A CHARACTERIZATION OF SHELLABLE AND SEQUENTIALLY COHEN-MACAULAY HYPERCYCLES

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Communicated by Jürgen Herzog

ABSTRACT. We consider a class of hypergraphs called hypercycles and we show that a hypercycle $C_n^{d,\alpha}$ is shellable or sequentially the Cohen–Macaulay if and only if $n \in \{3, 5\}$. Also, we characterize Cohen–Macaulay hypercycles. These results are hypergraph versions of results proved for cycles in graphs.

1. Introduction

Recently, monomial ideals have been extensively studied. In this context, finding relations between homological invariants of a squarefree monomial ideal and combinatorial data of the hypergraph associated to it, is of great interest. Among squarefree monomial ideals, edge ideals of graphs which first have been introduced in [13] have been studied more. Interesting classes of graphs like chordal graphs and bipartite graphs have been considered in several papers and some good algebraic results for these graphs have been proved. See, for example, [4, 5, 8, 9, 10]. Recently, also edge ideals of hypergraphs have been studied and some analogous results has been proved for hypergraphs; see [2, 3, 6, 12]. Our results here have inspiration of the work in [4], where shellable and sequentially Cohen–Macaulay cycles have been characterized. Here we will

generalize these results to analogous concepts in hypergraphs. We study hypercycles which were first introduced in [3] and classify shellable, sequentially Cohen–Macaulay and Cohen–Macaulay hypercycles.

Let $\mathcal{X}$ be a finite set and $\mathcal{E} = \{e_1, \ldots, e_s\}$ be a finite collection of non-empty subsets of $\mathcal{X}$. The pair $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is called a **hypergraph**. The elements of $\mathcal{X}$ are called **vertices** and the elements of $\mathcal{E}$ are called **edges** of the hypergraph.

A hypergraph is called **simple** if $|e_i| \geq 2$, for any $1 \leq i \leq s$, and $e_i \subseteq e_j$ implies $i = j$. Let $\mathcal{H}$ be a hypergraph. A subhypergraph $\mathcal{K}$ of $\mathcal{H}$ is a hypergraph such that $\mathcal{X}(\mathcal{K}) \subseteq \mathcal{X}(\mathcal{H})$ and $\mathcal{E}(\mathcal{K}) \subseteq \mathcal{E}(\mathcal{H})$. A hypergraph $\mathcal{H}$ is called d-**uniform** if $|e_i| = d$ for any $e_i \in \mathcal{E}(\mathcal{H})$. A **vertex cover** of $\mathcal{H}$ is a subset of vertices that contains at least one vertex from each edge. A vertex cover V is called minimal if no proper subset of V is a vertex cover of $\mathcal{H}$ and a **free vertex** is a vertex which belongs to exactly one edge of $\mathcal{H}$.

Throughout the paper, we denote by R the polynomial ring $k[x_1, \ldots, x_m]$ over some field k, where $\{x_1, \ldots, x_m\}$ is the set of vertices of a hypergraph considered at the moment. Also, the hypergraphs are simple and have no isolated vertices, i.e., for a hypergraph $\mathcal{H}$, $\mathcal{X}(\mathcal{H}) = \bigcup_{e \in \mathcal{E}(\mathcal{H})} e$. Let $\mathcal{H}$ be a hypergraph. For an edge e_i, we may consider $x^{e_i} = \prod_{x \in e_i} x$ as a monomial in R. The **edge ideal** $I(\mathcal{H})$ of a hypergraph H is defined as $I(\mathcal{H}) = (x^{e_i} : e_i \in \mathcal{E}(\mathcal{H})) \subseteq R$.

