parameters k yield algebras in different stable equivalence classes of the Morita type.*
  (2) *If A has two simple modules, then A is stably equivalent of the Morita type to one of the algebras $Q(2\mathcal{B})_1^{k,s}(a,c)$, for $k \geq 1, s \geq 3, a \neq 0$.*
  (3) *If A has three simple modules, then A is stably equivalent of the Morita type to one of the algebras $Q(3\mathcal{K})^{a,b,c}$, for $a \geq b \geq c \geq 1, b \geq 2, (a,b,c) \neq (2,2,1)$ or $Q(3\mathcal{A})_1^{2,2}(d)$, for $d \in K \setminus \{0,1\}$. Different parameters a,b,c yield algebras in different stable equivalence classes of the Morita type.*

One particularly nice consequence should be mentioned though.

Corollary 6.5. (*Zhou and Zimmermann* [52, Corollary 7.3]) *The Auslander -Reiten conjecture 5.3 is true for algebras of dihedral, semidihedral or quaternion type.*

6.2. **Białkowski-Erdmann-Skowroński deformation of preprojective algebras.** Recently, Białkowski, Erdmann and Skowroński classified in [5] all selfinjective algebras with the property that for all simple modules S the third syzygy of S is again isomorphic to S. A recent survey on the circle around these questions was given by Erdmann and Skowroński in [17].

The problem of classifying algebras so that $\Omega^2(S) \simeq S$, for all simple modules, was completely solved before and the next most interesting case is $\Omega^3(S) \simeq S$, for all simple modules. In order to formulate the result of Białkowski, Erdmann and Skowroński [5], we need to introduce deformed preprojective algebras as defined in [5].

The preprojective algebra of type A_n is given by the quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left, "a_1"] & \bullet \arrow[l, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & \bullet \arrow[l, shift left, "\overline{a}_2"] \arrow[r, shift left, "a_3"] & \bullet \arrow[l, shift left, "\overline{a}_3"] \arrow[r, shift left, "a_4"] & \cdots\cdots \arrow[l, shift left, "\overline{a}_4"] \arrow[r, shift left, "a_{n-1}"] & \bullet \arrow[l, shift left, "\overline{a}_{n-1}"] \end{tikzcd}$$

subject to the relations

$$a_1\overline{a}_1 = \overline{a}_{n-1}a_{n-1} = 0 \text{ and } \overline{a}_i a_i = a_{i+1}\overline{a}_{i+1}, \forall\ i \in \{1, 2, \ldots, n-2\}.$$

The deformed preprojective algebra of type D_{n+1} is given by the quiver

$$\begin{tikzcd} \bullet \arrow[dr, shift right, "a_0"'] & & & & & \\ & \bullet \arrow[ul, shift right, "\overline{a}_0"'] \arrow[dl, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & \bullet \arrow[l, shift left, "\overline{a}_2"] \arrow[r, shift left, "a_3"] & \bullet \arrow[l, shift left, "\overline{a}_3"] \arrow[r, shift left, "a_4"] & \cdots\cdots \arrow[l, shift left, "\overline{a}_4"] \arrow[r, shift left, "a_{n-1}"] & \bullet \arrow[l, shift left, "\overline{a}_{n-1}"] \\ \bullet \arrow[ur, shift left, "a_1"] & & & & & \end{tikzcd}$$

subject to the relations

$$\begin{aligned} a_0\overline{a}_0 &= a_1\overline{a}_1 = \overline{a}_{n-1}a_{n-1} = \overline{a}_1 a_1 + \overline{a}_0 a_0 + a_2\overline{a}_2 + f(\overline{a}_0 a_0, \overline{a}_1 a_1) \\ &= (\overline{a}_1 a_1 + \overline{a}_0 a_0)^{n-2} = 0 \end{aligned}$$

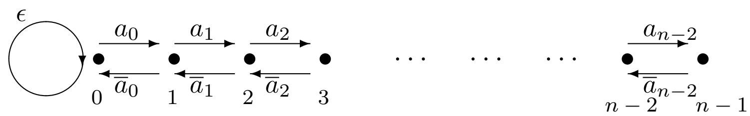
and

$$\overline{a}_i a_i = a_{i+1}\overline{a}_{i+1} \forall\ i \in \{2, \ldots, n-2\},$$

for an element

$$f(X,Y) \in rad^2(K < X,Y > /(X^2, Y^2, (X+Y)^{n-1}).$$

The algebra of type L_n, for $n \geq 2$, was already displayed in Example 3.19 and is given by the quiver

