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WEIGHTED FROBENIUS-PERRON AND KOOPMAN OPERATORS

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ABSTRACT. We introduce the weighted Frobenius-Perron operator P_φ^u on L^1 associated with the pair (u, φ) as a perdual of weighted Koopman operator $W = uC_\varphi$ on L^∞ and then investigate some fundamental properties of P_φ^u by the language of conditional expectation operator.

1. **Introduction and preliminaries**

Let (X, Σ, μ) be a complete σ-finite measure space and let $\varphi : X \to X$ be a non-singular transformation; i.e., φ is Σ-measurable and $\mu \circ \varphi^{-1}(A) := \mu(\varphi^{-1}(A)) = 0$, for all $A \in \Sigma$ such that $\mu(A) = 0$. This assumption about φ just says that the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to the measure μ (we write $\mu \circ \varphi^{-1} \ll \mu$, as usual), where $\mu \circ \varphi^{-1}(A) = \mu(\varphi^{-1}(A))$ for $A \in \Sigma$. We shall assume that the restriction of μ to σ-subalgebra $\varphi^{-1}(\Sigma)$ of Σ is σ-finite, and we denote by $(X, \varphi^{-1}(\Sigma), \mu)$ the completion of $(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$. We denote by h the Radon-Nikodym derivative, $h = d\mu \circ \varphi^{-1}/d\mu$. We will write $L^1(\varphi^{-1}(\Sigma))$ for $L^1(X, \varphi^{-1}(\Sigma), \mu_{|\varphi^{-1}(\Sigma)})$. $L^1(\varphi^{-1}(\Sigma))$ may then be viewed as a subspace of $L^1(\Sigma)$ and we denote its norm by $\|.\|_1$. Support of a measurable function f will be denoted by $\sigma(f) = \{x \in X; f(x) \neq 0\}$.

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Relationships between functions f and between sets are interpreted in the almost everywhere sense. For any non-negative Σ-measurable function f as well as for any $f \in L^p(\Sigma)$, by the Radon-Nikodym Theorem, there exists a unique $\varphi^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A E f d\mu = \int_A f d\mu, \qquad \text{for all } A \in \varphi^{-1}(\Sigma).$$

Hence, we obtain an operator E from $L^1(\Sigma)$ onto $L^1(\varphi^{-1}(\Sigma))$ which is called conditional expectation operator associated with the σ-algebra $\varphi^{-1}(\Sigma)$. It is easy to show that for each $f \in L^1(\Sigma)$, there exists a Σ-measurable function g such that $E(f) = g \circ \varphi$. We can assume that $\sigma(g) \subseteq \sigma(h)$, and there exists only one g with this property. We therefore write $g = E(f) \circ \varphi^{-1}$, though we make no assumptions regarding the invertibility of φ (see [1]). This operator will play a major role in our work, and we list here some of its useful properties:

- $E(fg) = E(f)g$, whenever g is $\varphi^{-1}(\Sigma)$-measurable and both conditional expectations are defined.
- $|E(f)| \leq E(|f|)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $E(|f|) = 0$, then $f = 0$.

Let f be a real-valued measurable function. Consider the set $B_f = \{x \in X : E(f^+)(x) = E(f^-)(x) = \infty\}$. The function f is said to be conditionable with respect to $\varphi^{-1}(\Sigma)$ if $\mu(B_f) = 0$. If f is complex-valued, then f is conditionable if the real and imaginary parts of f are conditionable and their respective expectations are not both infinite on the same set of positive measure. For more details on the properties of E see [11, 9]. Our aim here is to generalize some results obtained for the (classic) Frobenius-Perron operators in [4, 6, 7] to the weighted Frobenius-Perron operators.

2. Main results

Definition 2.1. Suppose $\varphi : X \to X$ is a non-singular transformation and let $u : X \to \mathbb{C}$ is a conditionable measurable function. If A is any Σ-measurable set for which $\int_{\varphi^{-1}(A)} u f d\mu$ exists, then the linear operator $\mathcal{P}_\varphi^u : L^1(\Sigma) \to L^1(\Sigma)$, defined by $\int_A \mathcal{P}_\varphi^u f d\mu = \int_{\varphi^{-1}(A)} u f d\mu$, is called the weighted Frobenius-Perron operator associated with the pair (u, φ).

