EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR
A PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. Here, using the fixed point theory in cone metric spaces, we prove the existence of a unique solution to a first-order ordinary differential equation with periodic boundary conditions in Banach spaces admitting the existence of a lower solution.

1. Introduction

Recently, some authors applied fixed point theory in partially ordered metric spaces to study the existence of a unique solution to periodic boundary value problems on real line [2,3,5,7–9]. Here, we consider the following periodic boundary value problem,

\begin{align}
\begin{cases}
u'(t) = f(t, u(t)), & \text{if } t \in I = [0, T] \\
 u(0) = u(T),
\end{cases}
\end{align}

where \( T \) is a positive real number, \((Y, \leq)\) is a Banach lattice and \( f : I \times Y \rightarrow Y \) is a continuous function.

Definition 1.1. A lower solution for (1.1) is a function \( \alpha \in C^1(I, Y) \) such that

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\[ \alpha'(t) \leq f(t, \alpha(t)), \text{ for } t \in I, \]
\[ \alpha(0) \leq \alpha(T). \]

To set up our results in the next section, we recall some definitions and facts. Throughout the paper, let \((Y, \leq)\) be a Banach lattice with the positive cone \(P\).

**Definition 1.2.** A Banach lattice \(Y\) is said to be
(a) order complete if every order bounded set in \(Y\) has a supremum;
(b) \(\sigma\)-order continuous, if, for every nonincreasing sequence \(\{y_n\}\) in
\(Y\) with \(\inf_n x_n = 0\), we have \(\lim_{n \to \infty} \|x_n\| = 0\).

**Theorem 1.3.** ([6], Proposition 1.a.8) Let \(Y\) be a Banach lattice. Suppose that every order bounded nondecreasing sequence in \(Y\) is convergent. Then, \(Y\) is order complete and \(\sigma\)-order continuous.

**Theorem 1.4.** ([6], Theorem 1.c.4) The following conditions are equivalent for any Banach lattice \(Y\):
(i) No subspace of \(Y\) is isomorphic to \(c_0\).
(ii) Every norm bounded nondecreasing sequence in \(Y\) is convergent.

**Lemma 1.5.** ([4]) Let \((X, d)\) be a cone metric space and \(\{x_n\}\) be a sequence in \(X\). Then,
(i) \(\{x_n\}\) converges to \(x\) if and only if \(\lim_{n \to \infty} d(x_n, x) = 0\). Moreover, the limit of a convergent sequence is unique.
(ii) \(\{x_n\}\) is a Cauchy sequence if and only if
\[ \lim_{m,n \to \infty} d(x_n, x_m) = 0. \]
(iii) If \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} y_n = y\), then \(\lim_{n \to \infty} d(x_n, y_n) = d(x, y)\).

The following result is a slightly improved version of Theorem 5 in [1], which we need in the next section.

**Theorem 1.6.** Let \((X, \leq)\) be a partially ordered set such that every pairs of elements of \(X\) has a lower bound or an upper bound. Suppose that there exists a cone metric \(d\) in \(X\) such that \((X, d)\) is a complete cone metric space with the normal cone. Let \(f : X \to X\) be a nondecreasing map such that there exists \(k \in [0, 1)\) with
\[ d(f(x), f(y)) \leq kd(x, y), \forall x \leq y. \]
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Suppose also that if a nondecreasing sequence \( \{x_n\} \) converges to \( x \) in \( X \), then \( x_n \preceq x \), for all \( n \). Then, \( f \) has a unique fixed point.

Proof. From Theorem 5 in [1], we get that \( f \) has a fixed point. To prove the uniqueness, let us suppose that \( x \) and \( y \) are fixed points of \( f \) and \( z \) is an upper or lower bound of \( x \) and \( y \), that is, there exists \( z \in X \) comparable to \( x \) and \( y \). Monotonicity of \( f \) implies that \( f^n(z) \) is comparable to \( f^n(x) = x \) and \( f^n(y) = y \), for all \( n \in \mathbb{N} \). Then,

\[
d(x, y) = d(f^n(x), f^n(y)) \leq d(f^n(x), f^n(z)) + d(f^n(z), f^n(y)) \leq k^n d(x, z) + k^n d(z, y).
\]

Since \( \lim_{n \to \infty} k^n = 0 \), from the above we get \( d(x, y) = 0 \), that is, \( x = y \). □

2. Main results

Lemma 2.1. Let \((Y, \preceq)\) be a Banach lattice with \( \text{int} P \neq \emptyset \). Then, \( A \subseteq Y \) is norm bounded if and only if \( A \) is order bounded.