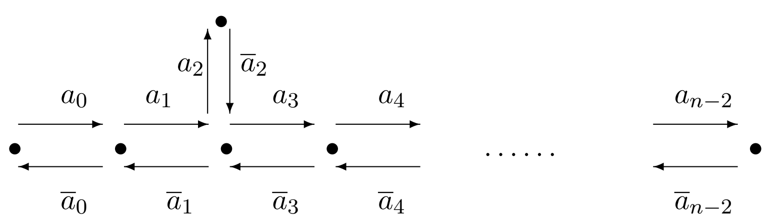


subject to the relations

$$a_i\overline{a}_i + \overline{a}_{i-1}a_{i-1} = 0 \text{ for all } i \in \{1, \dots, n-2\} \ ,$$
$$\overline{a}_{n-2}a_{n-2} = 0 \ , \ \epsilon^{2n} = 0 \ , \ \epsilon^2 + a_0\overline{a}_0 + \epsilon^3 p(\epsilon) = 0$$

for a polynomial $p(X) \in K[X]$. Denote by L_n^p the deformed preprojective algebra of type L with deformation polynomial $p(X)$ and abbreviate $L_n^j := L_n^{X^{2j}}$, for simplicity, when no confusion may occur.

The deformed preprojective algebra of type E_n for $n \in \{6, 7, 8\}$ is given by the quiver



subject to the relations

$$a_0\overline{a}_0 = \overline{a}_{n-2}a_{n-2} = \overline{a}_2a_2 = 0, \ \ \overline{a}_ia_i = a_{i+1}\overline{a}_{i+1} \forall \ i \in \{5, \dots, n-2\}$$
$$\overline{a}_1a_1 + a_2\overline{a}_2 + a_3\overline{a}_3 + f(\overline{a}_1a_1, a_2\overline{a}_2) = (\overline{a}_1a_1 + a_2\overline{a}_2)^{n-3} = 0$$

for

$$f \in rad^2(K < X,Y > /(X^3, Y^2, (X+Y)^{n-3})$$

so that

$$(X + Y + f(X,Y))^{n-3} = 0.$$

For all deformed preprojective algebras, we number the vertices by the condition that the vertex a_i starts at v_i and ends at a vertex of higher label. This convention numbers the vertices in a unique way.

Remark 6.6. *Observe that in* [17, page 238]*, for type E only "admissible deformations" may be applied, which is the condition that $(X+Y+f(X,Y))^{n-3}=0$. However, one relation is missing in* [17]*, for type D and type E, whereas the relation is correctly displayed in* [5]*. I am grateful to Karin Erdmann for a clarifying email on this subject.*

The result is the following.

Theorem 6.7. (*Białkowski, Erdmann and Skowroński* [5, Theorem 1.2]) *Let A be a finite dimensional selfinjective K-algebra. Then, $\Omega^3(S)\simeq S$, for every simple A-module, if and only if A is preprojective of type A_n, for $n\geq 1$, or deformed preprojective of type D_n^f, for $n\geq 4$, E_6^f, E_7^f, E_8^f or L_n^p, for $n\geq 1$.*

It is a non trivial task to determine when deformations f actually lead to non isomorphic algebras. In a lecture at the ICRA XIV conference in Tokyo in August 2010, Białkowski annonced that in characteristic 2 the deformed preprojective algebras of type $L_n^{X^{2j}}$, for $j\in\{0,1,\dots,n-1\}$, form a complete set of the Morita equivalence classes of these algebras, a fact that Skowroński pointed out in an email to the author on March 2007. Skowroński announced in another email to the author in October 2008 that all algebras L_n^p are symmetric and that moreover, in characteristic different from 2 the algebra L_n^p is Morita equivalent to $L_n^{X^{n-1}}$, the non deformed preprojective algebra of type L_n. The content of Białkowski's ICRA lecture are available in the conference abstract volume.

For type D_n^f, Białkowski, Erdmann and Skowroński [5, Proposition 6.2] show that the algebras $D_n^{(XY)^j}$ are not Morita equivalent for different values of j.

No statement is known for type E preprojective algebras.

In a joint work with Holm, we computed the Külshammer ideals for the algebras $L_n^{X^{2j}}$. One main difficulty was to determine the commutator subspace. It is not very difficult to get a generating set for the quotient of the algebra modulo the commutator space, but it is much more complicated to prove that the commutators one found really generate the commutator space. In order to do so, we apply the method described in Section 3.3 and Section 3.4.

