

**ON THE FIXED POINT THEOREMS FOR
GENERALIZED WEAKLY CONTRACTIVE MAPPINGS
ON PARTIAL METRIC SPACES**

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ABSTRACT. In this paper, we prove a fixed point theorem for a pair of generalized weakly contractive mappings in complete partial metric spaces. The are generalizations of very recent fixed point theorems due to Abdeljawad, Karapınar and Taş.

1. Introduction

The notion of partial metric spaces, a generalization of metric spaces, was first introduced by Matthews (see [15, 16]). This generalization is based on the fact that the condition $d(x, x) = 0$ is replaced with the inequality $d(x, x) \leq d(x, y)$, for all x, y in the definition of metric. The idea of the partial metric space has a number of applications in the fields of computer sciences such as computer domain and semantics. As a consequence, a number of authors have recently focused on partial metric spaces and its topological properties to generalize well-known fixed point theorems in the class of metric spaces to the class of partial metric spaces (see [2, 9–14] and the references given therein). The purpose of our work here is to prove fixed point results in partial metric spaces for generalized weakly contractive mappings by using control functions. The presented

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theorems are generalizations of the very recent fixed point theorems investigated by Abdeljawad, et al. [2] and Karapinar [10]. We state an example to show that our generalizations are very effective in partial metric spaces.

We first need to recall some basic definitions.

Definition 1.1. (See e.g., [9], [15]) Let X be a nonempty set. The mapping $p : X \times X \rightarrow [0, \infty)$ is said to be a partial metric on X , if for any $x, y, z \in X$, the followings conditions hold true:

(P1) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$.

(P2) $p(x, x) \leq p(x, y)$.

(P3) $p(x, y) = p(y, x)$.

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then called a partial metric space.

Let (X, p) be a partial metric space. Then, the functions $d_p, d_m : X \times X \rightarrow [0, \infty)$, given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$d_m(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\},$$

are metrics on X . It is easy to verify that d_p and d_m are equivalent. Recall that each partial metric p on X generates a T_0 -topology τ_p with a base of the family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$.

Definition 1.2. (See e.g., [2], [9]) Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x)$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{n \rightarrow \infty, m \rightarrow \infty} p(x_n, x_m)$ exists (and is finite).

(3) (X, p) is called to be complete, if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.

(4) A mapping $f : X \rightarrow X$ is said to be continuous at $x_0 \in X$, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

It follows from [2] that if the sequence $\{x_n\}$ converges to $x \in X$ and $p(x, x) = 0$, then $\lim_{n \rightarrow \infty} p(x_n, y) = p(x, y)$, for every $y \in X$.

Example 1.3. Let $X = [0, +\infty)$ and define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then, (X, p) is a complete partial metric space. Obviously, p is not a (usual) metric.

Proposition 1.4. (See e.g., [2], [9]) *Let (X, p) be a partial metric space.*

(1) *A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_p) .*

(2) *(X, p) is complete if and only if (X, d_p) is complete. Moreover,*

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_m, x_n).$$

The notion of (ψ, φ) -contraction was studied by several authors (see e.g., [8]). A self mapping T on a metric space (X, d) is called a (ψ, φ) -contraction, if

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad \forall x, y \in X,$$

where $\psi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$.

In 2009, Dorić [7] improved this contraction for two mappings and gave conditions for existence of a common fixed point. Very recently, Choudhury et al. [6] investigated more extensions of the notion of (ψ, φ) -contraction. They defined the concept of *generalized weakly contractive mappings* and proved the existence of fixed points for this class.

Here, we obtain a unique common fixed point for a pair of maps by using generalized weakly contractive mappings on complete partial metric spaces. Our result is an extension [2].

2. Main results

We start section with a lemma which is necessary in the proof of the main theorems.

Lemma 2.1. (See e.g., [2, 12]) *Let (X, p) be a complete PMS.*

- (A) *If $p(x, y) = 0$, then $x = y$,*
- (B) *If $x \neq y$, then $p(x, y) > 0$.*

We aim to prove the following result.

