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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. Here, using the fixed point theory in cone metric spaces, we prove the existence of a unique solution to a first-order ordinary differential equation with periodic boundary conditions in Banach spaces admitting the existence of a lower solution.

1. **Introduction**

Recently, some authors applied fixed point theory in partially ordered metric spaces to study the existence of a unique solution to periodic boundary value problems on real line [2, 3, 5, 7–9]. Here, we consider the following periodic boundary value problem,

$$\left\{ \begin{array}{l} u'(t) = f(t, u(t)), \quad \text{if } t \in I = [0, T] \\ u(0) = u(T), \end{array} \right. \tag{1.1}$$

where T is a positive real number, $(Y, \leq)$ is a Banach lattice and $f : I \times Y \longrightarrow Y$ is a continuous function.

Definition 1.1. *A lower solution for* (1.1) *is a function* $\alpha \in C^1(I, Y)$ *such that*

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$$\alpha'(t) \leq f(t, \alpha(t)), \;\; for \; t \in I,$$
$$\alpha(0) \leq \alpha(T).$$

To set up our results in the next section, we recall some definitions and facts.
Throughout the paper, let $(Y, \leq)$ be a Banach lattice with the positive cone P.

Definition 1.2. *A Banach lattice Y is said to be*

(a) *order complete if every order bounded set in Y has a supremum;*
(b) *σ-order continuous, if, for every nonincreasing sequence $\{y_n\}$ in Y with $\inf_n x_n = 0$, we have $\lim_{n\to\infty} \|x_n\| = 0$.*

Theorem 1.3. *([6], Proposition 1.a.8) Let Y be a Banach lattice. Suppose that every order bounded nondecreasing sequence in Y is convergent. Then, Y is order complete and σ-order continuous.*

Theorem 1.4. *([6], Theorem 1.c.4) The following conditions are equivalent for any Banach lattice Y:*

(i) *No subspace of Y is isomorphic to c_0.*
(ii) *Every norm bounded nondecreasing sequence in Y is convergent.*

Lemma 1.5. *([4]) Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X. Then,*

(i) *$\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. Moreover, the limit of a convergent sequence is unique.*
(ii) *$\{x_n\}$ is a Cauchy sequence if and only if*
$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$
(iii) *If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.*

The following result is a slightly improved version of Theorem 5 in [1], which we need in the next section.

Theorem 1.6. *Let $(X, \preceq)$ be a partially ordered set such that every pairs of elements of X has a lower bound or an upper bound. Suppose that there exists a cone metric d in X such that (X, d) is a complete cone metric space with the normal cone. Let $f : X \to X$ be a nondecreasing map such that there exists $k \in [0, 1)$ with*

$$d(f(x), f(y)) \leq kd(x, y), \;\; \forall \; x \preceq y.$$

Suppose also that if a nondecreasing sequence $\{x_n\}$ converges to x in X, then $x_n \preceq x$, for all n. Then, f has a unique fixed point.

Proof. From Theorem 5 in [1], we get that f has a fixed point. To prove the uniqueness, let us suppose that x and y are fixed points of f and z is an upper or lower bound of x and y, that is, there exists $z \in X$ comparable to x and y. Monotonicity of f implies that $f^n(z)$ is comparable to $f^n(x) = x$ and $f^n(y) = y$, for all $n \in \mathbb{N}$. Then,

$$d(x, y) = d(f^n(x), f^n(y)) \leq$$

$$d(f^n(x), f^n(z)) + d(f^n(z), f^n(y)) \leq k^n d(x, z) + k^n d(z, y).$$

Since $\lim_{n\to\infty} k^n = 0$, from the above we get $d(x, y) = 0$, that is, $x = y$. $\square$

2. **Main results**

Lemma 2.1. *Let $(Y, \leq)$ be a Banach lattice with int $P \neq \emptyset$. Then, $A \subseteq Y$ is norm bounded if and only if A is order bounded.*