A first step is the following lemma.

Lemma 6.8. [24, Lemma 3.9] *Let K be any field. Then, $L^p_{n+1}/[L^p_{n+1}, L^p_{n+1}]$ has a K-linear generating set*

$$\{e_0, e_1, \ldots, e_n, \epsilon, \epsilon^3, \epsilon^5, \epsilon^7, \ldots, \epsilon^{2n+1}\}.$$

We first need to fix a basis.

Proposition 6.9. [24, Proposition 3.1] *Let K be any field. A K-basis of L^p_n is given by the following paths between the vertices i and j, where $i, j \in \{0, 1, \ldots, n-1\}$.*

(1) $a_i a_{i+1} \ldots a_{j-1}$	*for* $i < j$
(2) $a_i a_{i+1} \ldots a_{j-1} a_j \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_j$	*for* $i < j$ *and some* $j \leq \ell \leq n-2$
(3) $\overline{a}_{i-1} \overline{a}_{i-2} \ldots \overline{a}_j$	*for* $i \geq j$
(4) $a_i a_{i+1} \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_i \ldots \overline{a}_j$	*for* $i \geq j$ *and some* $i \leq \ell \leq n-2$
(5) $\overline{a}_{i-1} \overline{a}_{i-2} \ldots \overline{a}_0 \epsilon a_0 a_1 \ldots a_{j-1}$	*for any* i, j
(6) $\overline{a}_{i-1} \overline{a}_{i-2} \ldots \overline{a}_0 \epsilon a_0 a_1 \ldots a_{\ell-1} a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_j$	*for* $i < j$ *and some* $j \leq \ell \leq n-2$
(7) $a_i a_{i+1} \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_1 \overline{a}_0 \epsilon a_0 a_1 \ldots a_{j-1}$	*for* $i \geq j$ *and some* $i \leq \ell \leq n-2$.

Now, we may define a Frobenius form with respect to this basis using Proposition 3.15. It turns out that this form is in fact symmetric, non degenerate and associative.

Theorem 6.10. [24, Theorem 3.5] *Let K be any field. Then, the algebra L^p_n is symmetric.*

We get many elements in the center of L^p_n by following lemma.

Lemma 6.11. [24, Lemma 3.13] *Let K be any field. Then,*

$$Z(L^p_n) \ni \epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{j=0}^{n-2} (-1)^{j+1} \overline{a}_j a_j.$$

Hence,

$$\left\{ \left(\epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{j=0}^{n-2} (-1)^{j+1} \overline{a}_j a_j \right)^\ell \;\middle|\; \ell \in \{0, 1, \ldots, n-1\} \right\} \subseteq Z(L^p_n)$$

is a K-free subset. Moreover, $soc(Z(L_n^p)) \subseteq Z(L_n^p)$.

Lemma 6.11 provides a large space in $Z(L_n^p)$ of dimension $2n$. Lemma 6.8 shows that $L_n^p/[L_n^p, L_n^p]$ has dimension at most $2n$, whereas Proposition 3.12 shows that the two vector spaces are isomorphic. Hence, the elements displayed in Lemma 6.8 form a basis of $L_n^p/[L_n^p, L_n^p]$.

Using this result, it is then possible to show the following result.

Theorem 6.12. *(Holm and Zimmermann* [24, Theorem 4.1]*) Let K be a perfect field of characteristic 2. Then, for $0 \leq j < n$, we get*

$$\dim\left(T_i(L_n^{X^{2j}})\right) - \dim\left([L_n^{X^{2j}}, L_n^{X^{2j}}]\right) = n - \max\left(\left\lceil \frac{2n - (2^{i+1}-2)j - (2^{i+1}-1)}{2^{i+1}} \right\rceil, \ 0\right).$$

The attentive reader may remark that for the theorem one needs to suppose that the field K is perfect, whereas this was not supposed in the auxiliary steps. The reason for this assumption comes from the technicalities in the proof of Theorem 6.12. We need to find elements x so that $x^{2^n} \in [L_n^p, L_n^p]$. By what preceded, and rather easy arguments, it is necessary to consider this question only for x being a K-linear combination of odd powers of ϵ. Now, the 2^n powers of x will give an element which is expressed in 2^n th powers of the original coefficients, and from there it is not hard to imagine that one needs to take 2^n th roots of the solutions, in order to get the original coefficients from some expression one obtains from some solution one got by a linear algebra argument.

