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ε-SIMULTANEOUS APPROXIMATIONS OF DOWNWARD SETS

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Abstract. We prove some results on characterization of ε-simultaneous approximations of downward sets in vector lattice Banach spaces. Also, we give some results about simultaneous approximations of normal sets.

1. **Introduction**

The theory of best simultaneous approximation has been studied by many authors (for example, [2, 9]). Singer [8] introduced the concept of ε-simultaneous approximation. Best simultaneous approximation is a generalization of best approximation and ε-simultaneous approximation in a sense is a generalization of best simultaneous approximation. Most studies about best simultaneous approximation have been done on convex sets. However, convexity is sometimes a very restrictive assumption. Here, we shall prove some results on characterization of ε-simultaneous approximations of downward sets in vector lattice Banach spaces.

There are many spaces along with an order $\leq$. The L^p and $C(X)$ spaces are some examples. The notion of an order in a vector space facilitates the study of the spaces in an abstract setting. First, let us give some basic preliminaries concerning vector lattices (see [1, 3]).

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Definition 1.1. *A lattice $(L, \leq)$ is said to be conditionally complete if it satisfies one of the following equivalent conditions:*

(1) *Every non-empty lower bounded set admits an infimum.*
(2) *Every non-empty upper bounded set admits an supremum.*
(3) *There exists a complete lattice $\bar{L} := L \cup \{\perp, \top\}$, which we call the minimal completion of L, with bottom element $\perp$ and top element $\top$, such that L is a sublattice of $\bar{L}$, $infL = \perp$ and $supL = \top$.*

A (real) vector lattice $(X, \leq, +, .)$ is a set X endowed with a partial order $\leq$ such that $(X, \leq)$ is a lattice, with a binary operation $+$ and a scalar product. A vector lattice $(X, \leq, +, .)$ such that $(X, \leq)$ is a conditionally complete lattice is called conditionally complete vector lattice. A conditionally complete lattice Banach space X is a real Banach space that is a conditionally complete vector lattice and $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, for all $x, y \in X$.

Let X be a normed space. For a non-empty subset W of X and a non-empty bounded set S in X, define $d(S, W) = \inf_{w \in W} \sup_{s \in S} \|s - w\|$. An element $w_0 \in W$ is called a best simultaneous approximation to S from W, if $d(S, W) = \sup_{s \in S} \|s - w_0\|$. The set of all best simultaneous approximation to S from W will be denoted by $S_W(S)$.

Definition 1.2. *Let X be a normed space, W a subset of X and S a bounded set in X. An element $w_0 \in W$ is called ε-simultaneous approximation, if*

$$\sup_{s \in S} \|s - w_0\| \leq d(S, W) + \varepsilon.$$

The set of all ε-simultaneous approximations to S from W will be denoted by $S_{W,\varepsilon}(S)$.

One advantage of considering the set $S_{W,\varepsilon}(S)$, instead of the set $S_W(S)$, is that the set $S_{W,\epsilon}(S)$ is always nonempty, for all $\varepsilon > 0$.

If for each bounded set S in X there exists at least one best simultaneous approximation to S from W, then W is called a simultaneous proximinal subset of X. If for each bounded set S in X there exists a unique best simultaneous approximation to S from W, then W is called a simultaneous Chebyshev subset of X.

Here, we study best simultaneous approximations in conditionally complete lattice Banach spaces with a strong unit **1**. Recall that an element $\mathbf{1} \in X$ is called a strong unit, if for each $x \in X$ there exists $\lambda > 0$ such that $x \leq \lambda\mathbf{1}$ (see [1]). We assume that X contains a strong unit $\mathbf{1}$. By using the strong unit $\mathbf{1}$, we can define a norm on X by $\|x\| = inf\{\lambda > 0 : |x| \leq \lambda\mathbf{1}\}$, for all $x \in X$. Also, we define

$$B(S, r) := \{y \in X : supS - r\mathbf{1} \leq y \leq infS + r\mathbf{1}\}, \tag{1.1}$$

where $r > 0$ and S is a bounded set in X. It is clear that $B(S, r)$ is a closed convex subset of X. We also have

$$|x| \leq \|x\|\mathbf{1}, \text{ for all } x \in X. \tag{1.2}$$

It is well known that X equipped with this norm is a conditionally complete lattice Banach space. Recall that a subset W of an ordered set X is said to be downward whenever for each $w \in W$ and $x \in X$ with $x \leq w$, we can conclude that $x \in W$. For each subset W of a normed space X, define the polar set of W by

$$W^0 := \{f \in X^* : f(w) \leq 0, \text{ for all } w \in W\},$$

where X^* is the dual space of X. If X is a lattice and there exists the least element of W, then we denote it by $minW$. Let $\varphi : X \times X \longrightarrow \mathbb{R}$ be a function defined by

$$\varphi(x, y) := sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + y,\} \text{ for all } x, y \in X. \tag{1.3}$$