Let $f \in L^1(\Sigma)$ be given. For the above u and φ, we define the measure,

$$\mu^u_{\varphi,f}(A) = \int_{\varphi^{-1}(A)} uf d\mu, \qquad A \in \Sigma.$$

The assumption $\mu \circ \varphi^{-1} \ll \mu$ implies $\mu^u_{\varphi,f} \ll \mu$. By the Radon-Nikodym Theorem, there exists a μ-unique function $\widetilde{f}^u_\varphi \in L^1(\Sigma)$ such that $\mu^u_{\varphi,f}(A) = \int_A \widetilde{f}^u_\varphi d\mu$, for any $A \in \Sigma$. This may be expressed alternatively as:

$$\int_A \widetilde{f}^u_\varphi d\mu = \int_{\varphi^{-1}(A)} uf d\mu, \qquad A \in \Sigma.$$

It follows that the mapping $\mathcal{P}^u_\varphi : f \mapsto \widetilde{f}^u_\varphi$ is well defined on $L^1(\Sigma)$.

We note that according to Proposition 2.3 (vi) below, to the same extent that the weighted Koopman operators are actual generalizations of the Koopman operators, the weighted Frobenius-Perron operators will be the actual generalizations of the (classic) Frobenius-Perron operators.

The weighted Koopman operator on $L^\infty(\Sigma)$ with respect to the pair (u, φ) is defined by $uU_\varphi(f) = u.f \circ \varphi$, for each $f \in L^\infty(\Sigma)$. Here, the non-singularity of φ guarantees that uU_φ is well defined as a mapping of equivalence classes of functions on $\sigma(u)$. Note that $uU_\varphi = M_u U_\varphi$ and $\mathcal{P}^u_\varphi = P_\varphi M_u$ where M_u is a multiplication operator, U_φ and P_φ are (classic) Koopman and Frobenius-Perron operators, respectively. It is easy to see that uU_φ is a bounded operator on $L^\infty(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$, and in this case $\|uU_\varphi\| = \|u\|_\infty$ (see [12]). For a bounded linear operator T on a Banach space, we use the symbols $\mathcal{N}(T)$ and $\mathcal{R}(T)$ to denote the kernel and the range of T, respectively.

Now, let $A \in \Sigma$ with $0 < \mu(A) < \infty$. As an application of the properties of the conditional expectation and using the change of variable formula, we have,

$$\int_A \mathcal{P}^u_\varphi f d\mu = \int_{\varphi^{-1}(A)} uf d\mu = \int_{\varphi^{-1}(A)} E(uf) d\mu = \int_A hE(uf) \circ \varphi^{-1} d\mu,$$

for all $f \in L^1(\Sigma)$. Since Σ is a σ-finite algebra, then it follows that $\mathcal{P}^u_\varphi f = hE(uf) \circ \varphi^{-1}$.

In the following theorem, we investigate the necessary and sufficient conditions for a weighted Frobenius-Perron operator $\mathcal{P}^u_\varphi$ to be bounded.

Theorem 2.2. *The weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ is a bounded operator on $L^1(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$ and its norm is given by $\|\mathcal{P}_\varphi^u\| = \|u\|_\infty$.*

Proof. Let $u \in L^\infty(\Sigma)$. Using the change of variable formula, we have,

$$\|\mathcal{P}_\varphi^u f\|_1 = \int_X |\mathcal{P}_\varphi^u f| d\mu = \int_X h|E(uf) \circ \varphi^{-1}| d\mu$$
$$\leq \int_X E(|uf|) d\mu = \int_X |uf| d\mu \leq \|u\|_\infty \|f\|_1,$$

for each $f \in L^1(\Sigma)$. Thus, $\|\mathcal{P}_\varphi^u\| \leq \|u\|_\infty$. Conversely, suppose that $\mathcal{P}_\varphi^u$ is a bounded operator on $L^1(\Sigma)$. Write uf as $w|uf|$, when $|w| = 1$. Then, we get,

$$\|M_u f\|_1 = \int_X |uf| d\mu = \int_X \overline{w} uf d\mu = \int_X hE(\overline{w} uf) \circ \varphi^{-1} d\mu$$
$$= \int_X \mathcal{P}_\varphi^u(\overline{w} f) d\mu = \|\mathcal{P}_\varphi^u(\overline{w} f)\|_1 \leq \|\mathcal{P}_\varphi^u\| \|\overline{w} f\|_1 = \|\mathcal{P}_\varphi^u\| \|f\|_1,$$

for each $f \in L^1(\Sigma)$. Hence, we conclude that the multiplication operator M_u is a bounded linear operator on $L^1(\Sigma)$. Therefore, $u \in L^\infty$ and $\|u\|_\infty = \|M_u\| \leq \|\mathcal{P}_\varphi^u\|$. The proof of the theorem is now complete. □

Some basic properties of $\mathcal{P}_\varphi^u$ are listed in the following proposition.