Moreover, we remark that the Morita equivalence classification of the algebras L_n^p is finer than what can be done by the Külshammer ideals. However, the Külshammer ideals distinguish algebras up to derived equivalences and even up to stable equivalences of the Morita type (cf. Theorem 5.13).

6.3. **Algebras of polynomial growth and domestic weakly symmetric algebras.** Let K be an algebraically closed field and let A be a finite dimensional K-algebra.

- The algebra A is called to be of *finite representation type* if A admits only a finite number of indecomposable A-modules up to isomorphism.
- The algebra A is called to be of *tame representation type* if A is not of finite representation type and if for every positive integer

d there are a finite number of $A \otimes_K K[X]$-modules $M_1(d), M_2(d), \ldots, M_{n(d)}(d)$, which are free as $K[X]$-modules and so that for each d, all but a finite number of d-dimensional indecomposable A-modules are isomorphic to a module of the form $M_i(d) \otimes_K K[X]/\ (X - \lambda)$, for some $\lambda \in K$ and some $i \in \{1, \ldots, n(d)\}$.

- The tame algebra A is called to be of *domestic representation type* if, taking $n(d)$ as small as possible, there is an integer m so that $n(d) \leq m$, for all d.
- The tame algebra A is called to be of *polynomial growth* if, taking $n(d)$ as small as possible, there is an integer m so that

$$\lim_{n \to \infty} \frac{n(d)}{d^m} = 0.$$

- The algebra A is called to be of *wild representation type* if for every algebra B there is a functor $B-mod \longrightarrow A-mod$ which is exact, preserves isomorphism classes and carries indecomposable objects to indecomposable objects.

Of course, tame domestic algebras are polynomial growth tame algebras.

A fundamental result of Drozd says that A is either tame or wild or of finite representation type. There is intensive research aiming at a possible classification of algebras of tame representation type, though the goal seems still to be very far. Nevertheless, Bocian, Holm and Skowroński classified tame domestic weakly symmetric algebras [9, 10, 21] and tame weakly symmetric algebras of polynomial growth [6, 7, 22] up to derived equivalences in a series of papers.

We present some of the details.

Definition 6.13. *A selfinjective algebra of tame representation type is called standard if its basic algebra admits simply connected Galois coverings. Otherwise, a selfinjective algebra is called non standard.*

Theorem 6.14. [9, Theorem 1] *For an algebra A, the following statements are equivalent:*

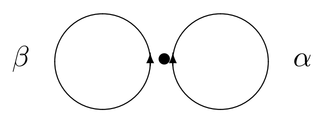
(1) *A is a representation-infinite domestic selfinjective algebra having simply connected Galois coverings and the Cartan matrix C_A is singular.*
(2) *A is derived equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*

(3) *A is stably equivalent to the trivial extension $T(C)$ of a canonical algebra C of Euclidean type.*

Moreover, the trivial extensions $T(C)$ and $T(C')$ of two canonical algebras C and C' of Euclidean type are derived equivalent (respectively, stably equivalent) if and only if the algebras C and C' are isomorphic.

In order to be able to formulate the Bocian, Holm and Skowroński's result for weakly symmetric algebras of domestic representation type with nonsingular Cartan matrices, we need to define the following algebras.

$A(\lambda)$
$\lambda \in K\backslash\{0\}$

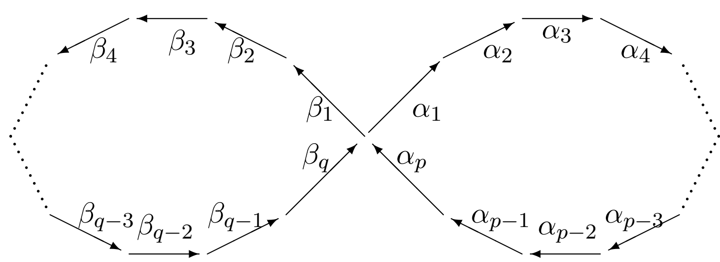


$\alpha^2 = 0, \beta^2 = 0, \alpha\beta = \lambda\beta\alpha$

$A(p,q)$

$1 \leq p \leq q$

$p + q \geq 3$



$$\alpha_1\alpha_2\cdots\alpha_p\beta_1\beta_2\cdots\beta_q = \beta_1\beta_2\cdots\beta_q\alpha_1\alpha_2\cdots\alpha_p$$