A **simplicial complex** consists of a finite set X of vertices and a collection Δ of subsets of X called **faces** such that

(i) If $x \in X$, then $\{x\} \in \Delta$.
(ii) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

The simplicial complex

$$\Delta_{\mathcal{H}} = \{F \subseteq \mathcal{X}(\mathcal{H}) : e \nsubseteq F, \forall e \in \mathcal{E}(\mathcal{H})\}$$

is called the **independence complex** of $\mathcal{H}$ and for any simplicial complex Δ with vertex set X, the **Alexander dual simplicial complex** $\Delta^{\vee}$ to Δ is defined as follows:

$$\Delta^{\vee} = \{F \subseteq X; X \setminus F \notin \Delta\}.$$

For a squarefree monomial ideal $I = (x_{11} \cdots x_{1n_1}, \ldots, x_{t1} \cdots x_{tn_t})$ of a polynomial ring, the **Alexander dual ideal** of I, which is denoted by $I^{\vee}$, is defined as:

$$I^{\vee} = (x_{11}, \ldots, x_{1n_1}) \cap \cdots \cap (x_{t1}, \ldots, x_{tn_t}).$$

Observe that for a hypergraph $\mathcal{H}$, with $I(\mathcal{H}) = (x_{i_1} \cdots x_{i_k} : \{x_{i_1}, \ldots, x_{i_k}\} \in E(\mathcal{H}))$, we have $I(\mathcal{H})^\vee = \cap_{\{x_{i_1}, \ldots, x_{i_k}\} \in E(\mathcal{H})}(x_{i_1}, \ldots, x_{i_k})$. It is then easy to see that x^F belongs to the minimal generating set of $I(\mathcal{H})^\vee$ if and only if F is a minimal vertex cover of $\mathcal{H}$.

Let Δ be a simplicial complex with vertex set $\{x_1, \ldots, x_n\}$. The **Stanley–Reisner ideal** I_Δ is an ideal in the polynomial ring $k[x_1, \ldots, x_n]$, generated by monomials $x_{i_1} \cdots x_{i_k}$, where $i_1 < \cdots < i_k$ and $\{x_{i_1}, \ldots, x_{i_k}\} \notin \Delta$. For a hypergraph $\mathcal{H}$, one can see that $I_{\Delta_\mathcal{H}} = I(\mathcal{H})$.

Given a simplicial complex Δ, we denote by $C_\cdot(\Delta)$ its reduced chain complex, and by $\widetilde{H}_n(\Delta, k) = Z_n(\Delta)/B_n(\Delta)$ its nth reduced homology group with coefficients in the field k.

Recall that a squarefree monomial ideal I of the polynomial ring $k[x_1, \ldots, x_n]$ has **linear quotients**, if there exists an order $f_1 < \cdots < f_m$ on the minimal generators of I such that the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by a subset of variables for all $2 \le i \le m$. This is equivalent to say that there is an order $f_1 < \cdots < f_m$ such that for any $i < j$, there exists a variable $x_k | (f_i : f_j)$ and $l < j$ such that $(f_l : f_j) = (x_k)$. Hypercycles, generalizing cycles in graphs, were defined in [3] as follows:

For positive integers n, α and $d \ge 2\alpha$, a **hypercycle** $C_n^{d,\alpha}$ is a hypergraph with edge set $\mathcal{E}(C_n^{d,\alpha}) = \{e_1, \ldots, e_n\}$ such that

(i.) for any $i \ne j$, we have $e_i \cap e_j \ne \emptyset$ if and only if $|j - i| \equiv 1 \mod (n)$.

(ii.) $|e_i \cap e_{i+1}| = \alpha$, for all i, $1 \le i \le n-1$, and $|e_1 \cap e_n| = \alpha$.

Sequentially Cohen–Macaulay cycle graphs are classified in [4].