Since $\mathbf{1}$ is a strong unit, the set $\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x + y\}$ is non-empty and bounded from above by $\|x + y\|$. Clearly, this set is closed. It follows from the definition of φ that the function enjoys the following properties:

$$(1.4) \quad -\infty < \varphi(x, y) \leq \|x + y\|, \text{ for all } x, y \in X$$
$$(1.5) \quad \varphi(x, y)\mathbf{1} \leq x + y, \text{ for all } x, y \in X$$
$$(1.6) \quad \varphi(x, y) = \varphi(y, x), \text{ for all } x, y \in X$$
$$(1.7) \quad \varphi(x, -x) = sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq x - x = 0\} = 0, \text{ for all } x \in X.$$

For each $y \in X$, define the function $\varphi_y : X \to \mathbb{R}$ by

$$\varphi_y(x) := \varphi(x, y), \text{ for all } x \in X. \tag{1.8}$$

A function$f : X \to \mathbb{R}$ is called topical if it is increasing. The function φ_y defined by (1.8) is topical and Lipschitz continuous (see [5]). In fact, we have

$$|\varphi_y(x) - \varphi_y(z)| \leq \|x - z\|, \text{ for all } x, z \in X. \tag{1.9}$$

Also, the function φ, defined by (1.3), is continuous.

2. ε-simultaneous approximations of downward sets

Let X be a conditionally complete lattice Banach space with a strong unit $\mathbf{1}$. In this section, we prove some results about ε-simultaneous approximation of downward sets. We start with the following results for easy citation.

Lemma 2.1. [4] *Let W be a downward subset of X and $x \in X$. Then, the following statements are true:*

(1) *If $x \in W$, then $x - \epsilon\mathbf{1} \in intW$, for all $\varepsilon > 0$,*
(2) *We have $intW = \{x \in X : x + \varepsilon\mathbf{1} \in W,$ for some $\varepsilon > 0\}$.*

Lemma 2.2. [4] *Let W be a downward subset of X and S be an arbitrary bounded subset of X. If $r = d(S, W)$, then $w_0 = \sup S - r1 \in S_W(S)$ and is the least element of $S_W(S)$. Thus, W is a simultaneous proximinal subset of X.*

Lemma 2.3. [5] *Let W be a closed downward subset of X, $y_0 \in bdW$ and φ be the function defined by (1.3). Then, $\varphi(w, -y_0) \leq 0$, for all $w \in W$.*

Let W be a closed subset of X and S be a bounded subset of X such that $S \cap W = \phi$. In addition, suppose that $w_0 \in intW \cap S_{W,\varepsilon}(S)$. Thus, there exists $\alpha > 0$ such that

$$V = \{y \in X : \|y - w_0\| < \alpha\} \subset W.$$

Lemma 2.4. *Let α be as above. Then, $\alpha \leq \varepsilon$.*

Proof. Assume that $r = d(S, W)$ and $\varepsilon < \alpha$. Let $\varepsilon_0 = \frac{\alpha}{r+\alpha}, s \in S$ and

$$w_s = w_0 + \varepsilon_0(s - w_0).$$

Note that $\|w_s - w_0\| = \varepsilon_0\|s - w_0\| \leq \varepsilon_0(r + \varepsilon) = \alpha\frac{r+\varepsilon}{r+\alpha} < \alpha$, because $\frac{r+\varepsilon}{r+\alpha} < 1$ and $\sup_{s\in\ S} \|s - w_0\| \leq r + \varepsilon$. Then, $w_s \in V$, for all $s \in S$ and

$$r = d(S, W) \leq \sup_{t\in S} \|t - w_s\|, \text{ for all } s \in S.$$

Thus, $r \leq \inf_{s\in S} \sup_{t\in S} \|t - w_s\|$. On the other hand, we have

$$\|t - w_s\| = \|(t - w_0) - \varepsilon_0(s - w_0)\|, \text{ for all } t, s \in S.$$

This implies that

$$r \leq \inf_{s\in S} \sup_{t\in S} \|t - w_s\| = \inf_{s\in S} \sup_{t\in S} \|(t - w_0) - \varepsilon_0(s - w_0)\|$$

$$\leq \sup_{t\in S} \|(t-w_0)-\varepsilon_0(t-w_0)\| = (1-\varepsilon_0)\sup_{t\in S}\|t-w_0\| \leq (1-\varepsilon_0)(r+\varepsilon) < r.$$

This contradiction completes the proof. □

By using Lemma 2.4, it is easy to prove the following result.