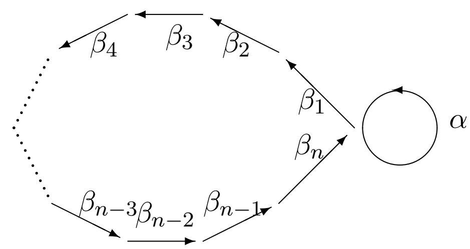
$$\alpha_p\alpha_1 = 0, \beta_q\beta_1 = 0$$

$$\alpha_i\alpha_{i+1}\cdots\alpha_p\beta_1\cdots\beta_q\alpha_1\cdots\alpha_{i-1}\alpha_i = 0, 2 \leq i \leq p$$

$$\beta_j\beta_{j+1}\cdots\beta_q\alpha_1\cdots\alpha_p\alpha_1\cdots\beta_{i-1}\beta_i = 0, 2 \leq j \leq q$$

$\Lambda(n)$

$n \geq 2$

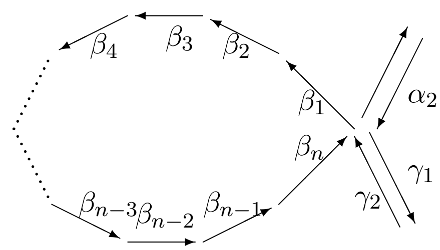


$$\alpha^2 = (\beta_1\beta_2\cdots\beta_n)^2, \alpha\beta_1 = 0, \beta_n\alpha = 0$$

$$\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\alpha_1\cdots\beta_{i-1}\beta_i = 0, 2 \leq j \leq n$$

$\Gamma(n)$

$n \geq 1$



$$\alpha_1\alpha_2 = (\beta_1\beta_2\cdots\beta_n)^2 = \gamma_1\gamma_2,$$

$$\alpha_2\beta_1 = 0, \gamma_2\beta_1 = 0, \beta_n\alpha_1 = 0$$

$$\beta_n\gamma_1 = 0, \alpha_2\gamma_1 = 0, \gamma_2\alpha_1 = 0$$

$$\beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\alpha_1\cdots\beta_{i-1}\beta_i = 0, 2 \leq j \leq n.$$

Theorem 6.15. (*Bocian, Holm and Skowroński* [9, Theorem 2]) *For a domestic standard selfinjective algebra A, the following statements are equivalent:*

(1) *A is weakly symmetric and the Cartan matrix C_A is nonsingular.*

(2) *A is derived equivalent to an algebra of the form* $A(\lambda), A(p,q), \Lambda(n), \Gamma(n)$.

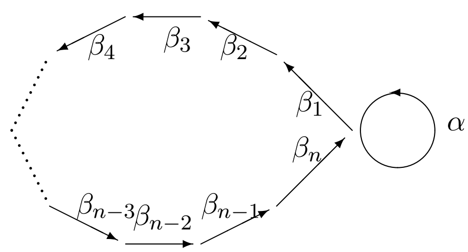
(3) *A is stably equivalent to an algebra of the form* $A(\lambda), A(p,q), \Lambda(n), \Gamma(n)$.

Moreover, two algebras of the forms $A(\lambda), A(p,q), \Lambda(n), \Gamma(n)$ *are derived equivalent (respectively, stably equivalent) if and only if they are isomorphic.*

For algebras of non standard type, we need to introduce the following algebra $\Omega(n)$. The quiver with relations of $\Omega(n)$ is as follows.

$\Omega(n)$

$n \geq 1$



$$\alpha^2 = \alpha\beta_1\beta_2\cdots\beta_n, \alpha\beta_1\beta_2\cdots\beta_n + \beta_1\beta_2\cdots\beta_n\alpha = 0,$$

$$\beta_n\beta_1 = 0, \beta_j\beta_{j+1}\cdots\beta_n\beta_1\cdots\beta_n\alpha_1\cdots\beta_{i-1}\beta_i = 0, 2 \leq j \leq n.$$

Theorem 6.16. (*Bocian, Holm and Skowroński* [10, Theorem 1]) *Any nonstandard representation infinite selfinjective algebra of domestic type is derived equivalent (respectively stably equivalent) to an algebra* $\Omega(n)$ *with* $n \geq 1$. *Moreover, two algebras* $\Omega(n)$ *and* $\Omega(m)$ *are derived equivalent* (*respectively, stably equivalent*) *if and only if* $m = n$.

Bocian, Holm and Skowroński showed that standard and non standard domestic algebras could not be derived equivalent. We are able to improve the result partially.