Theorem 2.2. *Let (X, p) be a complete partial metric space and $T, S : X \rightarrow X$ be self-mappings such that, for all $x, y \in X$,*

$$(2.1) \quad \psi(p(Tx, Sy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where,

(a) $\psi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions with $\psi(t) = 0$ if and only if $t = 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

(b) $M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Sy), \frac{p(x, Sy) + p(Tx, y)}{2}\}$, for every $x, y \in X$. Then, T and S have a unique common fixed point. Moreover, any fixed point of T is a fixed point of S and conversely.

Proof. Suppose that T and S have two common fixed points u and v , with $u \neq v$. Thus, $p(u, v) > 0$. From (2.1), we have

$$\begin{aligned} \psi(p(u, v)) &= \psi(p(Tu, Tv)) \\ &\leq \psi\left(\max\{p(u, v), p(u, Tu), p(v, Tv), \frac{p(u, Sv) + p(Tu, v)}{2}\}\right) \\ &\quad - \varphi\left(\max\{p(u, v), p(u, Tu), p(v, Tv), \frac{p(u, Sv) + p(Tu, v)}{2}\}\right), \end{aligned}$$

that is,

$$(2.2) \quad \psi(p(u, v)) \leq \psi(p(u, v)) - \varphi(p(u, v)).$$

Since $p(u, v) > 0$, by definition of φ, ψ , we have $\psi(p(u, v)) > 0$ and $\varphi(p(u, v)) > 0$. Thus, the inequality (2.2) is impossible. Hence, if the common fixed point exists, then it is unique. Now, suppose that u is a fixed point of T and $u \neq Su$. By (2.1), we get

$$\psi(p(u, Su)) \leq \psi(p(u, Su)) - \varphi(p(u, Su)),$$

which is a contradiction by virtue of a property of φ . Hence, $Su = u$. Using a similar argument, we infer that any fixed point of S is also a fixed point of T .

Now, let $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, for all $n = 0, 1, 2, 3, \dots$. If there exists a positive integer N such that $x_{2N} = x_{2N+1}$, then x_{2N} is a fixed point of T and hence a fixed point of S . A similar conclusion holds, if $x_{2N+1} = x_{2N+2}$, for some positive integer N . Therefore, we may assume that $x_n \neq x_{n+1}$,

for all n . From (2.1) again, we have

$$\begin{aligned}
 (2.3) \quad & \psi(p(x_{2n+1}, x_{2n+2})) \\
 &= \psi(p(Tx_{2n}, Sx_{2n+1})) \\
 &\leq \psi\left(\max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \right. \\
 &\quad \left. \frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{2}\}\right) \\
 &\quad - \varphi\left(\max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \right. \\
 &\quad \left. \frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{2}\}\right),
 \end{aligned}$$

for each $n = 0, 1, 2, \dots$.

Since

$$\begin{aligned}
 & \frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{2} \\
 & \leq \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) - p(x_{2n+1}, x_{2n+1}) + p(x_{2n+1}, x_{2n+1})}{2} \\
 & = \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2} \\
 & \leq \max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\},
 \end{aligned}$$

it follows that

$$(2.4) \quad \psi(p(x_{2n+1}, x_{2n+2})) \leq \psi(\max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}) - \varphi(\max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}).$$

Suppose that $p(x_{2n}, x_{2n+1}) \leq p(x_{2n+1}, x_{2n+2})$, for some positive integer n . Then, from (2.3), we obtain

$$\psi(p(x_{2n+1}, x_{2n+2})) \leq \psi(p(x_{2n+1}, x_{2n+2})) - \varphi(p(x_{2n+1}, x_{2n+2})),$$

that is, $\varphi(p(x_{2n+1}, x_{2n+2})) \leq 0$. By definition of the function φ , we have $p(x_{2n+1}, x_{2n+2}) = 0$. By Lemma 2.1, we have $x_{2n+1} = x_{2n+2}$, contradicting with our assumption that $x_n \neq x_{n+1}$, for all n . Therefore,

$$p(x_{2n+1}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}), \text{ for all } n.$$