Proposition 2.5. *Let W be a closed subset of X and S be a bounded subset of X such that $S \cap W = \phi$. Then, $S_{W,\varepsilon}(S) \subset V = \{w - \alpha\mathbf{1} :$ for some $w \in bdW$ and $0 \leq \alpha \leq \varepsilon\}$.*

Corollary 2.6. *Let W be a closed subset of X and S be a bounded subset of X such that $S \cap W = \phi$. Then, $S_W(S) \subset bdW$.*

Proposition 2.7. *Let W be a closed downward subset of X, and S be a bounded subset of X. Then, there exists the least element $w_0 := minS_{W,\varepsilon}(S)$.*

Proof. Put $r := d(S,W)$ and $w_0 = \sup S - (r+\varepsilon)\mathbf{1} \leq \sup S - r\mathbf{1}$. By Lemma 2.2, $\sup S - r\mathbf{1} \in W$. Since W is a downward set, $\sup S - (r + \epsilon)\mathbf{1} \in W$. Therefore, $w_0 \in S_{W,\varepsilon}(S)$, and so $\sup_{s\in S}\|s - w_0\| \leq r + \varepsilon$. Thus, $w \leq w_0$, for all $w \in S_{W,\varepsilon}(S)$. Hence, $w_0 := minS_{W,\varepsilon}(S)$. □

Proposition 2.8. *Let W be a closed downward subset of X, S be a bounded subset of X such that $S \cap W = \phi$, $w_0 \in S_{w,\varepsilon}(S)$ and φ be the function defined by (1.3). Then, $\varphi(w, -w_0) \leq \varepsilon$, for all $w \in W$.*

Proof. By Proposition 2.5, there exist $y_0 \in bdW$ and $0 \leq \alpha \leq \varepsilon$ such that $w_0 = y_0 - \alpha\mathbf{1}$. By Lemma 2.3, we have

$$\begin{aligned}
\varphi(w,-w_0) &= \varphi(w,\alpha\mathbf{1}-y_0)\\
&= \sup\{\lambda\in\mathbb{R} : \lambda\mathbf{1} \leq w + \alpha\mathbf{1} - y_0\}\\
&= \sup\{\lambda\in\mathbb{R} : (\lambda-\alpha)\mathbf{1} \leq w - y_0\}\\
&= \sup\{\beta+\alpha\in\mathbb{R} : \beta\mathbf{1} \leq w - y_0\}\\
&= \sup\{\beta\in\mathbb{R} : \beta\mathbf{1} \leq w - y_0\} + \alpha\\
&= \varphi(w,-y_0) + \alpha \leq \varepsilon.
\end{aligned}$$

This completes the proof. □

Theorem 2.9. *Let W be a closed downward subset of X, S be a bounded subset of X such that $S \cap W = \phi$, $y_0 \in W$, $r_0 = \sup_{s\in S}\|s - y_0\|$ and φ be the function defined by (1.3). Then, the following statements are equivalent:*

(1) $y_0 \in S_{W,\varepsilon}(S)$.
(2) *There exists* $l \in X$ *such that*

$$\varphi(w,l) \leq \varepsilon \leq \varphi(y,l), \text{ for all } w \in W, \ y \in B(S,r_0). \tag{2.1}$$

Moreover, if (2.1) holds with $l = -y_0$, *then* $y_0 = \min S_{W,\varepsilon}(S)$.

Proof. (1) $\Longrightarrow$ (2). Suppose that $y_0 \in S_{W,\epsilon}(S)$. Then, $r_0 = \sup_{s\in S} \|s - y_0\| \leq r + \epsilon$, where $r = d(S,W)$. Since W is a closed downward subset of X, by Lemma 2.2, the least element $\sup S - r\mathbf{1}$ of $S_W(S)$ exists. Let $w_0 := \sup S - (r_0 + \varepsilon)\mathbf{1}$. Note that

$$r \leq r_0 \Rightarrow (-r_0 - \varepsilon)\mathbf{1} \leq -r_0\mathbf{1} \leq -r\mathbf{1} \Rightarrow \sup S - (r_0 + \varepsilon)\mathbf{1} \leq \sup S - r\mathbf{1}.$$

