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BRANDT EXTENSIONS AND PRIMITIVE TOPOLOGICALLY PERIODIC INVERSE TOPOLOGICAL SEMIGROUPS

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Abstract. We find sufficient conditions on primitive inverse topological semigroup S under which the inversion $inv : (H(S))^* \to (H(S))^*$ is continuous; we show that every topologically periodic countable compact primitive inverse topological semigroups with closed H-classes is topologically isomorphic to an orthogonal sum $\sum_{i\in\Im} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups, for some finite cardinals $\lambda_i \geq 1$.

1. **Introduction**

Here, all topological spaces will be assumed to be Housdorff. We shall follow the terminology of [6, 7, 9, 20]. By ω, we means the first infinite ordinal. If Y is a subspace of a topological space X and $A \subset Y$, then by $cl_Y(A)$, we denote the topological closure of A in Y. A semigroup is a set with a binary associative operation. An element e of a semigroup S is called an idempotent, if $ee = e$. If S is a semigroup, then we denote the subset of all idempotents of S by $E(S)$. A semigroup S is called an

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inverse, if for any $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Such an element y is called inverse of x and is denoted by x^{-1}. If S is an inverse semigroup, then the map which takes $x \in S$ to the inverse element of x is called the inversion and will be denoted by inv. An inverse semigroup S is called an inverse Clifford semigroup, if $xx^{-1} = x^{-1}x$, for every $x \in S$. Let S be a semigroup and e be a idempotent element. Then, by $H(e)$, we denote the maximal subgroup in S with identity e. For a semigroup S, let $H(S) = \bigcup\{H(e); e \in E(S)\}$. A semigroup S equipped with a topology is called a topological semigroup, if its multiplication is jointly continuous. A topological inverse semigroup is a topological semigroup S that is algebraically an inverse semigroup with continuous inversion.

Koch and Wallace [18] and independently Kruming [19] proved that each compact inverse topological semigroup has a continuous inverse and thus is a topological inverse semigroup. The Koch-Wallace-Kruming Theorem cannot be extended to locally compact inverse topological semigroups: the space of non-negative real numbers with the Euclidean topology and the usual multiplication is a locally compact inverse topological semigroup but fails to be a topological inverse semigroup.

Also, Grant [10] and Yurèva [21] showed that a Hausdorff cancellative sequential countably compact topological semigroup is a topological group. Bokalo and Guran [5] established that an analogous theorem is true for cancellative sequentially compact semigroups. Banakh and Gutik [3] proved that any sequential countably compact inverse topological semigroup is a topological inverse semigroup. They also showed that if S is a topologically periodic Hausdorff inverse Clifford topological semigroup, then the inverse operation $i : S \to S$ is continuous provided S is inversely regular and countably compact or S is regular and $S \times S$ is countably compact. Gurik et al. [15] showed that for Tychonoff Clifford topological semigroup with the (countably compact) pseudocompact square $S \times S$, inversion is continuous on $H(S)$.

An idempotent that is minimal within the set of non-zero idempotent with a natural partial order by the rule $e \leq f$ if and only if $ef = fe = e$, is called primitive. A semigroup S without zero is called simple, if it has no proper ideals. A semigroup S with zero is called 0-simple, if 0 and S are the only ideals and $S^2 \neq 0$. A (0-)simple semigroup S with a primitive idempotent is called completely (0-)simple. The bicyclic semigroup $C(p, q)$ is the semigroup with the identity 1 generated by two elements p and q, subject only to the condition $pq = 1$. The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example, the well-known Andersen's result [1] states that a (0-) simple semigroup is completely (0-) simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete topology and a topological semigroup S can contain $C(p,q)$ only as an open subset [7]. Neither stable nor Γ-compact topological semigroups can contain a copy of the bicyclic semigroup [2, 16].