With a similar argument, it follows that

$$p(x_{2n+2}, x_{2n+3}) \leq p(x_{2n+1}, x_{2n+2}), \text{ for all } n.$$

Thus, $\{p(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there exists an $r \geq 0$ such that

$$(2.5) \quad \lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = r.$$

In the view of the facts above, it follows from (2.4) that

$$\psi(p(x_{2n+1}, x_{2n+2})) \leq \psi(p(x_{2n}, x_{2n+1})) - \varphi(p(x_{2n}, x_{2n+1})), \text{ for all } n.$$

Taking the limits as $n \rightarrow \infty$ in the inequality above and using the continuities of ψ and φ , we get

$$\psi(r) \leq \psi(r) - \varphi(r).$$

By virtue of a property of φ , we infer that $r = 0$. Hence,

$$(2.6) \quad \lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0.$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence. By using (2.6), it is sufficient to prove that $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\varepsilon > 0$ and a non-negative real number a , for which we can seek two positive sequences of integers $\{2m_k\}$ and $\{2n_k\}$ such that, for all positive integer k ,

$$(2.7) \quad \begin{cases} 2n_k > 2m_k > k \\ p(x_{2m_k}, x_{2n_k}) \geq a + \varepsilon \\ p(x_{2m_k}, x_{2n_k-2}) < a + \varepsilon, \end{cases}$$

or

$$(2.8) \quad \begin{cases} 2n_k > 2m_k > k \\ p(x_{2m_k}, x_{2n_k}) \leq a - \varepsilon \\ p(x_{2m_k}, x_{2n_k-2}) > a - \varepsilon. \end{cases}$$

Note that if $a = 0$, then we can reduce to the only case (2.7). Now, if (2.7) holds, then we have

$$\begin{aligned} a + \varepsilon &\leq p(x_{2m_k}, x_{2n_k}) \leq p(x_{2m_k}, x_{2n_k-2}) + p(x_{2n_k}, x_{2n_k-2}) \\ &\quad - p(x_{2n_k-2}, x_{n_k-2}) \\ &\leq p(x_{2m_k}, x_{2n_k-2}) + p(x_{2n_k}, x_{2n_k-2}) \\ &\leq p(x_{2m_k}, x_{2n_k-2}) + p(x_{2n_k}, x_{2n_k-1}) \\ &\quad + p(x_{2n_k-1}, x_{2n_k-2}) - p(x_{2n_k-1}, x_{2n_k-1}) \\ &\leq p(x_{2m_k}, x_{2n_k-2}) + p(x_{2n_k}, x_{2n_k-1}) \\ &\quad + p(x_{2n_k-1}, x_{2n_k-2}) \\ &\leq a + \varepsilon + p(x_{2n_k}, x_{2n_k-1}) + p(x_{2n_k-1}, x_{2n_k-2}). \end{aligned}$$

Letting $k \rightarrow \infty$ in the inequality above and using (2.6), we obtain

$$(2.9) \quad \lim_{k \rightarrow \infty} p(x_{2m_k}, x_{2n_k}) = a + \varepsilon.$$

By elementary computations, we get

$$p(x_{2m_k}, x_{2n_k}) \leq p(x_{2m_k}, x_{2m_k+1}) + p(x_{2m_k+1}, x_{2n_k+1}) + p(x_{2n_k+1}, x_{2n_k})$$

and

$$p(x_{2m_k+1}, x_{2n_k+1}) \leq p(x_{2m_k+1}, x_{2m_k}) + p(x_{2m_k}, x_{2n_k}) + p(x_{2n_k}, x_{2n_k+1}).$$

Letting $k \rightarrow \infty$ in the inequalities above and combining with (2.6) and (2.9), we have

$$(2.10) \quad \lim_{k \rightarrow \infty} p(x_{2m_k+1}, x_{2n_k+1}) = a + \varepsilon.$$