Proof. Let $A \subseteq Y$ be order bounded. Without loss of generality, we may assume that $A \subseteq P$. Then, there exists $\tau \in Y$ such that $0 \leq y \leq \tau$, for each $y \in A$. Then, for each $y \in A$, $\|y\| \leq \|\tau\|$, that is, A is norm bounded. Conversely, suppose that A is norm bounded. Then, there exists a constant $M > 0$ such that $\|y\| \leq M$, for each $y \in A$. Let $e \in \operatorname{int} P$. Then, there exists a positive number $r > 0$ such that $e+u \in P$, for $\|u\| < r$. Thus, $e - \lambda y \in P$ and $e + \lambda y \in P$, for each $y \in A$, where $0 < \lambda < \frac{r}{M}$. Therefore, $\frac{-e}{\lambda} \leq y \leq \frac{e}{\lambda}$, for each $y \in A$, and so A is order bounded. $\square$

Let $(Y, \leq)$ be a Banach lattice with int $P \neq \emptyset$. Let $C(I, Y)$ denote the set of all continuous maps $f : I \to Y$, where $I = [0, T]$, and $T > 0$. Then, $f(I)$ is a compact subset of Y and then by Lemma 2.1 is order bounded (note that $f(I)$ is norm bounded). Thus, Lemma 2.1 together with Theorem 1.3 and Theorem 1.4 yield that the set $f(I) = \{f(t) : t \in I\}$ has a supremum, for each $f \in C(I, Y)$. For each $x, y \in C(I, Y)$, set (note that $|x(.) - y(.)| \in C(I, Y)$)

$$d(x, y) := \sup_{t \in I} |x(t) - y(t)|.$$

Now, we are ready to prove the following lemma.

Lemma 2.2. *Let $(Y, \leq)$ be a Banach lattice with int $P \neq \emptyset$. Suppose that no subspace of Y is isomorphic to c_0. Then, $(C(I,Y), d)$ is a complete cone metric space with the normal cone.*

Proof. It is straightforward to see that $(C(I,Y), d)$ is a cone metric space. Now, we show that $(C(I,Y), d)$ is complete. Let $\{r_n\}$ be a numeration of the rationales of I and let

$$z_n = \sup_{1\leq k\leq n} |f(r_k) - g(r_k)|, \text{ where } f, g \in C(I,Y).$$

Since $\{z_n\}$ is a nondecreasing and norm bounded sequence, by Theorem 1.4, it is convergent. Since $\{r_n\}$ is dense in I, for each $t \in I$ there exists a subsequence $\{r_{k_n}\}_n$ such that $r_{k_n} \to t$. Thus,

$$|f(t) - g(t)| = \lim_{n\to\infty} |f(r_{k_n}) - g(r_{k_n})| \leq \lim_{n\to\infty} z_n, \ \forall\ t \in I.$$

Therefore,

$$\sup_{t\in I} |f(t) - g(t)| \leq \lim_{n\to\infty} z_n. \tag{2.1}$$

Now, let $\{f_n\}$ be a Cauchy sequence in $(C(I,Y), d)$. Then, by Lemma 1.5, we have $\lim_{m,n\to\infty} d(f_n, f_m) = 0$ and then $\lim_{m,n\to\infty} \|d(f_n, f_m)\| = 0$. Hence, for each positive number $\epsilon > 0$ there exists N such that for each $m \geq n \geq N$, we have

$$\epsilon > \|d(f_n, f_m)\| = \|\sup_{t\in I} |f_n(t) - f_m(t)|\| \geq$$
$$\sup_{t\in I} \|f_n(t) - f_m(t)\| = \|f_n - f_m)\|_\infty.$$