By Lemma 2.2, we get $w_0 \in W$. Let $l = -w_0$ and $y \in B(S,r_0)$ be arbitrary. Thus, by using (1.1), we have $-r_0\mathbf{1} \leq y - \sup S$. This implies

$$-r_0 \in \{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq y - \sup S\}.$$

Hence, we obtain

$$\begin{aligned}
\varphi(y,l) &= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y + l\} \\
&= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y - w_0\} \\
&= \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \leq y - (\sup S - (r_0 + \varepsilon)\mathbf{1})\} \\
&= \sup\{\lambda \in \mathbb{R} : (\lambda - r_0 - \varepsilon)\mathbf{1} \leq y - \sup S\} \\
&= \sup\{\alpha + r_0 + \varepsilon \in \mathbb{R} : \alpha\mathbf{1} \leq y - \sup S\} \\
&= \sup\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \leq y - \sup S\} + r_0 + \varepsilon \\
&\geq -r_0 + \varepsilon + r_0 = \varepsilon.
\end{aligned}$$

On the other hand, since $w_0 \in S_{W,\varepsilon}(S)$, by using Proposition 2.8, we get $\varphi(w, -w_0) \leq \varepsilon$, for all $w \in W$. Therefore, $\varphi(w,l) \leq \varepsilon$.

(2) $\Longrightarrow$ (1). Assume that there exists $l \in X$ such that $\varphi(w,l) \leq \varepsilon \leq \varphi(y,l)$, for all $w \in W$ and $y \in B(S,r_0)$. Since $B(S,r_0) = \{y \in X : supS - r_0\mathbf{1} \leq y \leq infS + r_0\mathbf{1}\}$, $supS - r_0\mathbf{1} \in B(S,r_0)$. Thus, we get $\varphi(supS - r_0\mathbf{1}, l) \geq \varepsilon \geq 0$. By definition of φ, we have $\varphi(\sup S, l) \geq r_0$. Hence, by using (1.5), we have

$$r_0\mathbf{1} \leq \varphi(\sup S, l)\mathbf{1} \leq \sup S + l. \tag{2.2}$$

Therefore, $-\sup S \leq l - r_0\mathbf{1}$. Now, let $w \in W$ and $t_w = \varphi(w, -\sup S)\mathbf{1} + \sup S \in X$. By (1.5), $\varphi(w, -\sup S)\mathbf{1} \leq w - \sup S$. Since W is a downward set and $w \in W$, $t_w \in W$ and so $\varphi(t_w, l) \leq \varepsilon$. Since $\varphi(t_w, .)$ is topical, by using (2.2), we have

$$\varphi(t_w, -\sup S) \leq \varphi(t_w, l - r_0\mathbf{1}) = \varphi(t_w, l) - r_0 \leq \varepsilon - r_0.$$

Since $\varphi(., -\sup S)$ is topical and $t_w = \varphi(w, -\sup S)\mathbf{1} + \sup S$, from (1.7) we get

$$\begin{aligned} \varepsilon - r_0 &\geq \varphi(t_w, -\sup S) = \varphi(\varphi(w, -\sup S)\mathbf{1} + \sup S, -\sup S) \\ &= \varphi(w, -\sup S) + \varphi(\sup S, -\sup S) = \varphi(w, -\sup S). \end{aligned}$$

Now, by using (1.7) and Lipschitz continuity of $\varphi_{-\sup S} := \varphi(., -\sup S)$, we obtain

$$\begin{aligned} \varepsilon + r_0 \quad &\leq \quad |\varphi(w, -\sup S)| \\ &= |\varphi(\sup S, -\sup S) - \varphi(w, -\sup S)| \\ &\leq \| \sup S - w\|. \end{aligned}$$