Again, by (P4), we have

$$\begin{aligned} p(x_{2n_k+2}, x_{2m_k+1}) &\leq p(x_{2m_k+2}, x_{2n_k+1}) + p(x_{2m_k+1}, x_{2n_k+1}) \\ &\quad - p(x_{2n_k+1}, x_{2n_k+1}) \\ &\leq p(x_{2n_k+2}, x_{2n_k+1}) + p(x_{2m_k+1}, x_{2n_k+1}) \end{aligned}$$

and

$$\begin{aligned} p(x_{2n_k+1}, x_{2m_k+1}) &\leq p(x_{2n_k+1}, x_{2n_k+2}) + p(x_{2n_k+2}, x_{2m_k+1}) \\ &\quad - p(x_{2n_k+2}, x_{2n_k+2}) \\ &\leq p(x_{2n_k+1}, x_{2n_k+2}) + p(x_{2n_k+2}, x_{2m_k+1}). \end{aligned}$$

Similarly,

$$p(x_{2m_k}, x_{2n_k+1}) \leq p(x_{2m_k}, x_{2n_k}) + p(x_{2n_k}, x_{2n_k+1})$$

and

$$p(x_{2m_k}, x_{2n_k}) \leq p(x_{2m_k}, x_{2n_k+1}) + p(x_{2n_k}, x_{2n_k+1}).$$

Furthermore, we have

$$\begin{aligned} p(x_{2m_k}, x_{2n_k+2}) &\leq p(x_{2m_k}, x_{2m_k+1}) + p(x_{2m_k+1}, x_{2n_k+1}) \\ &\quad + p(x_{2n_k+1}, x_{2n_k+2}) \end{aligned}$$

and

$$\begin{aligned} p(x_{2m_k+1}, x_{2n_k+1}) &\leq p(x_{2m_k+1}, x_{2m_k}) \\ &\quad + p(x_{2m_k}, x_{2n_k+2}) + p(x_{2n_k+2}, x_{2n_k+1}). \end{aligned}$$

Taking limits as $k \rightarrow \infty$ in the six inequalities above and using (2.6), (2.9) and (2.10), we get

$$(2.11) \quad \lim_{k \rightarrow \infty} p(x_{2n_k+2}, x_{2m_k+1}) = a + \varepsilon,$$

$$(2.12) \quad \lim_{k \rightarrow \infty} p(x_{2m_k}, x_{2n_k+2}) = a + \varepsilon,$$

and

$$(2.13) \quad \lim_{k \rightarrow \infty} p(x_{2m_k}, x_{2n_k+1}) = a + \varepsilon.$$

Applying (2.1) with $x = x_{2m_k}, y = x_{2n_k+1}$, we have

$$\begin{aligned} & \psi(p(x_{2m_k+1}, x_{2n_k+2})) \\ &= \psi(p(Tx_{2m_k}, Sx_{2n_k+1})) \\ &\leq \psi\left(\max\{p(x_{2m_k}, x_{2n_k+1}), p(x_{2m_k}, x_{2m_k+1}), p(x_{2n_k+1}, x_{2n_k+2}), \right. \\ &\quad \left. \frac{p(x_{2m_k+1}, x_{2n_k+2}) + p(x_{2n_k+1}, x_{2n_k+2})}{2}\}\right) \\ &\quad - \varphi\left(\max\{p(x_{2m_k}, x_{2n_k+1}), p(x_{2m_k}, x_{2m_k+1}), p(x_{2n_k+1}, x_{2n_k+2}), \right. \\ &\quad \left. \frac{p(x_{2m_k+1}, x_{2n_k+2}) + p(x_{2n_k+1}, x_{2n_k+2})}{2}\}\right). \end{aligned}$$