This shows that $\{f_n\}$ is a Cauchy sequence in $(C(I,Y), \|.\|_\infty)$. Since $(C(I,Y), \|.\|_\infty)$ is complete, there exists a $f_0 \in C(I,Y)$ such that $\lim_{n\to\infty}$ $\|f_n - f_0\|_\infty = 0$. Now, we prove that $\lim_{n\to\infty} \|d(f_n, f_0)\| = 0$ and then by Lemma 1.5 we are done. On the contrary, assume that there exist a positive number ϵ_0 and a subsequence $\{f_{n_i}\}$ such that $\|d(f_{n_i}, f_0)\| > \epsilon_0$, for each $i \in \mathbb{N}$. From (2.1), we have

$$\sup_{t\in I} |f_{n_i}(t) - f_0(t)| \leq \lim_{p\to\infty} \sup_{1\leq k\leq p} |f_{n_i}(r_k) - f_0(r_k)|.$$

Thus, for each $i \in \mathbb{N}$, we get

$$\epsilon < \|d(f_{n_i}, f_0)\| = \|\sup_{t\in I} |f_{n_i}(t) - f_0(t)|\|$$
$$\leq \lim_{p\to\infty} \|\sup_{1\leq k\leq p} |f_{n_i}(r_k) - f_0(r_k)|\|.$$

Then, there exists a $p_0 \in \mathbb{N}$ such that

$$\epsilon < \| \sup_{1 \le k \le p_0} |f_{n_i}(r_k) - f_0(r_k)| \|, \ \forall\, i \in \mathbb{N}.$$

Since $\{f_{n_i}\}$ is uniformly convergent to f_0 on I, from the above, we get

$$\epsilon \le \lim_{i \to \infty} \| \sup_{1 \le k \le p_0} |f_{n_i}(r_k) - f_0(r_k)| \| = 0,$$

a contradiction. Therefore, $(C(I,Y), d)$ is a complete cone metric space. Now, let $0 \le x \le y$. Then, $\|x\| \le \|y\|$ and thus P is normal. □

Now, we prove the existence and uniqueness of the solution for the problem (1.1) in presence of a lower solution.

Theorem 2.3. *Let $(Y, \le)$ be a Banach lattice with int $P \ne \emptyset$ and suppose that no subspace of Y is isomorphic to c_0. Consider problem* (1.1) *with $f : I \times Y \to Y$ continuous and suppose that there exist $\lambda > 0$ and $\mu > 0$ with $\mu < \lambda$ such that for x, $y \in Y$ with $y \ge x$,*

$$0 \le f(t,y) + \lambda y - [f(t,x) + \lambda x] \le \ \mu(y - x).$$

Then, the existence of a lower solution for (1.1) *provides the existence of an unique solution of* (1.1).

Proof. Problem (1.1) can be written as

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), \ t \in I \\ \qquad\qquad u(0) = u(T), \end{cases}$$

and equivalently as the integral equation

$$u(t) = \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s)]ds,$$

where

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, \ 0 \le s < t \le T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}. \ 0 \le t < s \le T. \end{cases}$$

Define $F : C(I,Y) \longrightarrow C(I,Y)$ by

$$(Fu)(t) = \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s)]ds.$$

Note that $u \in C(I,Y)$ is a fixed point of F if and only if $u \in C^1(I,Y)$ is a solution of (1.1). Now, we check that the hypotheses in Theorem 1.6 are satisfied. From Lemma 2.2, we have $C(I,Y)$ is a complete cone metric space with the normal cone. Indeed, the complete cone metric space $X = C(I, Y)$ is a partially ordered set, if we define the following order relation in X:

$$x, y \in C(I, Y),\ x \preceq y \text{ if and only if } x(t) \leq y(t),\ \forall\ t \in I.$$

For each $x, y \in C(I, Y)$, $z(t) = |x(t)| + |y(t)| \in C(I, Y)$ is an upper bound of x and y. Note that the mapping F is nondecreasing, since, by the hypothesis, for $u \succeq v$,

$$f(t, u(t)) + \lambda u(t) \geq f(t, v(t)) + \lambda v(t)$$

which implies, for $t \in I$, using that $G(t, s) > 0$, for $(t, s) \in I \times I$, that

$$(Fu)(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds \geq$$

$$\int_0^T G(t, s)[f(s, v(s)) + \lambda v(s)]ds = (Fv)(t),$$

that is, $Fu \succeq Fv$. Besides, for $u \succeq v$ (note that $(Fu)(t) - (Fv)(t) \geq 0$),

$$d(Fu, Fv) = \sup_{t \in I} |(Fu)(t) - (Fv)(t)| = \sup_{t \in I}\ [(Fu)(t) - (Fv)(t)].$$