Proof. Let \( A \subseteq Y \) be order bounded. Without loss of generality, we may assume that \( A \subseteq P \). Then, there exists \( \tau \in Y \) such that \( 0 \preceq y \preceq \tau \), for each \( y \in A \). Then, for each \( y \in A \), \( \|y\| \preceq \|\tau\| \), that is, \( A \) is norm bounded. Conversely, suppose that \( A \) is norm bounded. Then, there exists a constant \( M > 0 \) such that \( \|y\| \preceq M \), for each \( y \in A \). Let \( e \in \text{int} P \). Then, there exists a positive number \( r > 0 \) such that \( e + u \in P \), for \( \|u\| < r \). Thus, \( e - \lambda y \in P \) and \( e + \lambda y \in P \), for each \( y \in A \), where \( 0 < \lambda < \frac{r}{M} \). Therefore, \( \frac{e}{M} \preceq y \preceq \frac{e}{M} \), for each \( y \in A \), and so \( A \) is order bounded. □

Let \((Y, \preceq)\) be a Banach lattice with \( \text{int} P \neq \emptyset \). Let \( C(I, Y) \) denote the set of all continuous maps \( f : I \to Y \), where \( I = [0, T] \), and \( T > 0 \). Then, \( f(I) \) is a compact subset of \( Y \) and then by Lemma 2.1 is order bounded (note that \( f(I) \) is norm bounded). Thus, Lemma 2.1 together with Theorem 1.3 and Theorem 1.4 yield that the set \( f(I) = \{f(t) : t \in I\} \) has a supremum, for each \( f \in C(I, Y) \). For each \( x, y \in C(I, Y) \), set (note that \( |x(.) - y(.)| \in C(I, Y) \))

\[
d(x, y) := \sup_{t \in I} |x(t) - y(t)|.
\]

Now, we are ready to prove the following lemma.
Lemma 2.2. Let \((Y, \leq)\) be a Banach lattice with \(\text{int } P \neq \emptyset\). Suppose that no subspace of \(Y\) is isomorphic to \(c_0\). Then, \((C(I,Y), d)\) is a complete cone metric space with the normal cone.

Proof. It is straightforward to see that \((C(I,Y), d)\) is a cone metric space. Now, we show that \((C(I,Y), d)\) is complete. Let \(\{r_n\}\) be a numeration of the rationales of \(I\) and let

\[ z_n = \sup_{1 \leq k \leq n} |f(r_k) - g(r_k)|, \quad \text{where } f, g \in C(I,Y). \]

Since \(\{z_n\}\) is a nondecreasing and norm bounded sequence, by Theorem 1.4, it is convergent. Since \(\{r_n\}\) is dense in \(I\), for each \(t \in I\) there exists a subsequence \(\{r_{k_n}\}_n\) such that \(r_{k_n} \to t\). Thus,

\[ |f(t) - g(t)| = \lim_{n \to \infty} |f(r_{k_n}) - g(r_{k_n})| \leq \lim_{n \to \infty} z_n, \quad \forall t \in I. \]

Therefore,

\[ \sup_{t \in I} |f(t) - g(t)| \leq \lim_{n \to \infty} z_n. \]

(2.1)

Now, let \(\{f_n\}\) be a Cauchy sequence in \((C(I,Y), d)\). Then, by Lemma 1.5, we have \(\lim_{m,n \to \infty} d(f_n, f_m) = 0\) and then \(\lim_{m,n \to \infty} \|d(f_n, f_m)\| = 0\). Hence, for each positive number \(\epsilon > 0\) there exists \(N\) such that for each \(m \geq n \geq N\), we have

\[ \epsilon > \|d(f_n, f_m)\| = \| \sup_{t \in I} f_n(t) - f_m(t) \| \geq \sup_{t \in I} \| f_n(t) - f_m(t) \| = \| f_n - f_m \|_\infty. \]