Theorem A [4, Proposition 4.1] *Let G be an n-cycle for some $n \ge 3$. Then, G is sequentially Cohen–Macaulay if and only if $n \in \{3, 5\}$.*

It is known that any shellable hypergraph is sequentially Cohen–Macaulay. From this fact and Theorem A, one can see that a cycle C_n is shellable if and only if $n \in \{3, 5\}$. In this paper it is shown that the hypercycles $C_n^{d,\alpha}$ is shellable or sequentially Cohen–Macaulay if and only if $n \in \{3, 5\}$. As a corollary we see that $C_n^{d,\alpha}$ is Cohen–Macaulay if and only if $n \in \{3, 5\}$ and $d = 2\alpha$.

2. Main Results

We begin by recalling the relevant definition of a shellable simplicial complex.

Definition 2.1. A simplicial complex Δ is **shellable** if the facets (maximal faces) of Δ can be ordered as $F_1, \ldots, F_s$ such that for all $1 \le i < j \le s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ a shelling for Δ.

The above concept is often referred to as *non-pure shellability* and is due to Björner and Wachs [1]. Here, we will drop the adjective "non-pure". A hypergraph $\mathcal{H}$ is called shellable, if the simplicial complex $\Delta_{\mathcal{H}}$ is shellable.

The following result relates squarefree monomial ideals with linear quotients and shellable simplicial complexes.

Theorem B [9, Theorem 1.4] *The simplicial complex Δ is shellable if and only if $I_\Delta^\vee$ has linear quotients.* To prove Theorem given below, we need the following easy lemma.

Lemma 2.2. *For arbitrary pairwise disjoint sets $P_i = \{a_{i1}, \ldots, a_{in_i}\}$, $1 \le i \le m$, there exists an order of linear quotients for the ideal $I = (a_{1r_1} \cdots a_{mr_m} : 1 \le r_i \le n_i)$, which comes from the lex order on the vertices as follows:*

$$a_{ij} < a_{rs} \text{ if } i < r \text{ or } i = r \text{ and } j < s.$$

Proof. Let $a_{1r_1} \cdots a_{mr_m}$ and $a_{1s_1} \cdots a_{ms_m}$ be two monomials in the minimal generating set of I such that $a_{1r_1} \cdots a_{mr_m} <_{lex} a_{1s_1} \cdots a_{ms_m}$. Let t be the integer for which $s_t > r_t$ and for any $t' > t$, $s_t = r_t$. Then, $a_{tr_t} | a_{1r_1} \cdots a_{mr_m} : a_{1s_1} \cdots a_{ms_m}$ and $(f : a_{1s_1} \cdots a_{ms_m}) = (a_{tr_t})$, where $f = a_{1s_1} \cdots a_{t-1s_{t-1}} a_{tr_t} a_{t+1s_{t+1}} \cdots a_{ms_m}$. Since $f <_{lex} a_{1s_1} \cdots a_{ms_m}$, the result holds. $\qquad\square$

Let $P_i = \{a_{i1}, \ldots, a_{in_i}\}$, $1 \le i \le m$, be pairwise disjoint sets. We use the notation $\overline{a_1 \cdots a_m}$ to denote the order of linear quotients for the ideal $(a_{1r_1} \cdots a_{mr_m} : 1 \le r_i \le n_i)$, which comes from the lex order that is described in the above Lemma 2.2.

Theorem 2.3. *The hypercycle $C_n^{d,\alpha}$ is shellable if and only if $n \in \{3, 5\}$.*

Proof. **"Only if"** Assume that $C_n^{d,\alpha}$ is shellable and $\mathcal{E}(C_n^{d,\alpha}) = \{e_1, \ldots, e_n\}$. Then, by Theorem B, $I(C_n^{d,\alpha})^\vee$ has linear quotients. Let $I(C_n^{d,\alpha})^\vee = (x^{F_1}, \ldots, x^{F_r})$ and $x^{F_1} < \cdots < x^{F_r}$ be an order of linear quotients. We know that $\{F_1, \ldots, F_r\}$ is the set of minimal vertex covers of $C_n^{d,\alpha}$. For any i, $1 \le i \le n-1$, let $x_i \in e_i \cap e_{i+1}$ and $x_n \in e_1 \cap e_n$. Then, $\{x_1, \ldots, x_n\}$ is a vertex cover of $C_n^{d,\alpha}$. Let $F_{l_1}, \ldots, F_{l_k}$ be all minimal vertex covers of $C_n^{d,\alpha}$ which are contained in $\{x_1, \ldots, x_n\}$ and $F_{l_1} <$