For each $t \in I$, we have

$$\begin{aligned}
&[(Fu)(t) - (Fv)(t)] \\
&= \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s) - f(s, v(s)) - \lambda v(s)]ds \\
&\leq \int_0^T G(t, s)\ \mu(u(s) - v(s))ds \\
&\leq \mu d(u, v) \int_0^T G(t, s)ds.
\end{aligned}$$

Thus,

$$d(Fu, Fv) = \sup_{t \in I}[(Fu)(t) - (Fv)(t)] \leq$$

$$\mu d(u, v) \sup_{t \in I} \int_0^T G(t, s)ds =$$

$$\mu d(u, v) \sup_{t \in I} \frac{1}{e^{\lambda T} - 1}\left(\frac{1}{\lambda} e^{\lambda(T+s-t)}\Big|_0^t\ +\ \frac{1}{\lambda} e^{\lambda(s-t)}\Big|_t^T\right) =$$

$$\mu d(u, v) \frac{1}{\lambda(e^{\lambda T} - 1)}(e^{\lambda T} - 1) = \frac{\mu}{\lambda} d(u, v).$$

Then, for $u \succeq v$ (notice $\frac{\mu}{\lambda} < 1$),

$$d(Fu, Fv) \leq \frac{\mu}{\lambda}\, d(u, v).$$

Finally, let $\alpha(t)$ be a lower solution for (1.1) and we will show that $\alpha \leq F\alpha$. Indeed,

$$\alpha'(t) + \lambda\alpha(t) \leq f(t, \alpha(t)) + \lambda\alpha(t) \ for \ t \in I.$$

Multiplying by $e^{\lambda t}$, we get

$$(\alpha(t)e^{\lambda t})' \leq [f(t, \alpha(t)) + \lambda\alpha(t)]e^{\lambda t}, \ for \ t \in I,$$

and this gives

$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t [f(s, \alpha(s)) + \lambda\alpha(t)]e^{\lambda s} ds, \ for \ t \in I, \tag{2.2}$$

which implies

$$\alpha(0)e^{\lambda T} \leq \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T [f(s, \alpha(s)) + \lambda\alpha(s)]e^{\lambda s} ds,$$

and thus,

$$\alpha(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds.$$

From this inequality and (2.2), we obtain

$$\begin{aligned} \alpha(t)e^{\lambda t} \leq & \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds \\ & + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds, \end{aligned}$$

and consequently,

$$\begin{aligned} \alpha(t) \leq & \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds \\ & + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}[f(s, \alpha(s)) + \lambda\alpha(s)]ds. \end{aligned}$$

Hence,

$$\alpha(t) \leq \int_0^T G(t, s)[f(s, \alpha(s)) + \lambda\alpha(s)]ds = (F\alpha)(t) \ for \ t \in I.$$

Thus, $\alpha \preceq F\alpha$. Now, suppose that $\{x_n\}$ is a nondecreasing sequence convergent to x in $C(I, Y)$. For each $t_0 \in I$, we have

$$|x_n(t_0) - x(t_0)| \leq d(x_n, x),$$

and thus $\lim_{n\to\infty} x_n(t_0) = x(t_0)$. Since $\{x_n\}$ is a nondecreasing sequence, we have

$$x_1(t_0) \leq x_2(t_0) \leq \ldots \leq x_n(t_0) \leq x_{n+1}(t_0) \leq \cdots .$$

Since Y is σ-order continuous, we get

$$\sup_n x_n(t_0) = \lim_{n\to\infty} x_n(t_0) = x(t_0),$$

and so $x_n(t_0) \leq x(t_0)$, for each n. Finally, Theorem 1.6 gives that F has a unique fixed point. □

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