Let S be a semigroup and I_λ be a non-empty set of cardinality λ. We define the semigroup operation "." on the set $B_\lambda(S) = I_\lambda \times S^1 \times I_\lambda \cup \{0\}$ as follows:

$$(\alpha, a, \beta).(\gamma, b, \delta) = \begin{cases} (\alpha, ab, \beta), & if \beta = \gamma \\ 0, & if \beta \neq \gamma \end{cases}$$

where $(\alpha, a, \beta).0 = 0.(\alpha, a, \beta) = 0.0 = 0$, for $\alpha, \beta, \gamma, \delta \in I_\lambda$, and $a, b \in S^1$. The semigroup $B_\lambda(S)$ is called a Brandt λ-extension of the semigroup S [11]. Gutik and Pavlyk [13] proved that any compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(G)$ of G in the class of topological inverse semigroups such that G is compact topological group and λ is finite. They also showed that the subsemigroup of idempotents of a compact 0-simple topological inverse semigroup is finite, and hence the topological space of a compact 0-simple topological inverse semigroup is homeomorphic to a finite topological sum of compact topological group and a single point. Gutik and Repovs [12] showed that if S is a countably compact 0-simple topological inverse semigroup, then S has a structure similar to a compact 0-simple topological inverse semigroup. Berezovski et al. [4] proved that every primitive countably compact topological inverse semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i \in \Im} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups, for some finite cardinals $\lambda_i \geq 1$.

Here, we shall find sufficient conditions on inverse topological semigroup S under which the inversion, $inv : (H(S))^* \to (H(S))^*$, is continuous; we will show that every topologically periodic countable compact primitive inverse topological semigroups, with closed H-classes is topologically isomorphic to an orthogonal sum $\sum_{i \in \Im} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups, for some finite cardinals $\lambda_i \geq 1$.

Lemma 1.1. *Let S be a primitive inverse topological semigroup. Then, eSe is a closed subset of S, for each e in $E(S)$.*

Proof. We know that $eSe = \{s \in S | es = se = s\} \cup \{0\}$. Suppose, on the contrary, that $t \in \overline{eSe} \setminus eSe$ such that $et \neq t$ or $te \neq t$.
Suppose $et \neq t$. Let $O(et)$ and $O(t)$ be open neighborhoods of the points et and t, respectively, such that $O(et) \cap O(t) = \emptyset$, since S is a topological semigroup there exist open neighborhoods $O_1(t)$ and $O(e)$ of the points e and t, respectively, such that $O(e)O_1(t) \subset O(et)$. Since $t \in \overline{eSe}$, then $O_1(t) \cap O(t) \cap eSe \neq \emptyset$. Let $s \in O_1(t) \cap O(t) \cap eSe$. Hence, $es = s \in O(et) \cap O(t)$, giving a contradiction. Therefore, eSe is closed. □

2. **Main result**

An element x of a topological semigroup S is called topologically periodic, if for any open neighborhood $U(x)$ of x there exists an integer $n > 1$ such that $x^n \in U(x)$. A topological semigroup S is called topologically periodic, if any element of S is topologically periodic. In paratopological group G, this is equivalent to: for each $x \in G$ and every open neighborhood $U(e)$ of the identity e, there exists an integer such that $x^n \in U(e)$.

Theorem 2.1. *Let S be a topologically periodic countable compact primitive inverse topological semigroup. Then*
(i) the inversion $inv : (H(S))^* \to (H(S))^*$ is continuous;
(ii) the inversion $inv : S \setminus \{0\} \to S \setminus \{0\}$ is continuous.