This shows that \(\{f_n\}\) is a Cauchy sequence in \((C(I,Y), \| \cdot \|_\infty)\). Since \((C(I,Y), \| \cdot \|_\infty)\) is complete, there exists a \(f_0 \in C(I,Y)\) such that \(\lim_{n \to \infty} \|f_n - f_0\|_\infty = 0\). Now, we prove that \(\lim_{n \to \infty} \|d(f_n, f_0)\| = 0\) and then by Lemma 1.5 we are done. On the contrary, assume that there exist a positive number \(\epsilon_0\) and a subsequence \(\{f_{n_i}\}\) such that \(\|d(f_{n_i}, f_0)\| > \epsilon_0\), for each \(i \in \mathbb{N}\). From (2.1), we have

\[ \sup_{t \in I} |f_{n_i}(t) - f_0(t)| \leq \lim_{p \to \infty} \sup_{1 \leq k \leq p} |f_{n_i}(r_k) - f_0(r_k)|. \]

Thus, for each \(i \in \mathbb{N}\), we get

\[ \epsilon < \|d(f_{n_i}, f_0)\| = \| \sup_{t \in I} f_{n_i}(t) - f_0(t) \| \leq \lim_{p \to \infty} \| \sup_{1 \leq k \leq p} f_{n_i}(r_k) - f_0(r_k) \|. \]
Then, there exists a $p_0 \in \mathbb{N}$ such that
$$
\epsilon < \| \sup_{1 \leq k \leq p_0} | f_{n_i} (r_k) - f_0 (r_k) |\|, \forall i \in \mathbb{N}.
$$
Since $\{f_{n_i}\}$ is uniformly convergent to $f_0$ on $I$, from the above, we get
$$
\epsilon \leq \lim_{i \to \infty} \| \sup_{1 \leq k \leq p_0} | f_{n_i} (r_k) - f_0 (r_k) |\| = 0,
$$
a contradiction. Therefore, $(C(I, Y), d)$ is a complete cone metric space.

Now, let $0 \leq x \leq y$. Then, $\| x \| \leq \| y \|$ and thus $P$ is normal. □

Now, we prove the existence and uniqueness of the solution for the problem (1.1) in presence of a lower solution.

**Theorem 2.3.** Let $(Y, \leq)$ be a Banach lattice with $\text{int} \ P \neq \emptyset$ and suppose that no subspace of $Y$ is isomorphic to $c_0$. Consider problem (1.1) with $f : I \times Y \to Y$ continuous and suppose that there exist $\lambda > 0$ and $\mu > 0$ with $\mu < \lambda$ such that for $x, y \in Y$ with $y \geq x$,
$$
0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \mu (y - x).
$$
Then, the existence of a lower solution for (1.1) provides the existence of an unique solution of (1.1).

**Proof.** Problem (1.1) can be written as
\[
\begin{cases}
  u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), & t \in I \\
  u(0) = u(T),
\end{cases}
\]
and equivalently as the integral equation
$$
u(t) = \int_0^T G(t, s) [f(s, u(s)) + \lambda u(s)] ds,
$$
where
$$
G(t, s) = \begin{cases}
  e^{\lambda (T + s - t)} - 1, & 0 \leq t < s \leq T \\
  e^{\lambda (s - t)} - 1, & 0 \leq s < t \leq T.
\end{cases}
$$
Define $F : C(I, Y) \to C(I, Y)$ by
$$
(Fu)(t) = \int_0^T G(t, s) [f(s, u(s)) + \lambda u(s)] ds.
$$
Note that $u \in C(I, Y)$ is a fixed point of $F$ if and only if $u \in C^1(I, Y)$ is a solution of (1.1). Now, we check that the hypotheses in Theorem 1.6 are satisfied. From Lemma 2.2, we have $C(I, Y)$ is a complete cone metric space with the normal cone. Indeed, the complete cone metric
space \( X = C(I,Y) \) is a partially ordered set, if we define the following order relation in \( X \):

\[
x, y \in C(I,Y), \quad x \preceq y \text{ if and only if } x(t) \leq y(t), \quad \forall \ t \in I.
\]