$\cdots < F_{l_k}$ be the order induced from the order of linear quotients for $I(C_n^{d,\alpha})^\vee$. Then, $\{F_{l_1}, \ldots, F_{l_k}\}$ is the set of minimal vertex covers of the cycle $C_n : x_1, \ldots, x_n$. We claim that $I(C_n)^\vee$ has linear quotients with the ordering $x^{F_{l_1}} < \cdots < x^{F_{l_k}}$. Let $F_{l_i} < F_{l_j}$. Then, from the definition of linear quotients, there exists $t < l_j$ and $u \in \mathcal{X}(C_n^{d,\alpha})$ such that $u | x^{F_{l_i}} : x^{F_{l_j}}$ and $x^{F_t} : x^{F_{l_j}} = (u)$. Since $u \in F_{l_i}$ and $F_t \setminus \{u\} \subseteq F_{l_j}$, then $F_t \subseteq \{x_1, \ldots, x_n\}$. Therefore, $t = l_s$, for some $1 \leq s \leq k$, and the claim is proved. It means that C_n is shellable. Thus, by Theorem A, $n \in \{3, 5\}$.

  "If" In order to show that $C_3^{d,\alpha}$ is shellable, it is enough to show that $I(C_3^{d,\alpha})^\vee$ has linear quotients. A monomial x^F is a minimal generator of $I(C_3^{d,\alpha})^\vee$ if and only if F is a minimal vertex cover of $C_3^{d,\alpha}$. Let $e_1 = \{x_1, \ldots, x_\alpha, u_1, \ldots, u_{d-2\alpha}, y_1, \ldots, y_\alpha\}$, $e_2 = \{y_1, \ldots, y_\alpha, v_1, \ldots, v_{d-2\alpha}, z_1, \ldots, z_\alpha\}$, $e_3 = \{z_1, \ldots, z_\alpha, w_1, \ldots, w_{d-2\alpha}, x_1, \ldots, x_\alpha\}$. The set of minimal vertex covers of $C_3^{d,\alpha}$ is equal to $\{\{x_i, y_j\}, \{x_i, z_j\}, \{y_i, z_j\}, \{x_i, v_k\}, \{y_i, w_k\}, \{z_i, u_k\}, \{u_k, v_l, w_t\} : 1 \leq i, j \leq \alpha, 1 \leq k, l, t \leq d - 2\alpha\}$. One can check that the following ordering is an order of linear quotients for $I(C_3^{d,\alpha})^\vee$:

$$\overline{xy} < \overline{xz} < \overline{yz} < \overline{xv} < \overline{yw} < \overline{zu} < \overline{uvw}.$$

For example, consider two monomials $x_i z_j$ and $y_l w_t$, for some $1 \leq i, j, l \leq \alpha$ and $1 \leq t \leq d - 2\alpha$. Since $x_i | x_i z_j : y_l w_t$ and $x_i y_l : y_l w_t = (x_i)$ and $x_i y_l < y_l w_t$, for these monomials the property of linear quotients holds.