Proof. First, we prove $inv : H(e) \to H(e)$ is a continuous. Let $U(e)$ be any open neighborhood of e in S. Since the semigroup operation in $H(e)$ is continuous, we can find $\{V_i; i \in \omega\}$, a family of neighborhoods of e, such that $V_0 = U$ and $V_{i+1}^2 \subset V_i$, for each $i \in \omega$. By Lemma 1 of [10], we have $(cl_{H(e)}(V_{i+1}^{-1}))^{-1} \subset V_i$. Since every open subgroup of a topologically periodic semigroup is topologically periodic, by the proof of Theorem 3 of [5] we can prove $F = \bigcap\{cl_{H(e)}(V_{i+1}^{-1}); i \in \omega\} \subset U$. Since eSe is a closed subset of S, by Theorem of 3.10.4 [9], eSe is countable compact, and there exist $i_1, i_2, \cdots, i_k$ such that $\cap\{V_{i_j}^{-1}; j = 1, \cdots, k\} \subset U$. Hence, $inv : H(e) \to H(e)$ is continuous. Now, let $U(x^{-1})$ be an open neighborhood of x^{-1} in $(H(S))^*$. Then, there exists a maximal subgroup $H(e)$ such that $x^{-1} \in H(e)$. Since $H(e) \cap U(x^{-1})$ is open neighborhood of x^{-1} in $H(e)$ with subspace topology and $H(e)$ is a topological

group, there exists an open neighborhood $V(x)$ in $(H(S))^*$ such that $(V(x) \cap H(e))^{-1} \subset U(x^{-1}) \cap H(e) \subset U(x^{-1})$. Since $H(e)$ is an open in $(H(S))^*$, $inv : (H(S))^* \to (H(S))^*$ is continuous.
(ii) By Theorem 2.4.3 of [20], the semigroup S is an orthogonal sum of Brandt semigroups (S is an orthogonal sum $\sum_{i \in \Im} B_{\lambda_i}(G_i)$ of Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of groups G_i). Then, each $0 \neq x^{-1}$ belongs to an H-class $(\alpha, G_{\lambda_i}, \beta)$ such that $\alpha, \beta \in I_{\lambda_i}$. Since H-classes are open in S, it is sufficient to show that

$$\begin{aligned} inv : (\alpha, G_{\lambda_i}, \beta) &\to (\beta, G_{\lambda_i}, \alpha) \\ (\alpha, g, \beta) &\to (\alpha, g^{-1}, \beta) \end{aligned}$$

is continuous. But, G_{λ_i} is topologically isomorphic, $(\alpha, G_{\lambda_i}, \alpha) = H(e)$ and $inv : H(e) \to H(e)$ is continuous, and hence $inv : S \setminus \{0\} \to S \setminus \{0\}$ is continuous. □

Since every completely 0-simple semigroup is a primitive semigroup [20], by Theorem 2.1, we have the following result.

Corollary 2.2. *Let S be a completely 0-simple topologically periodic countable compact inverse topological semigroup. Then*
(i) the inversion $inv : (H(S))^* \to (H(S))^*$ is continuous; and
(ii) the inversion $inv : S \setminus \{0\} \to S \setminus \{0\}$ is continuous.

Theorem 2.3. *Every primitive topologically periodic countably compact inverse topological semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i \in \Im} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of topological groups G_i in the class of inverse topological semigroups, for some finite cardinals $\lambda_i \geq 1$.*

Proof. Since every G_i is topologically isomorphic to $H(e)$, by Theorem 2.1, G_i is a topological group with induced topology. Now, it is sufficient to show that each λ_i is finite. Since inversion is continuous on $S \setminus \{0\}$, we can prove that $B_{\lambda_i}(e_{G_i})$ is a closed subset of S (similar to the proof of Theorem 14 in [12]), and hence, by Theorem 3.10.4 of [9], $B_{\lambda_i}(e_{G_i})$ is a countably compact topological space. Therefore, Theorem 6 of [12] implies that $B_{\lambda_i}(e_{G_i})$ is a finite discrete subsemigroup of S, and hence the set I_{λ_i} is finite. □

Theorem 2.3 implies the following result.

Corollary 2.4. *Every completely* 0*-simple topologically periodic countably compact inverse topological semigroup* S *is topologically isomorphic to* $B_\lambda(G)$ *of topological groups* G_i *in the class of inverse topological semigroups, for some finite cardinals* $\lambda \geq 1$.