For each \( x, y \in C(I,Y) \), \( z(t) = |x(t)| + |y(t)| \in C(I,Y) \) is an upper bound of \( x \) and \( y \). Note that the mapping \( F \) is nondecreasing, since, by the hypothesis, for \( u \preceq v \),

\[
f(t, u(t)) + \lambda u(t) \geq f(t, v(t)) + \lambda v(t)
\]

which implies, for \( t \in I \), using that \( G(t,s) > 0 \), for \( (t,s) \in I \times I \), that

\[
(Fu)(t) = \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s)]ds \\
\int_0^T G(t,s)[f(s,v(s)) + \lambda v(s)]ds = (Fv)(t),
\]

that is, \( Fu \preceq Fv \). Besides, for \( u \preceq v \) (note that \( (Fu)(t) - (Fv)(t) \geq 0 \),

\[
d(Fu,Fv) = \sup_{t \in I} |(Fu)(t) - (Fv)(t)| = \sup_{t \in I} [(Fu)(t) - (Fv)(t)].
\]

For each \( t \in I \), we have

\[
[(Fu)(t) - (Fv)(t)] \\
= \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s) - f(s,v(s)) - \lambda v(s)]ds \\
\leq \int_0^T G(t,s) \mu(u(s) - v(s))ds \\
\leq \mu d(u,v) \int_0^T G(t,s)ds.
\]

Thus,

\[
d(Fu,Fv) = \sup_{t \in I} [(Fu)(t) - (Fv)(t)] \leq \\
\mu d(u,v) \int_0^T G(t,s)ds = \\
\mu d(u,v) \sup_{t \in I} \frac{1}{e^{\lambda T} - 1} \left( \frac{1}{\lambda} e^{\lambda(T+s-t)} \big|_0^t + \frac{1}{\lambda} e^{\lambda(s-t)} \big|_t^T \right) = \\
\mu d(u,v) \frac{1}{\lambda(e^{\lambda T} - 1)} (e^{\lambda T} - 1) = \frac{\mu}{\lambda} d(u,v).
\]
Then, for $u \geq v$ (notice $\frac{d}{\lambda} < 1$),
$$d(Fu, Fv) \leq \frac{\mu}{\lambda} d(u, v).$$

Finally, let $\alpha(t)$ be a lower solution for (1.1) and we will show that $\alpha \leq F\alpha$. Indeed,
$$\alpha'(t) + \lambda \alpha(t) \leq f(t, \alpha(t)) + \lambda \alpha(t) \text{ for } t \in I.$$ 

Multiplying by $e^{\lambda t}$, we get
$$(\alpha(t)e^{\lambda t})' \leq [f(t, \alpha(t)) + \lambda \alpha(t)]e^{\lambda t}, \text{ for } t \in I,$$
and this gives
(2.2)  
$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t [f(s, \alpha(s)) + \lambda \alpha(s)]e^{\lambda s}ds, \text{ for } t \in I,$$
which implies
$$\alpha(0)e^{\lambda T} \leq \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T [f(s, \alpha(s)) + \lambda \alpha(s)]e^{\lambda s}ds,$$
and thus,
$$\alpha(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s)]ds.$$

From this inequality and (2.2), we obtain
$$\alpha(t)e^{\lambda t} \leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s)]ds$$
$$+ \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s)]ds,$$
and consequently,
$$\alpha(t) \leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s)]ds$$
$$+ \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} [f(s, \alpha(s)) + \lambda \alpha(s)]ds.$$ 

Hence,
$$\alpha(t) \leq \int_0^T G(t, s)[f(s, \alpha(s)) + \lambda \alpha(s)]ds = (F\alpha)(t) \text{ for } t \in I.$$ 

Thus, $\alpha \leq F\alpha$. Now, suppose that $\{x_n\}$ is a nondecreasing sequence convergent to $x$ in $C(I, Y)$. For each $t_0 \in I$, we have
$$|x_n(t_0) - x(t_0)| \leq d(x_n, x),$$
and thus \( \lim_{n \to \infty} x_n(t_0) = x(t_0) \). Since \( \{x_n\} \) is a nondecreasing sequence, we have
\[
x_1(t_0) \leq x_2(t_0) \leq \cdots \leq x_n(t_0) \leq x_{n+1}(t_0) \leq \cdots.
\]
Since \( Y \) is \( \sigma \)-order continuous, we get
\[
\sup_n x_n(t_0) = \lim_{n \to \infty} x_n(t_0) = x(t_0),
\]
and so \( x_n(t_0) \leq x(t_0) \), for each \( n \). Finally, Theorem 1.6 gives that \( F \) has a unique fixed point. \( \square \)

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