  Now, let $\mathcal{E}(C_5^{d,\alpha}) = \{e_1, \ldots, e_5\}$, where $e_1 = \{x_1, \ldots, x_\alpha, a_1, \ldots, a_{d-2\alpha}, y_1, \ldots, y_\alpha\}$, $e_2 = \{y_1, \ldots, y_\alpha, b_1, \ldots, b_{d-2\alpha}, z_1, \ldots, z_\alpha\}$, $e_3 = \{z_1, \ldots, z_\alpha, c_1, \ldots, c_{d-2\alpha}, s_1, \ldots, s_\alpha\}$, $e_4 = \{s_1, \ldots, s_\alpha, e_1, \ldots, e_{d-2\alpha}, t_1, \ldots, t_\alpha\}$, $e_5 = \{t_1, \ldots, t_\alpha, f_1, \ldots, f_{d-2\alpha}, x_1, \ldots, x_\alpha\}$. Then, with the above notation, $\overline{xys}$
$< \overline{xzs} < \overline{xzt} < \overline{yzt} < \overline{yst} < \overline{xze} < \overline{xsb} < \overline{ysf} < \overline{ytc} < \overline{zta} <$
$\overline{xyec} < \overline{xtbc} < \overline{stab} < \overline{zsaf} < \overline{yzef} < \overline{xbce} < \overline{ycef} < \overline{zefa} <$
$\overline{sabf} < \overline{tabc} < \overline{abcef}$ is an order of linear quotients for $I(C_5^{d,\alpha})^\vee$. For example, consider two monomials $x_i z_j e_k$ and $s_{i'} a_{j'} b_{k'} f_l$, for some $1 \leq i, j, i' \leq \alpha$ and $1 \leq k, j', k', l \leq d - 2\alpha$. For the monomial $z_j s_{i'} a_{j'} f_l$, observe that $z_j s_{i'} a_{j'} f_l < s_{i'} a_{j'} b_{k'} f_l$ and $z_j s_{i'} a_{j'} f_l : s_{i'} a_{j'} b_{k'} f_l = (z_j)$, $z_j | x_i z_j e_k : s_{i'} a_{j'} b_{k'} f_l$. □

Definition 2.4. Let $R = k[x_1, \ldots, x_n]$. A graded R-module M is called **sequentially Cohen–Macaulay** (over k) if there exists a finite filteration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen–Macaulay and

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A hypergraph $\mathcal{H}$ is called sequentially Cohen–Macaulay if $R/I(\mathcal{H})$ is sequentially Cohen–Macaulay.

For a monomial ideal I, let $(I)_j$ be the ideal generated by all monomials of degree j belonging to I. Then, I is called **componentwise linear** if $(I)_j$ has a linear resolution for all j. To prove Theorem below, we need the following result.

Theorem C [7, Theorem 2.1] Let I be a squarefree monomial ideal. Then, R/I is sequentially Cohen–Macaulay if and only if $I^\vee$ is componentwise linear.

Theorem 2.5. *The hypercycle $C_n^{d,\alpha}$ is sequentially Cohen–Macaulay if and only if $n \in \{3, 5\}$.*

Proof. From Theorem 2.3, we have that $C_3^{d,\alpha}$ and $C_5^{d,\alpha}$ are shellable. Thus, from the fact that every shellable hypergraph is sequentially Cohen–Macaulay, one implication is clear. Now, let $n \notin \{3, 5\}$, $\mathcal{E}(C_n^{d,\alpha}) = \{e_1, \ldots, e_n\}$ and $I = I(C_n^{d,\alpha})$. First, assume that $n = 2r$, for some $r \geq 2$. Since we have $2r$ edges to cover, each minimal vertex cover has at least r elements. For any $1 \leq i \leq n-1$, let $e_i \cap e_{i+1} = \{x_{i,1}, \ldots, x_{i,\alpha}\}$ and $e_1 \cap e_n = \{x_{n,1}, \ldots, x_{n,\alpha}\}$. A minimal vertex cover of $C_n^{d,\alpha}$ of cardinality r is of the form $\{x_{1,l_1}, x_{3,l_3}, \ldots, x_{2r-1,l_{2r-1}}\}$ or $\{x_{2,l_2}, x_{4,l_4}, \ldots, x_{2r,l_{2r}}\}$, for some $1 \leq l_i \leq \alpha$, $i \in \{1, \ldots, 2r\}$. Let

$$J = (x_{1,l_1} x_{3,l_3} \cdots x_{2r-1,l_{2r-1}} : \ 1 \leq l_1, \ldots, l_{2r-1} \leq \alpha),$$

$$K = (x_{2,l_2} x_{4,l_4} \cdots x_{2r,l_{2r}} : \ 1 \leq l_2, \ldots, l_{2r} \leq \alpha).$$