We want to extent this continuity on S. We need another condition on the primitive inverse topological semigroups.

Theorem 2.5. *Let* S *be a primitive topologically periodic countable compact inverse topological semigroup with the closed* H*-classes. Then,* $inv : S \to S$ *is continuous.*

Proof. Since $S \setminus \{0\}$ is an open subset of S, by Theorem 2.1, we only need to show that inversion is continuous at 0. Since each $B_{\lambda_i}(G_i)$ is clopen subset of S, we may repeat the argument of the proof of Theorem 13 of [4] and show that the family $\mathbf{B}(0) = \{S \setminus (B_{\lambda_1}(G_{i_1}) \cup B_{i_2}(G_{i_2} \cup \ldots \cup B_{\lambda_n}(G_{i_n}))^* | i_1, i_2, \ldots, i_n \in \Im, n \in \omega\}$ determines a base of the topology at zero 0 of S. Now, by Proposition 1.4.1 of [9], it is sufficient to show that $B^{-1}_{i_1,i_2\ldots,i_n} = (S \setminus (B_{\lambda_1}(G_{i_1}) \cup B_{i_2}(G_{i_2} \cup \ldots \cup B_{\lambda_n}(G_{i_n}))^*)^{-1}$ is open, for finitely many indexes $i_1, i_2, \ldots, i_n \in \Im$. But, $B^{-1}_{i_1,i_2,\ldots,i_n} = B_{i_1,i_2,\ldots,i_n}$ implies that U is open. Therefore, inversion is continuous on S. □

Since inversion is continuous on primitive topologically periodic countably compact inverse topological semigroup S with closed H-classes, Theorem 13 of [4] and Theorem 2.5 give the following result.

Corollary 2.6. *Every primitive topologically periodic countably compact inverse topological semigroup* S *with closed* H*-classes is topologically isomorphic to an orthogonal sum* $\sum_{i\in\Im} B_{\lambda_i}(G_i)$ *of topological Brandt* λ_i*-extensions* $B_{\lambda_i}(G_i)$ *of countably compact topological groups* G *in the class of topological inverse semigroups, for some finite cardinals* $\lambda_i \geq 1$.

Theorem 2.6 gives the following corollary.

Corollary 2.7. *Let* S *be a topologically periodic countable compact completely* 0*-simple inverse topological semigroup with the closed* H*-classes. Then,* $inv : S \to S$ *is continuous.*

Theorem 2.8. *A topologically periodic topological semigroup cannot contain the bicyclic semigroup. Therefore, every (*0*-)simple topologically periodic topological semigroup is completely (*0*-)simple.*

Proof. Suppose, on the contrary, that S contain a copy of $C(p,q)$. Let $q^l p^k$, $l > k$. Then, by Corollary I.2 of [8], the bicyclic semigroup $C(p,q)$ endowed with the topology induced from S is a discrete topological space. So, we can find a neighborhood $O(q^l p^k) \subset S$ containing no other points of the semigroup $C(p,q)$. Thus, by Theorem I.3 of [9], $C(p,q)$ is an open subspace of S, and hence $C(p,q)$ is topologically periodic. Therefore, there exists $n \in \omega$ such that $q^l p^k \neq q^{l+n(l-k)} = (q^l p^k)^{n+1} \in O(q^l p^k)$, which is a contradiction. This contradiction shows that the simple semigroup S contains no copy of $C(p,q)$, and hence is completely (0-)simple by the Andersens Theorem [1, *Theorem* 2.54].

□

Theorem 2.8 implies the following.

Corollary 2.9. *Let S be a topologically periodic countable compact 0-simple inverse topological semigroup with the closed H-classes. Then, $inv : S \to S$ is continuous.*

Since any 0-simple topologically periodic topological semigroup is completely 0-simple topological semigroup, Theorem 2 of [13] and Theorem 2.8 give the following result.