Proposition 2.3. *Let φ_i be a measurable transformation of X such that $\mu \circ \varphi_i^{-1}$ is absolutely continuous with respect to μ and $h_i = d\mu \circ \varphi_i^{-1}/d\mu \in L^\infty(\Sigma)$, for $i = 1, 2$. Put $\varphi_3 = \varphi_1 \circ \varphi_2$ and $E(.|\varphi_i^{-1}(\Sigma)) = E_i$. Then the following assertions hold.*

(i) $\mu \circ \varphi_3^{-1} \ll \mu$ and $h_3 = d\mu \circ \varphi_3^{-1}/d\mu = h_1 E_1(h_2) \circ \varphi_1^{-1}$.

(ii) $P_{\varphi_1} \mathcal{P}_{\varphi_2}^u = \mathcal{P}_{\varphi_3}^u$.

(iii) $\mathcal{P}_{\varphi_1}^u \mathcal{P}_{\varphi_2}^u = P_{\varphi_1} P_{\varphi_2} M_{u.u \circ \varphi_2}$.

(iv) $(\mathcal{P}_\varphi^u)^n = (\prod_{i=0}^{n-1} u \circ \varphi^i) P_\varphi^n$.

(v) Let $u \geq 0$. Then, $\mathcal{P}_\varphi^u f \geq 0$ if $f \geq 0$ and $(uU_\varphi) g \geq 0$ if $g \geq 0$.

(vi) $(\mathcal{P}_\varphi^u)^* = uU_\varphi$.

Proof. *(i)* The assumption $\mu\circ\varphi_i^{-1}\ll\mu$ implies that for each $A\in\Sigma$ with $\mu(A)=0$, $\mu(\varphi_1^{-1}(A))=0$, and so $\mu(\varphi_2^{-1}(\varphi_1^{-1}(A))=0$. Hence, $\mu\circ\varphi_3^{-1}\ll\mu$. Also, by use of conditional expectation operator and change of variables formula, we have,

$$\int_A h_3 d\mu=\int_A \frac{d\mu\circ(\varphi_1\circ\varphi_2)^{-1}}{d\mu}d\mu=\int_A d\mu\circ\varphi_2^{-1}\circ\varphi_1^{-1}$$
$$=\int_{\varphi_1^{-1}(A)} d\mu\circ\varphi_2^{-1}=\int_{\varphi_1^{-1}(A)} h_2 d\mu=\int_{\varphi_1^{-1}(A)} E_1(h_2)d\mu$$
$$=\int_A E_1(h_2)\circ\varphi_1^{-1}d\mu\circ\varphi_1^{-1}=\int_A h_1E_1(h_2)\circ\varphi_1^{-1}d\mu.$$

Since (X,Σ,μ) is a σ-finite measure space, then the proof is complete.

(ii) Since $P_{\varphi_i}f=h_iE_i(f)\circ\varphi_i^{-1}$, then for any $A\in\Sigma$ and $f\in L^1(\Sigma)$ we get,

$$\int_A \mathcal{P}^u_{\varphi_3}fd\mu=\int_A h_3E_3(uf)\circ\varphi_3^{-1}d\mu=\int_A E_3(uf)\circ\varphi_3^{-1}d\mu\circ\varphi_3^{-1}$$
$$=\int_{\varphi_3^{-1}(A)} E_3(uf)d\mu=\int_{\varphi_2^{-1}(\varphi_1^{-1}(A))} ufd\mu=\int_{\varphi_2^{-1}(\varphi_1^{-1}(A))} E_2(uf)d\mu$$
$$=\int_{\varphi_1^{-1}(A)} h_2E_2(uf)\circ\varphi_2^{-1}d\mu=\int_A h_1E_1(h_2E_2(uf)\circ\varphi_2^{-1})\circ\varphi_1^{-1}d\mu$$
$$=\int_A P_{\varphi_1}(h_2E_2(uf)\circ\varphi_2^{-1})d\mu=\int_A P_{\varphi_1}(\mathcal{P}^u_{\varphi_2}f)d\mu\ .$$

Now, since (X,Σ,μ) is a σ-finite measure space, then the proof is complete.