Then, $(I^\vee)_r = J + K$. We show that $\beta_{1,2r}((I^\vee)_r) \neq 0$. Let M be a monomial ideal and

$$K^{\mathbf{b}}(M) = \{ \text{ squarefree vectors } \tau : x^{\mathbf{b}-\tau} \in M\}$$

be the Koszul simplicial complex of M for a vector $\mathbf{b} \in \mathbb{N}^m$. Then, the multigraded Betti numbers of M can be computed from the formula

$$\beta_{i,\mathbf{b}}(M) = \dim_k \widetilde{H}_{i-1}(K^{\mathbf{b}}(M), k);$$

see [11, Theorem 1.34].

We have $\beta_{1,\mathbf{b}}((I^\vee)_r) = \dim_k \widetilde{H}_0(K^{\mathbf{b}}((I^\vee)_r), k) = \lambda - 1$, where λ is the number of connected components of $K^{\mathbf{b}}((I^\vee)_r)$ (see [14, Proposition 5.2.3]). Let $\mathbf{b}$ be the squarefree vector whose support (see for example, [11, Theorem 1.13]) corresponds to $\{x_{1,i_1}, \ldots, x_{2r,i_{2r}}\}$ (with the above notation), for some $1 \leq i_j \leq \alpha$, $1 \leq j \leq 2r$. Consider a face F of $K^{\mathbf{b}}((I^\vee)_r)$. Then, F is contained in $\{x_{1,i_1}, x_{3,i_3}, \ldots, x_{2r-1,i_{2r-1}}\}$ or in $\{x_{2,i_2}, x_{4,i_4}, \ldots, x_{2r,i_{2r}}\}$. In other words, F does not contain x_{k,i_k} and x_{l,i_l}, for some odd integer k and even integer l. Otherwise, $x_{1,i_1} x_{2,i_2} x_{3,i_3} \cdots \hat{x}_{k,i_k} \cdots \hat{x}_{l,i_l} \cdots x_{2r,i_{2r}} \in (I^\vee)_r$, a contradiction, since $x_{1,i_1} x_{2,i_2} x_{3,i_3} \cdots \hat{x}_{k,i_k} \cdots \hat{x}_{l,i_l} \cdots x_{2r,i_{2r}} \notin J, K$. Therefore, $\lambda \geq 2$ and $\beta_{1,\mathbf{b}}((I^\vee)_r) \neq 0$. Thus, $\beta_{1,2r}((I^\vee)_r) \neq 0$. This shows that the minimal free resolution of $(I^\vee)_r$ is not linear. Thus, $I^\vee$ is not componentwise linear and then I is not sequentially Cohen–Macaulay.