Letting $k \rightarrow \infty$ in the above inequality, using (2.6), (2.10)-(2.13), and using continuities of ψ and φ , we get

$$\psi(a + \varepsilon) \leq \psi(a + \varepsilon) - \varphi(a + \varepsilon).$$

By virtue of a property of φ , we obtain a contradiction. In the case (2.8), we may assume that $a - \varepsilon > 0$. By the same computation as in the previous case, we derive

$$\psi(a - \varepsilon) \leq \psi(a - \varepsilon) - \varphi(a - \varepsilon),$$

and get a contradiction. Therefore, $\{x_{2n}\}$ is a Cauchy sequence, and so is $\{x_n\}$. From the completeness of X , there exists a $u \in X$ such that $x_n \rightarrow u$, as $n \rightarrow \infty$, and

$$(2.14) \quad p(u, u) = \lim_{m, n \rightarrow \infty} p(x_m, x_n) = \lim_{n \rightarrow \infty} p(x_n, u).$$

For $x = x_{2n}, y = u$ in (2.1), we have

$$\begin{aligned} \psi(p(x_{2n+1}, Su)) &= \psi(p(Tx_{2n}, Su)) \\ &\leq \psi\left(\max\{p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(u, Su), \right. \\ &\quad \left. \frac{p(x_{2n}, Su) + p(u, x_{2n+1})}{2}\}\right) \\ &\quad - \varphi\left(\max\{p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(u, Su), \right. \\ &\quad \left. \frac{p(x_{2n}, Su) + p(u, x_{2n+1})}{2}\}\right). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the above inequality, using (2.6) and (2.14), and using the continuities of ψ and φ , we have

$$\psi(p(u, Su)) \leq \psi(p(u, Su)) - \varphi(p(u, Su)),$$

which implies $Su = u$, by Lemma 2.1. By what we have already shown, we can conclude that u is the common fixed point of T and S . \square

In Theorem 2.2, if we choose $\psi(t) = t$, for all $t \in [0, \infty)$, then we can get the main result of [2].

Corollary 2.3. (See [2]) *Let (X, p) be a complete partial metric space and $T, S : X \rightarrow X$ be self-mappings such that, for all $x, y \in X$,*

$$(2.15) \quad p(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)),$$

where φ and $M(x, y)$ are defined as in Theorem 2.2. Then, T and S have a unique common fixed point. Moreover, any fixed point of T is a fixed point of S and conversely.

As in the cases of metric spaces (see [7]), the effectiveness of generalization with respect to the previous result may be seen from the fact in the cases $\varphi(t) = t$, Theorem 2.2 still holds, while condition (2.15) is useless.

Example 2.4. Let $X := [0, 1] \cup [2, 3]$ and define $p : X \times X \rightarrow [0, \infty)$ as

$$p(x, y) = \begin{cases} |x - y|, & \text{if } \{x, y\} \subset [0, 1] \\ \max\{x, y\}, & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset. \end{cases}$$

It is easy to check that (X, p) is a complete partial space. Let $T, S : X \rightarrow X$ be defined by

$$Tx = \frac{x}{3} \text{ and } Sx = 0 \text{ for all } x \in X.$$

It is easy to see that 0 is the unique fixed point of T and S . By an elementary computation, we obtain

$$p(Tx, Sy) = \frac{x}{3}, \text{ for all } x, y \in X,$$

and for all $x, y \in [0, 1]$,

$$(2.16) \quad \begin{aligned} M(x, y) &= \max\{|x - y|, \frac{2x}{3}, y, \frac{1}{2}(|y - \frac{x}{3}| + x)\} \\ &= \begin{cases} x - y, & \text{if } 0 \leq y \leq \frac{x}{3} \\ \frac{2x}{3}, & \text{if } \frac{x}{3} \leq y \leq \frac{2x}{3} \\ y, & \text{if } \frac{2x}{3} < y \leq 1. \end{cases} \end{aligned}$$