(iii) Since $P_{\varphi_1}P_{\varphi_2}=P_{\varphi_1\circ\varphi_2}$, then for any $A\in\Sigma$ and $f\in L^1(\Sigma)$ we have,

$$\int_A P_{\varphi_1}P_{\varphi_2}M_{u.u\circ\varphi_2}fd\mu=\int_A P_{\varphi_1\circ\varphi_2}(u.u\circ\varphi_2)fd\mu$$
$$=\int_{\varphi_3^{-1}(A)} h_3E_3(u.u\circ\varphi_2 f)\circ\varphi_3^{-1}d\mu=\int_{\varphi_3^{-1}(A)} u.u\circ\varphi_2 fd\mu$$
$$=\int_{\varphi_1^{-1}(A)} h_2uE_2(uf)\circ\varphi_2^{-1}d\mu=\int_A h_1E_1(h_2uE_2(uf)\circ\varphi_2^{-1})\circ\varphi_1^{-1}d\mu$$
$$=\int_A h_1E_1(u\mathcal{P}^u_{\varphi_2}f)\circ\varphi_1^{-1}d\mu=\int_A \mathcal{P}^u_{\varphi_1}(\mathcal{P}^u_{\varphi_2}f)d\mu.$$

Again, since (X, Σ, μ) is a σ-finite measure space, then the proof is complete.

(iv) It follows from *(iii)*.

(v) It is trivial.

(vi) It is well-known that $L^\infty(\Sigma)$ is the dual space of $L^1(\Sigma)$; that is, $f \in L^\infty(\Sigma)$ is viewed as a bounded linear functional f^* on $L^1(\Sigma)$, defined by $f^*(g) = (g, f) = \int_X gfd\mu$. First, suppose that $f = \chi_A$, $A \in \Sigma$ ($\mu(A) = +\infty$ is possible). Then, for each $g \in L^1(\Sigma)$, we have,

$$(g, (\mathcal{P}^u_\varphi)^*\chi_A) = (\mathcal{P}^u_\varphi g, \chi_A) = \int_A E(ug) \circ \varphi^{-1} d\mu \circ \varphi^{-1} = \int_{\varphi^{-1}(A)} E(ug) d\mu$$

$$= \int_{\varphi^{-1}(A)} ugd\mu = \int_X gu\chi_{\varphi^{-1}(A)} d\mu = \int_X g(u\chi_A \circ \varphi) d\mu = (g, (uU_\varphi)\chi_A).$$

Hence, $(\mathcal{P}^u_\varphi)^*\chi_A = (uU_\varphi)\chi_A$. It follows that the result holds if f is a simple function. Now, since the simple functions are dense in $L^\infty(\Sigma)$, then we get $(\mathcal{P}^u_\varphi)^* f = (uU_\varphi)f$, for all $f \in L^\infty(\Sigma)$. This completes the proof.

Many problems in ergodic theory and physical sciences are related to the problem of existance and computation of absolutely continuous invariant measures (see [2]). Let $u \in L^\infty(\Sigma)$ and $f \in L^1(\Sigma)$. Define $\nu_f(A) = \int_A ufd\mu$, for all $A \in \Sigma$. It is easy to see that $\nu_f \ll \mu$.

Proposition 2.4. *Let $u \in L^\infty(\Sigma)$ and $f \in L^1(\Sigma)$. Then, $f \in \mathcal{N}(\mathcal{P}^u_\varphi - M_u)$ if and only if the measure ν_f is invariant under φ (here, the invariance of the measure ν_f means that $\nu_f \circ \varphi^{-1} = \nu_f$).*

Proof. Since Σ is σ-finite, then for all $A \in \Sigma$ we have,

$$f \in \mathcal{N}(\mathcal{P}^u_\varphi - M_u) \Longleftrightarrow \mathcal{P}^u_\varphi f = uf \Longleftrightarrow$$

$$\nu_f \circ \varphi^{-1}(A) = \int_{\varphi^{-1}(A)} ufd\mu = \int_A \mathcal{P}^u_\varphi f d\mu = \int_A ufd\mu = \nu_f(A).$$