Lemma 6.17. (*Zhou and Zimmermann* [53, Lemma 2.3]) *A standard weakly symmetric algebra of domestic type cannot be stably equivalent to a nonstandard one.*

The method of proof is to compare stable the Auslander-Reiten quivers and to use the fact that the class of special biserial algebras is closed under stable equivalences, a result due to Pogorzały [42, Theorem 7.3].

Observe that a Morita equivalence classification of standard selfinjective domestic algebras is not given in the results of Bocian, Holm and Skowroński. This is the reason why we suppose that the domestic algebras are weakly symmetric and not only selfinjective.

We show the following.

Theorem 6.18. (*Zhou and Zimmermann* [53, Theorem 2.5])

(1) *Two weakly symmetric algebras of domestic representation type are derived equivalent if and only if they are stably equivalent.*
(2) *The class of weakly symmetric algebras of domestic representation type is closed under stable equivalences.*

Bocian, Holm and Skowroński give a classification of symmetric algebras of polynomial growth up to derived equivalences [6], [7], [22]. They get a finite list of algebras which are defined by quivers and relations, and where the relations involve some parameters. We are not completely able to distinguish the algebras with the same quiver and relations and different parameters. We call this problem the scalar problem.

As for symmetric algebras of polynomial growth, we get the following result.

Theorem 6.19. (*Zhou and Zimmermann* [53, Theorem 3.5]) *The classification of indecomposable non-domestic weakly symmetric algebras of polynomial growth up to stable equivalences of the Morita type coincide with the derived equivalence classification, modulo the scalar problems.*

The method of proof uses, among other general arguments, a computation of the Külshammer ideals.

Concerning the Auslander-Reiten conjecture we get the following result.

Corollary 6.20. (*Zhou and Zimmermann* [53, Corollary 2.7, Theorem 3.6, Theorem 3.7])

- *Let A be an indecomposable algebra stably equivalent to an indecomposable symmetric algebra B of domestic type. Then, A and B have the same number of isomorphism classes of simple modules.*
- *Let A be an indecomposable algebra stably equivalent to an indecomposable weakly symmetric standard algebra B of domestic type. Then, A and B have the same number of isomorphism classes of simple modules.*

- *Let A and B be indecomposable algebras which are stably equivalent of the Morita type. If A is tame symmetric with only Ω-periodic modules, then A and B have the same number of isomorphism classes of simple modules.*
- *Let A be an indecomposable algebra and suppose that A is stably equivalent of the Morita type to an indecomposable non-domestic symmetric algebra B of polynomial growth. Then, A and B have the same number of isomorphism classes of simple modules.*
- *Let A and B be two indecomposable algebras which are standard non-domestic weakly symmetric algebra of polynomial growth or non-standard non-domestic selfinjective algebra of polynomial growth. If they are stably equivalent of the Morita type, then A and B have the same number of isomorphism classes of simple modules.*

7. **Hochschild's (co-)homology**

It is well-known (cf. e.g., Loday [37, Paragraphs 1.1.6 and 1.5.2]), and actually we already used this fact implicitly in Section 3.3 and Section 5.3, that $A/[A,A] = HH_0(A)$ is the degree 0 Hochschild's homology and $Z(A) = HH^0(A)$ is the degree 0 Hochschild's cohomology. A natural question becomes now to generalise Külshammer's ideals to higher Hochschild's (co-)homology. This was done by the author in [57] and [58].

The symmetrising form on A induces a non degenerate pairing $HH^0(A) \times HH_0(A) \longrightarrow K$. Hence, in order to generalize to higher Hochschild's (co-)homology, we first need to produce a non degenerate bilinear form,

$$HH^n(A) \times HH_n(A) \longrightarrow K,$$

for symmetric algebras induced by the symmetrising form. Let $\mathbb{B}A$ be the bar resolution (i.e., a specific projective resolution; and actually projective resolution will do at this place) of A as $A \otimes_K A^{op}$-modules:

$$\begin{aligned} Hom_K(HH_n(A),K) &= Hom_K(H_n(\mathbb{B}A \otimes_{A\otimes A^{op}} A),K) \\ &= H^n(Hom_K(\mathbb{B}A \otimes_{A\otimes A^{op}} A,K)) \\ &= H^n(Hom_{A\otimes_K A^{op}}(\mathbb{B}A, Hom_K(A,K)) \\ &\simeq H^n(Hom_{A\otimes_K A^{op}}(\mathbb{B}A,A)) \\ &= HH^n(A), \end{aligned}$$

where the second last isomorphism is induced by the symmetrising form $A \simeq Hom_K(A,K)$, as $A \otimes_K A^{op}$ modules.