Corollary 2.10. *Any 0-simple countable compact inverse topological semigroup with closed H-classes is topologically isomorphic to a topological Brandt λ-extension $B_\lambda(G)$ of G in the class of topological inverse semigroups such that G is compact topological group and λ is finite. Hence, the topologically periodic countable compact 0-simple inverse topological semigroup is homeomorphic to a finite topological sum of compact topological group and a single point.*

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References

[1] O. Andersen, *Ein Bericht über die Struktur Abstrakter Halbgruppen*, Ph.D. Thesis, Hamburg, 1952.

[2] L. W. Anderson, R. P. Hunter and R. J. Koch, Some results on stability in semigroups, *Trans. Amer. Math. Soc.* **117** (1965) 521–529.

[3] T. O. Banakh and O. V. Gutik, On the continuity of inversion in countably compact inverse topological semigroups, *Semigroup Forum* **68** (2004), no. 3, 411–418.

[4] T. Berezovski, O. Gutik and K. Pavlk, Brandt extensions and primitive topological inverse semigroups, *Int. J. Math. Math. Sci.* **2010** (2010), Article ID 671401, 13 pages.

[5] B. M. Bokalo and I. Guran, Sequentially compact Hausdorff cancellative semigroup is a topological group, *Mat. Stud.* **6** (1996) 39–40.

[6] J. H. Carruth, J. A. Hildebrant and R. J. Koch, The Theory of Topological Semigroups, Monographs and Textbooks in Pure and Applied Mathematics, 75, Marcel Dekker, Inc., New York, 1983.

[7] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Mathematical Surveys, 7, Amer. Math. Soc., Providence, R.I., 1967.

[8] C. Eberhart and J. Selden, On the closure of the bicyclic semigroup, *Trans. Amer. Math. Soc.* **144** (1969) 115–126.

[9] R. Engelking, General Topology, Mir, Moscow, 1986.

[10] D. L. Grant, Sequentially Compact Cancellative Topological Semigroups: some progress on the Wallace problem, *Papers on General Topology and Applications (Madison, WI, 1991)*, 150–154, 704, New York Acad. Sci., New York, 1993.

[11] O. Gutik and K. P. Pavlyk, H-closed topological semigroups and Brandt λ-extensions, *Mat. Metodi Fiz.-Mekh. Polya* **44** (2001), no. 3, 20–28.

[12] O. Gutik and K. P. Pavlyk, On topological semigroups of matrix units, *Semigroup Forum* **71** (2005), no. 3, 389–400.

[13] O. Gutik and D. Repovš, On countably compact 0-simple topological inverse semigroups, *Semigroup Forum* **75** (2007), no. 2, 464–469.

[14] O. Gutik, D. Repovš, On the Brandt λ^0-extensions of monoids with zero, *Semigroup Forum* **80** (2010), no. 1, 8–32.

[15] O. Gutik, D. Pagon and D. Repovš, The continuity of the inversion and the structure of maximal subgroups in countably compact topological semigroups, *Acta Math. Hungar.* **124** (2009), no. 3, 201–214.

[16] J. A. Hildebrant and R. J. Koch, Swelling actions of Γ-compact semigroups, *Semigroup Forum* **33** (1986), no. 1, 65–85.

[17] K. H. Hofmann and P. S. Mostert, Elements of Compact Semigroups, Charles E. Merr II Books, Inc., Ohio, 1966.

[18] R. J. Koch, and A. D. Wallace, Notes on inverse semigroups, *Rev. Roumaine Math. Pures Appl.* **9** (1964) 19–24.

[19] P. D. Kruming, Lattice-ordered semigroups, *Izv. Vysš Ućebn. Zaved. Matematika (in Russian)* **43** (1964), no. 6, 78–87.

[20] M. Petrich, Inverse Semigroups, Pure and Applied Mathematics, John Wiley & sons, Inc., New York, 1984.

[21] A. A. Yurèva, A countably compact sequential topological semigroup with two-sided cancellations is a topological group, L' *vīv. Mat. Tov.* **2** (1993) 23–24.

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