□

Corollary 2.5. *The function $f \in L^1(\Sigma)$ is a fixed point of the Frobenius-Perron operator P_φ if and only if $\mu_f \circ \varphi^{-1} = \mu_f$, where $\mu_f(A) = \int_A fd\mu$ ($A \in \Sigma$).*

It is well-known that each $\nu \in ba(X, \Sigma, \mu)$, the space of all bounded complex charges on Σ which vanish on all sets of μ-measure 0, defines a bounded linear functional F_ν on $L^\infty(\Sigma)$ by $F_\nu(f) = \int_X f d\nu$. Moreover the mapping $\nu \to F_\nu$ is an isometric isomorphism from $ba(X, \Sigma, \mu)$ onto $(L^\infty(\Sigma))^*$; see [3, 10]. For $\nu \in ba(X, \Sigma, \mu)$ and $u \in L^\infty(\Sigma)$, we define the measure Λ_ν by

$$\Lambda_\nu(A) = \int_{\varphi^{-1}(A)} u d\nu, \qquad A \in \Sigma.$$

Since $\mu \circ \varphi^{-1} \ll \mu$, then we see that $\Lambda_\nu \in ba(X, \Sigma, \mu)$. Now, we compute the dual of $W := uU_\varphi$. Take $f \in L^\infty(\Sigma)$ and $\nu \in ba(X, \Sigma, \mu)$. As an application of the properties of the conditional expectation operator E and using the change of variable formula, we have,

$$W^*(F_\nu)(f) = F_\nu(Wf) = \int_X uf \circ \varphi d\nu = \int_X E_\nu(u) f \circ \varphi d\nu$$
$$= \int_X f E_\nu(u) \circ \varphi^{-1} d\nu \circ \varphi^{-1} = \int_X f d\Lambda_\nu = F_{\Lambda_\nu}(f).$$

After identifying $(L^\infty(\Sigma))^*$ with $ba(X, \Sigma, \mu)$ and ν with F_ν, we can write $W^*(\nu) = \Lambda_\nu$.

Let $ca(X, \Sigma, \mu)$ be the subspace of $ba(X, \Sigma, \mu)$ consisting of all complex measures absolutely continuous with respect to σ-finite measure μ. Since for each $f \in L^1(X, \Sigma, \mu)$, $\mu_f \ll \mu$, then we have $\mu_f \in ca(X, \Sigma, \mu)$. Define a mapping $\Psi : L^1(X, \Sigma\mu) \longrightarrow ca(X, \Sigma, \mu)$ by $\Psi(f) = \mu_f$, with inverse $\Psi^{-1}(\nu) = \frac{d\nu}{d\mu}$ (see [4]). Now, for any $A \in \Sigma$ and $f \in L^1(X, \Sigma, \mu)$, we get,

$$\Lambda_{\mu_f}(A) = \int_{\varphi^{-1}(A)} u d\mu_f = \int_{\varphi^{-1}(A)} uf d\mu = \int_A \mathcal{P}_\varphi^u(f) d\mu.$$

Hence, $\frac{d\Lambda_{\mu_f}}{d\mu} = \mathcal{P}_\varphi^u(f)$. On the other hand, we have,

$$\Psi^{-1} W^* \Psi(f) = \Psi^{-1} W^*(\mu_f) = \psi^{-1}(\Lambda_{\mu_f}) = \frac{d\Lambda_{\mu_f}}{d\mu} = \mathcal{P}_\varphi^u(f).$$

Therefore, W^* is the natural extension of the weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ on $(L^1(X, \Sigma, \mu))^{**}$ (see [4]).

In the following theorem, we give a sufficient condition for $\mathcal{P}_\varphi^u$ to have closed range on $L^1(\Sigma)$.