This isomorphism yields a non degenerate bilinear form,

$$\langle\ ,\ \rangle_n : HH^n(A) \times HH_n(A) \longrightarrow K,$$

which extends the symmetrising form on A.

In order to define Külshammer's ideals, we used the p-power map on $HH_0(A) = A/[A,A]$. However, how to get a p-power map on $HH_*(A)$ is not completely clear. Nevertheless, the multiplicative structure on $HH^*(A)$, defined by the cup product, can be used instead. By adjointness with respect to the bilinear form, we then get an analogue of the Külshammer ideal structure on Hochschild's homology instead of cohomology. This dual construction on $A/[A,A] = HH_0(A)$ was studied by Külshammer [33] as well.

For the p^n-power mapping by the cup product $HH^m(A,A) \longrightarrow HH^{p^n m}(A,A)$, one gets a right adjoint, $\kappa_n^{(m),A} : HH_{p^n m}(A,A) \longrightarrow HH_m(A,A)$, with respect to $\langle\ ,\ \rangle_m$ and $\langle\ ,\ \rangle_{p^n m}$. Observe that if p is odd, then the cup product square is 0 in odd degree cohomology. Hence, for p odd, the p-power map as well as the adjoint is 0 in odd degree cohomology.

Proposition 7.1. [57, Lemma 2.6, the beginning remarks of Section 3] *Let A be a finite dimensional symmetric K algebra over a perfect field K of characteristic $p > 0$. Then, for all $n \in \mathbb{N}$ and for all $x \in HH_{p^n m}(A,A)$, there is a unique $\kappa_n^{(m)}(x) \in HH_m(A,A)$ so that for all $f \in HH^m(A,A)$, one has*

$$\left\langle f^{p^n}, x \right\rangle_{p^n m} = \left(\left\langle f, \kappa_n^{(m)}(x) \right\rangle_m \right)^{p^n}.$$

Using the mapping $\kappa_n^{(m)}$, one gets the compatibility with derived equivalences.

Theorem 7.2. [57, Theorem 1] *Let A be a finite dimensional symmetric K-algebra over a perfect field K of characteristic $p > 0$. Let B be a second algebra so that $D^b(A) \simeq D^b(B)$, as triangulated categories. Let p be a prime and let $m \in \mathbb{N}$. Then, there is a standard equivalence $F : D^b(A) \simeq D^b(B)$, and any such standard equivalence induces an isomorphism $HH_m(F) : HH_m(A,A) \longrightarrow HH_m(B,B)$ of all Hochschild's homology groups satisfying*

$$HH_m(F) \circ \kappa_n^{(m),A} \circ HH_{p^n m}(F)^{-1} = \kappa_n^{(m),B}.$$

The proof is very much like the in degree 0 for Hochschild's cohomology. Nevertheless, a clear definition of an isomorphism on Hochschild's homology induced by a standard derived equivalence was not published explicitly before. The construction was somewhat implicit in Rickard's work, but an explicit clarification seems to appear in [57, Section 1.2] for the first time.

One should notice however that a derived equivalence may be non standard, and then it is not clear how to define an induced mapping on the category of bimodules, and in the sequel on the Hochschild's homology. One needs the standard equivalence in order to control the way it acts on Hochschild's homology.

What happens if A is not symmetric? Already for the non degenerate pairing between Hochschild's homology and cohomology it is not clear how to to define it properly. We may again use trivial extension algebras. Then, there are ring homomorphisms $A \longrightarrow \mathbb{T}A$ and $\mathbb{T}A \longrightarrow A$ by projection, injection from and to the second component. Hochschild's homology is functorial, contrary to Hochschild's cohomology. Hence, we get mappings

$$HH_n(\iota_A) : HH_n(A) \longrightarrow HH_n(\mathbb{T}A)$$

and

$$HH_n(\pi_A) : HH_n(\mathbb{T}A) \longrightarrow HH_n(A).$$

Now, defining

$$\hat{\kappa}_n^{(m)} := HH_m(\pi_A) \circ \kappa_n^{(m)} \circ HH_{p^n m}(\iota_A) : HH_{p^n m}(A) \longrightarrow HH_m(A),$$

one obtains an invariant under derived equivalences.