Thus, $-\varepsilon + r_0 \leq |\sup S - w\| \leq \sup_{s\in S} |s - w\|$, for all $w \in W$ and so we obtain $-\varepsilon + r_0 \leq r = d(S, W)$. Consequently, $r_0 \leq r + \varepsilon$ and $y_0 \in S_{W,\varepsilon}(S)$. Finally, suppose that (2.1) holds with $l = -y_0$. Then, by the implication (1) $\Longrightarrow$ (2), we have $y_0 \in S_{W,\varepsilon}(S)$. Let $w_1 \in S_{W,\varepsilon}(S)$ be arbitrary. If $r_1 = \sup_{s\in S} \|s - w_1\|$, then by the implication (1) $\Longrightarrow$ (2) we have $\varphi(w, l) \leq \varepsilon \leq \varphi(y, l)$, for all $w \in W$ and $y \in B(S, r_1)$, where $l = -\sup S + (r_1 + \varepsilon)\mathbf{1}$. Since $y_0 \in W$, $\varphi(y_0, l) = \varphi(y_0, -\sup S + (r_1 + \varepsilon)\mathbf{1}) \leq \varepsilon$. Thus, from definition of φ, we get $y_0 - \sup S + (r_1 + \varepsilon)\mathbf{1} \leq \varepsilon\mathbf{1}$. Hence, $y_0 \leq \sup S - r_1\mathbf{1}$. Therefore, $\sup S - r_1\mathbf{1} \leq w_1$ and so $y_0 \leq w_1$. Thus, $y_0 = \min S_{W,\varepsilon}(S)$. □

Here, we recall that a downward set W is called strictly downward, if for each boundary point w_0 of W, the inequality $w_0 < w$ implies $w \notin W$. For example, the level sets of a continuous strictly increasing real function give rise to strictly downward sets ([5, 6, 7]).

Theorem 2.10. *Let W be a closed downward subset of X and S be a bounded subset of X such that $S \cap W = \phi$. Then, the following statements are equivalent:*

(1) *W is a strictly downward subset of X.*
(2) *W is a simultaneous Chebyshev subset of X.*

Proof. (1) $\Rightarrow$ (2). Since W is downward set, by using Lemma 2.2, W is simultaneous proximinal. We claim $S_W(S) = \{\sup S - r'\mathbf{1}\}$, where $r' = d(S, W)$. Let there exist $w_0 \in S_W(S)$ such that $w_0 \neq \sup S - r'\mathbf{1}$. In this case, by Corollary 2.6, $\sup S - r'\mathbf{1} \in bdW$. Also, by Lemma

2.2, $\sup S - r'\mathbf{1} < w_0$. Since W is a strictly downward set, this implies that $w_0 \notin W$, which is a contradiction. Therefore, W is a simultaneous Chebyshev set of X.

(2) $\Rightarrow$ (1). Let W be a simultaneous Chebyshev subset of X. If W is not a strictly downward, then there exists $w_0 \in bdW$, such that $w_0 < w$, for all $w \in W$. Let $r \geq \|w - w_0\| > 0$. It follows from (1.2) that

$$w - w_0 \leq |w - w_0| \leq \|w - w_0\|\mathbf{1} \leq r\mathbf{1},$$

and so $w \leq w_0 + r\mathbf{1}$. Let $S = \{w_0 + r\mathbf{1}\}$. Then, $\sup_{s\in S} \|s - w_0\| = \|r\mathbf{1}\| = r$. We claim that $d(S, W) = r$. Suppose that this does not hold. Then, there exists $y \in W$ such that $\|w_0 + r\mathbf{1} - y\| < r$ (note that $w_0 + r\mathbf{1} \neq y$, because if $w_0 + r\mathbf{1} = y \in W$, then by Lemma 2.1, $w_0 \in intW$, which is a contradiction). Thus, there exists $r_0 \in (0, r)$ such that $\|w_0 + r\mathbf{1} - y\| \leq r_0$. Hence, by using (1.2), we have $w_0 + r\mathbf{1} \leq y + r_0\mathbf{1}$, and so

$$w_0 + \lambda_0\mathbf{1} \leq y, \text{ where } \lambda_0 = (r - r_0) > 0.$$

Since W is a downward set and $y \in W$, $w_0 + \lambda_0\mathbf{1} \in W$. Hence, by Lemma 2.1, $w_0 \in intW$. This is a contradiction. Therefore, $d(S, W) = r = \sup_{s\in S} \|s - w_0\|$, that is, $w_0 \in S_W(S)$. On the other hand, we have $w < w_0 + r\mathbf{1}$. Since $w_0 < w$, we have $0 \leq (w_0 + r\mathbf{1}) - w < w_0 + r\mathbf{1} - w_0 = r\mathbf{1}$. Hence,

$$\sup_{s\in S} \|s - w\| = \|w_0 + r\mathbf{1} - w\| \leq \|r\mathbf{1}\| = r = d(S, W) \leq \sup_{s\in S} \|s - w\|.$$

Thus, $\sup_{s\in S} \|s - w\| = d(S, W)$, and so $w \in S_W(S)$, where $w \neq w_0$. This is impossible, because W is a simultaneous Chebyshev subset of X. □

3. Downward hulls and simultaneous approximation

As known, the downward hull U_* of the set $U \subseteq X$ is the intersection of all downward sets containing U. Recall that a subset G of the positive cone

$$X^+ = \{x \in X : x \geq 0\}$$

is called normal whenever $g \in G, x \in X^+$ and $x \leq g$ imply that $x \in G$. For a subset A, we shall use the notation $A^+ = \{a^+ : a \in A\}$, where $a^+ = \sup(a, 0)$. We also use the notation $a^- = -\inf(a, 0)$.