Theorem 2.6. *Let $\mathcal{P}_\varphi^u$ be the weighted Frobenius-Perron operator and $W = uU_\varphi$ be the weighted Koopman operator with respect to the pair (u, φ). If there exists a constant $\delta > 0$ such that $|u| \geq \delta$ on $\sigma(u)$, then $\mathcal{R}(\mathcal{P}_\varphi^u)$ and $\mathcal{R}(W)$ are closed in $L^1(\Sigma)$ and $L^\infty(\Sigma)$, respectively.*

Proof. First, we show that the range of W is closed. Let $\{Wf_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{R}(W)$, which converges to some $g \in L^\infty(\Sigma)$. Hence, $\{Wf_n\}_{n\in\mathbb{N}}$ converges to $\frac{g}{u} \in L^\infty(\sigma(u), \Sigma_{|\sigma(u)}, \mu_{|\sigma(u)})$. In fact, since $|U_\varphi f_n - \frac{g}{u}| = |\frac{1}{u}||Wf_n - g| \leq \frac{1}{\delta}|Wf_n - g|$ on $\sigma(u)$, then it follows that $\|U_\varphi f_n - \frac{g}{u}\|_{L^\infty(\sigma(u))} \longrightarrow 0$, as $n \to \infty$. On the other hand, since P_φ and so $P_\varphi^* = U_\varphi$ always have a closed range (see [7]), then we obtain a function $f \in L^\infty(\Sigma)$ such that $U_\varphi f = \frac{g}{u}$ on $\sigma(u)$. Since $g = 0$ on $X \backslash \sigma(u)$ and $L^\infty(X, \Sigma, \mu) = L^\infty(\sigma(u)) \oplus L^\infty(\sigma(u)^c)$, then we deduce that $g = Wf \in L^\infty(\Sigma)$. By the Banach closed range theorem, this implies that the range of W^* is also closed. Now, we show that $\Psi^{-1}W^*\Psi = \mathcal{P}_\varphi^u$ has a closed range. Suppose $W^*(\mu_{f_n}) = (W^*\Psi)f_n \longrightarrow \Psi g$, for some $g \in L^1(X, \Sigma, \mu)$. So, there exists $\nu \in ca(X, \Sigma, \mu)$ such that $\Psi g = W^*(\nu)$. Hence, $g = \Psi^{-1}W^*(\nu) = \Psi^{-1}W^*\Psi(\frac{d\nu}{d\mu})$. This completes the proof. □

As a consequence of the above theorem and the Banach closed range theorem (see [13]), we have the following corollary.

Corollary 2.7. *Under the same assumptions as in Theorem 2.6, we have:*
(a) $\mathcal{P}_\varphi^u$ is one-to-one if and only if W is onto.
(b) W is one-to-one if and only if $\mathcal{P}_\varphi^u$ is onto.

The proofs are similar to the proofs of the similar results in [7].

In the following, we show that the weighted Frobenius-Perron operator $\mathcal{P}_\varphi^u$ is the product of two linear operators. This is a generalization of the work done in [6]. Define $T_1 : L^1(\varphi^{-1}(\Sigma)) \longrightarrow L^1(\Sigma)$ and $T_2 : L^1(\Sigma) \longrightarrow L^1(\varphi^{-1}(\Sigma))$ by

$$T_1 f = h.f \circ \varphi^{-1}, \qquad f \in L^1(\varphi^{-1}(\Sigma))$$

and

$$T_2 f = E(uf), \qquad f \in L^1(\Sigma),$$

respectively. It follows that

$$\|T_1 f\|_1 = \int_X h|f \circ \varphi^{-1}| d\mu = \int_X |f| \circ \varphi^{-1} d\mu \circ \varphi^{-1} = \int_X |f| d\mu = \|f\|_1.$$

Hence, T_1 is an isometry. Note that $T_1 \circ T_2 = \mathcal{P}_\varphi^u$. Thus, if $u \in L^\infty(\Sigma)$, then $\mathcal{P}_\varphi^u$ is actually the product of two bounded linear operators and

$$\|\mathcal{P}_\varphi^u f\|_1 = \|T_2 f\|_1, \qquad f \in L^1(\Sigma). \tag{2.1}$$

Therefore, $\|T_2\| = \|\mathcal{P}_\varphi^u\| = \|u\|_\infty$. Also, equality (2.1) shows that the operator $\mathcal{P}_\varphi^u$ is compact if and only if T_2 is a compact operator. On the other hand, since $(\mathcal{P}_\varphi^u)^* = W$ and $\mathcal{P}_\varphi^u = \Psi^{-1} W^* \Psi$, then compactness of $\mathcal{P}_\varphi^u$ is equivalent to compactness of W.

Recall that an atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atom is called non-atomic. It is a well-known fact that every σ-finite measure space (X, Σ, μ) can be partitioned uniquely as follows:

$$X = \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cup B, \tag{2.2}$$

where $\{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_n, is non-atomic (see [14]).