Theorem 7.3. [58, Theorem 2] *Let K be a perfect field of characteristic $p > 0$, A and B be finite dimensional K-algebras and suppose that $D^b(A) \simeq D^b(B)$ as triangulated categories. Let F be an explicit standard equivalence between $D^b(A)$ and $D^b(B)$. Then, F induces a sequence of isomorphisms $HH_m(F) : HH_m(A) \longrightarrow HH_m(B)$ so that*

$$HH_m(F) \circ \hat{\kappa}_n^{(m);A} = \hat{\kappa}_n^{(m);B} \circ HH_{p^n m}(F).$$

Obviously, Theorem 7.3 generalizes Theorem 7.2 to non symmetric algebras. Since Hochschild's homology is often better understood than Hochschild's cohomology, we expect that this generalization will bear use in future.

Nevertheless, there is a p-power map available in some cases coming from the Gerstenhaber's structure on Hochschild's homology. The construction is due to Stasheff and Quillen.

Let

$$\begin{aligned} Coder(\mathbb{B}(A), \mathbb{B}(A)) := \{D \in End_{A\otimes A^{op}}(\mathbb{B}(A)) | \Delta \circ D \\ = (id_{\mathbb{B}(A)} \otimes D + D \otimes id_{\mathbb{B}(A)}) \circ \Delta\} \end{aligned}$$

be the coderivations. Since $\mathbb{B}(A)$ is graded, $Coder(\mathbb{B}(A), \mathbb{B}(A))$ is graded as well. Denote by $Coder^n(\mathbb{B}(A), \mathbb{B}(A))$, the degree n coderivations. The vector space $Coder(\mathbb{B}(A), \mathbb{B}(A))$ is a graded Lie algebra with Lie bracket being the commutator. Moreover, (cf. Stasheff [48, Proposition]),

$$Coder(\mathbb{B}(A), \mathbb{B}(A)) \simeq Hom_{A\otimes A^{op}}(\mathbb{B}(A), A)[1]\ .$$

The key observation is the following.

Lemma 7.4. [57, Lemma 4.1] (*Keller, personal communications*)

- *Suppose K is a field. Then,*

  $D \in Coder^{2n+1}(\mathbb{B}(A), \mathbb{B}(A)) \Rightarrow D^2 \in Coder^{2\cdot(2n+1)}(\mathbb{B}(A), \mathbb{B}(A))$.
- *Suppose K is a field of characteristic $p > 0$. Then,*

  $D \in Coder^{2n}(\mathbb{B}(A), \mathbb{B}(A)) \Rightarrow D^p \in Coder^{2pn}(\mathbb{B}(A), \mathbb{B}(A))$.

This p-power structure carries over to Hochschild's cohomology.

Lemma 7.5. [57, Lemma 4.2] *Let K be a field of characteristic $p > 0$ and $D \in Coder^n(\mathbb{B}(A), \mathbb{B}(A))$.*

(1) *If $p = 2$ and $n \in \mathbb{N}$, then the mapping $D \mapsto D^2$ induces a mapping*

$$HH^{n+1}(A, A) \longrightarrow HH^{2n+1}(A, A).$$

(2) *If $p > 2$ and $n = 2m \in 2\mathbb{N}$, then the mapping $D \mapsto D^p$ induces a mapping*

$$HH^{2m+1}(A, A) \longrightarrow HH^{2pm+1}(A, A).$$

Hence, for $p = 2$, the Hochschild's cohomology becomes a 2-restricted Lie algebra with the Gerstenhaber's bracket and the 2-power map. For $p > 2$, the odd degree Hochschild's cohomology becomes a p-restricted Lie algebra with the Gerstenhaber's bracket and the p-power mapping.

Using these constructions we get the following result.

Theorem 7.6. [57, Proposition 4.4] *Let A and B be K-algebras over a field K. Suppose $D^b(A) \simeq D^b(B)$, as triangulated categories.*

- *If the characteristic of K is 2, then $HH^*(A, A)$ and $HH^*(B, B)$ are isomorphic, as restricted Lie algebras.*

- *If the characteristic of K is $p > 2$, then the Lie algebras consisting of odd degree Hochschild cohomologies $\bigoplus_{n\in\mathbb{N}} HH^{2n+1}(A, A)$ and $\bigoplus_{n\in\mathbb{N}} HH^{2n+1}(B, B)$ are isomorphic, as restricted Lie algebras.*

However, we fail to prove that the so-defined Gerstenhaber's p-power map is additive, neither semilinear. Hence, it does not seem to be clear how a Külshammer's structure could be built from there.

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