Now, let $n = 2r+1$, for some $r \geq 3$. We will show that $\beta_{2,2r+1}((I^\vee)_{r+1}) \neq 0$, which implies that the minimal free resolution of $(I^\vee)_{r+1}$ is not linear, since $2r + 1 > r + 3$. Each minimal vertex cover of $C_n^{d,\alpha}$ has at least $r + 1$ elements. One can see that $\{x_{1,1}, x_{2,1}, x_{4,1}, \ldots, x_{2r,1}\}$ is a minimal vertex cover of $C_n^{d,\alpha}$. Consider the cycle graph C_n : $x_{1,1}, x_{2,1}, x_{3,1}, \ldots, x_{2r+1,1}$ and let $J = (I(C_n)^\vee)_{r+1}$. Then, as shown in the proof of [4, Proposition 4.1], $\beta_{2,2r+1}(J) \neq 0$. From Hochster's formula (see [10, Theorem 1.5.21]), we have $\beta_{2,2r+1}(J) = \sum_{|V|=2r+1, V \subseteq X} \dim_k \widetilde{H}_{2r-3}(\Delta_V, k)$, where Δ is the Stanley–Reisner simplicial complex of J and $X = \{x_{1,1}, x_{2,1}, x_{3,1}, \ldots, x_{2r+1,1}\}$. Thus, $\dim_k \widetilde{H}_{2r-3}(\Delta, k) \neq 0$. Also, $\beta_{2,2r+1}((I^\vee)_{r+1}) = \sum_{|V|=2r+1, V \subseteq Y} \dim_k \widetilde{H}_{2r-3}(\Delta'_V, k)$, where Δ' is the Stanley–Reisner simplicial complex of $(I^\vee)_{r+1}$ on some vertex set Y. We show that $\Delta'_X = \Delta$. Any $F \subseteq X$ is a vertex cover of C_n if and only if F is a vertex cover of $C_n^{d,\alpha}$. Let F be a face of Δ. Then, $\Pi_{x \in F} x \notin J$. Since J is generated by all minimal vertex covers of C_n, then F is not a vertex cover of C_n. Thus, F is not a vertex cover of $C_n^{d,\alpha}$ and then $\Pi_{x \in F} x \notin (I^\vee)_{r+1}$. This means that $F \in \Delta'_X$. The proof of the other inclusion is similar. Therefore, $\dim_k \widetilde{H}_{2r-3}(\Delta'_X, k) = \dim_k \widetilde{H}_{2r-3}(\Delta, k) \neq 0$. Since $|X| = 2r + 1$, we

have $\beta_{2,2r+1}((I^\vee)_{r+1}) \geq \dim_k \widetilde{H}_{2r-3}(\Delta'_X, k) > 0$. Therefore, $I^\vee$ is not componentwise linear. □

Corollary 2.6. *The hypercycle $C_n^{d,\alpha}$ is shellable if and only if it is sequentially Cohen–Macaulay.*

Corollary 2.7. *The hypercycle $C_n^{d,\alpha}$ is Cohen–Macaulay if and only if $n \in \{3, 5\}$ and $d = 2\alpha$.*

Proof. Assume that $C_n^{d,\alpha}$ is Cohen–Macaulay. Then, it is sequentially Cohen–Macaulay. Thus by the Theorem 2.5 $n = 3$ or $n = 5$. Also, all minimal vertex covers of $C_n^{d,\alpha}$ have the same cardinality. Let $\mathcal{E}(C_3^{d,\alpha}) = \{e_1, e_2, e_3\}, x_1 \in e_1 \cap e_2$ and $x_2 \in e_2 \cap e_3$. Then, $\{x_1, x_2\}$ is a minimal vertex cover of $C_3^{d,\alpha}$. If $d > 2\alpha$, then each edge of $C_3^{d,\alpha}$ has a free vertex. Therefore, $\{u_1, u_2, u_3\}$, where $u_i \in e_i$ is a free vertex, is also a minimal vertex cover, a contradiction. Thus, $d = 2\alpha$. In the case $n = 5$, if $d > 2\alpha$, then similarly one can find two minimal vertex covers $\{x_1, x_2, x_4\}$, $x_i \in e_i \cap e_{i+1}$, and $\{u_1, \ldots, u_5\}$, where $u_i \in e_i$ is a free vertex to get a contradiction. Conversely, if $d = 2\alpha$, then each minimal vertex cover of $C_3^{d,\alpha}$ has cardinality two and each minimal vertex cover of $C_5^{d,\alpha}$ is of cardinality three. Since they are sequentially Cohen–Macaulay, the result holds. □

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