In the sequel, we investigate compact weighted Frobenius-Perron operator on $L^1(\Sigma)$. Recall that a linear operator T on a Banach space $\mathcal{B}$ is compact if it maps every bounded sequence $\{x_n\}$ in $\mathcal{B}$ onto a sequence $\{Tx_n\}$ in $\mathcal{B}$ which has a convergent subsequence.

Theorem 2.8. *Let (X, Σ, μ) be a non-atomic σ-finite measure space. Then, no bounded weighted Frobenius-Perron operator on $L^1(\Sigma)$ is compact unless it is the zero operator.*

Proof. Recall that the operator $\mathcal{P}_\varphi^u$ is compact if and only if T_2 is a compact operator. Hence, it suffices to show that the non-zero bounded operator T_2 is not compact. Consider the set $F = \{x \in X : |u(x)|^2 > \frac{1}{2}\|u\|_\infty^2\}$. Obviously, $\mu(F) > 0$. Since Σ is non-atomic and σ-finite, then there are measurable sets $\{A_n\}_{n=1}^\infty$ such that $A_{n+1} \subseteq A_n \subseteq A_0 \subseteq F$,

$\mu(A_0) < \infty$, and $0 < \mu(A_{n+1}) = \frac{1}{2}\mu(A_n)$. For all $n \in \mathbb{N}$, define:

$$f_n = \frac{\bar{u}}{\|u\|_\infty^2 \mu(A_n)} \chi_{A_n}.$$

Then, $\|f_n\|_1 \leq 1/\|u\|_\infty$. Now, since E is a positive operator, then for any $m, n \in \mathbb{N}$ with $m > n$, we have,

$$\|T_2 f_m - T_2 f_n\|_1 = \int_X |E(uf_m) - E(uf_n)| d\mu = \int_X E(u(f_n - f_m)) d\mu$$

$$= \int_X u(f_n - f_m) d\mu = \int_X \frac{|u|^2}{\|u\|_\infty^2} \left(\frac{\chi_{A_n}}{\mu(A_n)} - \frac{\chi_{A_m}}{\mu(A_m)} \right) d\mu$$

$$\geq \int_{A_n \setminus A_m} \frac{d\mu}{2\mu(A_n)} = \frac{1}{2} \frac{\mu(A_n \setminus A_m)}{\mu(A_n)} = \frac{1}{2} \left(1 - \frac{\mu(A_m)}{\mu(A_n)} \right).$$

Since $\mu(A_m) < \frac{1}{2}\mu(A_n)$, then we get $\|T_2 f_m - T_2 f_n\|_1 \geq \frac{1}{4}$, which shows that the sequence $\{T_2 f_n\}$ dose not contain a convergent subsequence. $\square$

In the following theorem, we give the sufficient conditions for the compactness of $\mathcal{P}_\varphi^u$ on $L^1(\Sigma)$.

Theorem 2.9. *Let $\mathcal{P}_\varphi^u$ be a bounded Frobenius-Perron operator on $L^1(\Sigma)$ and let (X, Σ, μ) be partitioned as (2.2). Suppose that $u(\varphi^{-1}(B)) = 0$ and for any $\varepsilon > 0$, there exist finite disjoint atoms $A_\varepsilon^1, \ldots, A_\varepsilon^n$ such that $\mu(\{x \in \varphi^{-1}(\cup_{i=1}^n A_\varepsilon^i) : |u(x)| > \varepsilon\}) > 0$, and $\mu(\{x \in \varphi^{-1}(X \setminus \cup_{i=1}^n A_\varepsilon^i) : |u(x)| > \varepsilon\}) = 0$. Then, $\mathcal{P}_\varphi^u$ is a compact operator.*

Proof. Take $\varepsilon > 0$ arbitrarily. Put $B_\varepsilon = \varphi^{-1}(\cup_{i=1}^n A_\varepsilon^i)$ and $v = \chi_{B_\varepsilon} u$. It is easy to see $u = v = 0$ on $\varphi^{-1}(B)$ and $u = v$ on B_ε. Then, for each $f \in L^1(\Sigma)$, we have,

$$\|(\mathcal{P}_\varphi^u - \mathcal{P}_\varphi^v) f\|_1 = \int_{X \setminus (\varphi^{-1}(B) \cup B_\varepsilon)} |hE(uf) \circ \varphi^{-1}| d\mu$$

$$\leq \int_{X \setminus (\varphi^{-1}(B) \cup B_\varepsilon)} E(|uf|) \circ \varphi^{-1} d\mu \circ \varphi^{-1} = \int_{\varphi^{-1}(X \setminus (\varphi^{-1}(B) \cup B_\varepsilon))} E(|uf|) d\mu$$

$$= \int_{\varphi^{-1}(X \setminus (\varphi^{-1}(B) \cup B_\varepsilon))} |uf| d\mu \leq \varepsilon \int_X |f| d\mu = \varepsilon \|f\|_1.$$