Remark 3.1. *Let W be a downward set, S be a bounded subset such that $S \cap W = \phi$, $w \in W$ and $s \in S$. If $0 \leq w - s$, then $s < w$, and so*

$s \in W$, which is a contradiction. After here, we suppose that $0 \leq s - w$, for all $w \in W$ and $s \in S$.

We start with the following result for easy citation.

Proposition 3.2. [4] *Let G be a a normal subset of X^+ and $G_* \subset X$ be the downward hull of G. Then, the following statements hold:*

(1) $G_* = \{x \in X : x^+ \in G\}$.
(2) $G = G_* \cap X^+$.
(3) *G is closed if and only if G_* is closed.*
(4) $(G_*)^+ = G$.

Proposition 3.3. *Let S be a bounded set of X, G be a normal subset of X^+ and G_* be the downward hull of the set G. If $S \cap G_* = \phi$, then, for each $g \in G_*$, we have*

$$\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|.$$

Proof. Let $s \in S$ and $g = g^+ - g^-$. Then, $s - g^+ \leq s - g$ and by Remark 3.1, $s - g^+ \geq 0$. Therefore, $s - g^+ = |s - g^+| \leq |s - g|$. It follow that $\|s - g^+\| \leq \|s - g\|$, for all $s \in S$. Hence, for each $g \in G_*$, we obtain $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|$. □

Proposition 3.4. *Let G be a normal subset of X^+. Then, G is a simultaneous proximinal subsets of X.*

Proof. By Lemma 2.2, G_* is simultaneous proximinal. Thus, $S_{G_*}(S) \neq \phi$ for all bounded subsets S with $S \cap G_* = \phi$. If $g_0 \in S_{G_*}(S)$, then $g_0 \in G_*$ and $g_0^+ \in G$, by Proposition 3.2. By using Proposition 3.3, for each $g \in G_*$, we have

$$\sup_{s\in S} \|s - g_0^+\| \leq \sup_{s\in S} \|s - g_0\| \leq \sup_{s\in S} \|s - g\|.$$

Since $g_0 \in S_{G_*}(S)$ and $G \subset G_*$, $\sup_{s\in S} \|s - g_0^+\| \leq \sup_{s\in S} \|s - g\|$, for all $g \in G$. Therefore, $g_0^+ \in S_G(S)$. □

In the following corollaries, G is a normal subset of X^+, S is a bounded subset of X such that $S \cap G_* = \phi$, where G_* is the downward hull of G.

Corollary 3.5. $S_{G_*}(S) = S_G(S)$.

Proof. Let $g \in S_{G_*}(S)$. By Proposition 3.3, $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|$, for all $g \in G_*$. Since $G \subseteq G_*$, by using Proposition 3.2, we have $g^+ \in G_*$. Thus, again by using Proposition 3.2, $g^+ = g$, and so $S_{G_*}(S) \subseteq S_G(S)$. Now, let $g_0 \in G_*$ and $g_0 \notin S_{G_*}(S)$. Then, there exists $g \in G_*$ such that $\sup_{s\in S} \|s - g\| \leq \sup_{s\in S} \|s - g_0\|$. By using Proposition 3.3, we obtain $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g\|$. Hence, we get $\sup_{s\in S} \|s - g^+\| \leq \sup_{s\in S} \|s - g_0\|$. Since $g^+ \in G$, $g_0 \notin S_G(S)$. □

Corollary 3.6. $d(S, G_*) = d(S, G)$.

Proof. Since $G \subseteq G_*$, $d(S, G_*) \leq d(S, G)$. The equality holds by Proposition 3.3. □

Corollary 3.7. $\min S_{G_*}(S) = \min S_G(S)$.

Proof. By Lemma 2.2, $w_0 = \min S_{G_*}(S)$ exists. Now, the equality follows from Corollary 3.5. □

Corollary 3.8. *G is simultaneous proximinal.*

Proof. The result follows from Lemma 2.2 and Corollary 3.5. □

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