For $x, y \in X \cap [2, 3]$, we may assume that $x \geq y$. Then,

$$(2.17) \quad \begin{aligned} M(x, y) &= \max\{p(x, y), p(x, Tx), p(y, Sy), \frac{p(x, Sy) + p(Tx, y)}{2}\} \\ &= \max\{p(x, y), p(x, \frac{x}{3}), p(y, 0), \frac{p(x, 0) + p(\frac{x}{3}, y)}{2}\} \\ &= x. \end{aligned}$$

For $\psi(t) = 3t$ and $\varphi(t) = t$, we have

$$\begin{aligned} \psi(p(Tx, Sy)) &= x, \\ \psi(M(x, y)) - \varphi(M(x, y)) &= \begin{cases} 2(x - y), & \text{if } 0 \leq y \leq \frac{x}{3} \\ \frac{4x}{3}, & \text{if } \frac{x}{3} \leq y \leq \frac{2x}{3} \\ 2y, & \text{if } \frac{2x}{3} < y \leq 1. \end{cases} \end{aligned}$$

for every $x, y \in [0, 1]$, and

$$\psi(M(x, y)) - \varphi(M(x, y)) = 2x,$$

for $x \in X \cap [2, 3]$ or $y \in X \cap [2, 3]$ with $x \geq y$. This implies that T and S satisfy the condition (2.1) in Theorem 2.2. Note that, if we fix $\varphi(t) = t$, then T and S do not satisfy Corollary 2.3.

As a corollary, we immediately have the following result.

Theorem 2.5. *Let (X, p) be a complete partial metric space and $T : X \rightarrow X$ be a self-mapping such that, for all $x, y \in X$,*

$$(2.18) \quad \psi(p(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where,

(a) $\psi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions with $\psi(t) = 0$ if and only if $t = 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

(b) $M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(Tx, y)}{2}\}$, for every $x, y \in X$. Then, T has a unique common fixed point.

The following result is another corollary of Theorem 2.2.

Corollary 2.6. Let (X, p) be a complete partial metric space and $T, S : X \rightarrow X$ be self-mappings such that, for all $x, y \in X$, and some positive integer m and n ,

$$(2.19) \quad \psi(p(T^m x, S^n y)) \leq \psi(M_{m,n}(x, y)) - \varphi(M_{m,n}(x, y)),$$

where,

(a) $\psi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions with $\psi(t) = 0$ if and only if $t = 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

(b) $M_{m,n}(x, y) = \max\{p(x, y), p(x, T^m x), p(y, S^n y), \frac{p(x, S^n y) + p(T^m x, y)}{2}\}$, for every $x, y \in X$. Then, T and S have a unique common fixed point. Moreover, any fixed point of T is a fixed point of S and vice versa.

Proof. By Theorem 2.2, we may deduce that T^m and S^n have a unique common fixed point u . Any fixed point of T^m is a fixed point of S^n and vice versa. By the fact that every fixed point of T is a fixed point of T^m , it is sufficient to prove that u is a common fixed point of T and S . From (2.19), we have

$$\begin{aligned} \psi(p(Tu, u)) &= \psi(p(TT^m u, S^n u)) = \psi(p(T^m Tz, S^n u)) \\ &\leq \psi(M_{m,n}(Tu, u)) - \varphi(M_{m,n}(Tu, u)), \end{aligned}$$

where

$$\begin{aligned} M_{m,n}(Tu, u) &= \max\{p(T^m Tu, Tu), p(S^n u, u), p(Tu, u), \\ &\quad \frac{p(T^m Tu, u) + p(S^n u, Tu)}{2}\} \\ &= \max\{p(Tu, Tu), p(Tu, u)\}. \end{aligned}$$

By the fact that $p(Tu, Tu) \leq p(Tu, u)$, we obtain

$$\psi(p(Tu, u)) \leq \psi(p(Tu, u)) - \varphi(p(Tu, u)),$$

that is, $p(Tu, u) = 0$ or $Tu = u$. By the same argument, u is a fixed point of S . The proof is complete. \square

Remark 2.7. *We noticed that Abdeljawad [3] published very recently a paper on this subject. However, the results in [3] are weaker than ours.*

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