On the other hand, we have,

$$\mathcal{P}_\varphi^v f = hE\left((\sum_{i=1}^n \chi_{A_\varepsilon^i} \circ \varphi) uf \right) \circ \varphi^{-1} = \sum_{i=1}^n (\mathcal{P}_\varphi^u f)(A_\varepsilon^i) \chi_{A_\varepsilon^i}.$$

Therefore, $\mathcal{P}_\varphi^v$ has a finite rank and hence $\mathcal{P}_\varphi^u$ is compact. $\square$

Example 2.10. Let $w = \{m_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Consider the space $l^p(w) = L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $u = \{u(n)\}_{n=1}^\infty$ be a sequence of nonnegative real numbers. Suppose that the restriction of μ to σ-subalgebra $\varphi^{-1}(2^{\mathbb{N}})$ is σ-finite, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a non-singular measurable transformation. Direct computations show that for all $f = \{f(n)\}_{n=1}^\infty \in l^1(w)$, we have,

$$h(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} m_j \ ,$$

$$(E(f))(k) = \frac{\sum_{j \in \varphi^{-1}(\varphi(k))} f(j) m_j}{\sum_{j \in \varphi^{-1}(\varphi(k))} m_j} \ ,$$

$$(E(f) \circ \varphi^{-1})(k) = \frac{\sum_{j \in \varphi^{-1}(k)} f(j) m_j}{\sum_{j \in \varphi^{-1}(k)} m_j} \ ,$$

$$\mathcal{P}_\varphi^u(f)(k) = h(k)(E(uf) \circ \varphi^{-1})(k) = \frac{1}{m_k} \sum_{j \in \varphi^{-1}(k)} u(j) f(j) m_j \ .$$

Example 2.11. Let $X = [0,1]$, $d\mu = dx$, and Σ be the Lebesgue sets. A mapping $\varphi : [0,1] \to [0,1]$ is called piecwise monotonic if there exists a partition $0 < a_0 < a_1 < \ldots < a_n = 1$ of $[0,1]$ such that $\varphi_j := \varphi\,|_{(a_{j-1}, a_j)}$ is a c^1-function, which can be extended to a c^1-function on $A_j = [a_{j-1}, a_j]$ and $|\varphi_j'(x)| > 0$ on (a_{j-1}, a_j), $j = 1, \ldots, n$. Put $\Sigma_j = \Sigma_{|A_j}$, $E(.|\varphi^{-1}(\Sigma_j)) = E_j$ and $\mu_{|\Sigma_j} = \mu_j$. It is easy to see that $\mu_j \circ \varphi_j^{-1} \ll \mu_j$ and $\varphi^{-1}(\Sigma_j) = \Sigma_j$. Thus, $E_j = I$ on $L^1(A_j, \Sigma_j, \mu_j)$ and $h_j(x) = (d\mu_j \circ \varphi^{-1}/d\mu_j)(x) = (\varphi_j^{-1})'(x) = 1/\varphi_j'(\varphi_j^{-1}(x))$, for all $x \in (a_{j-1}, a_j)$. Note that, in general, one does not have $h_j = h_{|A_j}$ (see [1]). Then, for all $f \in L^1(\Sigma)$ and $x \in [0,1]$, we get,

$$(P_\varphi f)(x) = \mathcal{P}_\varphi^1(f)(x) = h(x)(E(f) \circ \varphi^{-1})(x)$$

$$= \sum_{j=1}^n h_j(x)(E_j(\chi_{A_j} f) \circ \varphi_j^{-1})(x) = \sum_{j=1}^n \frac{f(\varphi_j^{-1}(x))}{\varphi_j'(\varphi_j^{-1}(x))} \chi_{A_j}(\varphi_j^{-1}(x)) \ .$$

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