A CHARACTERIZATION OF L-DUAL FRAMES AND L-DUAL RIESZ BASES

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Abstract. This paper is an investigation of L-dual frames with respect to a function-valued inner product, the so called L-bracket product on $L^2(G)$, where G is a locally compact abelian group with a uniform lattice $L$. We show that several well known theorems for dual frames and dual Riesz bases in a Hilbert space remain valid for L-dual frames and L-dual Riesz bases in $L^2(G)$.

1. Introduction

In [2], the bracket product is defined as a function valued inner product on $L^2(\mathbb{R})$ and, in [9], the $\phi$-bracket product is defined as its extension to $L^2(G)$, where G is a locally compact abelian group (LCA) and $\phi$ is a topological isomorphism on G. As a new inner product on $L^2(G)$, we define the $L-$bracket product which can be applied to extend several ideas and constructions from the theory of shift invariant spaces, factorable operators and Wely-Heisenberg frames on $\mathbb{R}^n$ to the setting of LCA groups. These extensions are, in a more general and different way, using various tools in abstract harmonic analysis.

Dual frames and Riesz Bases for Hilbert spaces are defined in [1, 3].
The present paper deals with characterizing $L$-dual frames and $L$-dual Riesz bases on $L^2(G)$, and consists of four sections. In the first section some definitions and preliminaries related to locally compact abelian groups and $L$-bracket products are introduced. In Section 2, we state some definitions and notations related to $L$-frames. In Section 3, we define and characterize $L$-dual frames and, finally, in Section 4, we define $L$-dual Riesz basis.

2. Preliminaries

In this section we give a brief review of definitions and notations from LCA groups and $L$-bracket product. For more details on LCA groups we refer to the book [4] and an extensive study of the $L$-bracket product theory can be found in [7].

**Definition 2.1.** A subgroup $L$ of $G$ is called a uniform lattice, if it is discrete and co-compact; i.e., $G/L$ is compact.

**Definition 2.2.** Let $f, g \in L^2(G)$. The $L$-bracket product of $f$ and $g$ is defined as the mapping $[,]_L: L^2(G) \times L^2(G) \rightarrow L^1(G/L)$ given by

$$[f, g]_L(\hat{x}) = \sum_{k \in L} f(k^{-1}) \overline{g(xk)} \quad \text{for all } \hat{x} \in G/L.$$ 

We define the $L$-norm of $f$ as $\|f\|_L(\hat{x}) = ([f, f]_L(\hat{x}))^{\frac{1}{2}}$.

The above definition appears in [7] with the following formula

$$[f, g]_\phi(\hat{x}) = \sum_{\phi(k) \in \phi(L)} f(\phi(k^{-1})) \overline{g(xk^{-1})} \quad \text{for all } \hat{x} \in G/\phi(L),$$

where $\phi$ is any topological isomorphism on $G$.

**Note 2.3.** If $\phi: G \rightarrow G$ is a topological automorphism and $L$ is a uniform lattice in $G$, then $\phi(L)$ is also a uniform lattice in $G$ [3]. Thus, we assume that $G$ is a LCA group with uniform lattice $L'$ and we set $L = \phi(L')$. In the present paper we always assume that $G/L$ is normalized; i.e., $|G/L|=1$.

Let $L$ be a uniform lattice in $G$. Choosing the counting measure on $L$, a relation between the Haar measure $dx$ on $G$ and $d\hat{x}$ on $\frac{G}{L}$ is given by the following case of Weil’s formula [4]. For $f \in L^1(G)$, we have
\[ \sum_{k \in L} f(xk^{-1}) \in L^1(G) \text{ and} \]
\[ \int_{G} f(x)dx = \int_{G/L} \sum_{k \in L} f(xk^{-1})dx. \]

**Example 2.4.** Examples of \( L \)-bracket product:

1. Consider \( G = \mathbb{R}^n \) with the uniform lattice \( L' = \mathbb{Z}^n \). Let the topological automorphism \( \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \), given by \( \phi(x) = Ax \), where \( A \) is an invertible \( n \times n \) matrix. The \( L \)-bracket product is the \( A \)-bracket product defined as \( [f, g]_L(x) = \sum_{n \in \mathbb{Z}^n} f\overline{g}(x - An) \), for \( f, g \in L^2(\mathbb{R}^n) \) (see [5]).

   In particular, let \( n = 1 \) and \( G = \mathbb{R} \) with the uniform lattice \( L' = \mathbb{Z} \). Fix \( a \in \mathbb{R}^+ \), we define: \( \phi(x) = ax \), for \( x \in \mathbb{R} \). The mapping \( [.,.]_L : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1([0,1]) \), defined by \( [f, g]_L(x) = \sum_{n \in \mathbb{Z}} f\overline{g}(x - na) \) is the \( a \)-pointwise inner product of \( f \) and \( g \) (see [2]).

2. Consider the LCA group \( G = \mathbb{R} \times \Delta_p \), where \( p \) is a prime and \( \Delta_p \) denote the group of \( p \)-adic integers as defined in [6]. and let \( L \) be the subgroup \( \{(n,nu) \}_{n \in \mathbb{Z}} \) of \( \mathbb{R} \times \Delta_p \), where \( u = (1,0,0,...) \). Then, \( L \) is a uniform lattice in \( \mathbb{R} \times \Delta_p \) (obviously \( L \) is discrete and by Theorem 10.13 in [6], \( \mathbb{R} \times \Delta_p)/L \) is compact). Let \( a := (1/p,0,0,...) \in \Delta_p \). Then, the mapping \( \phi : \mathbb{R} \times \Delta_p \rightarrow \mathbb{R} \times \Delta_p \) defined by \( (x,v) \in \mathbb{R} \times \Delta_p \), by \( \phi(x,v) = (2x,av) \), is a topological isomorphism on \( \mathbb{R} \times \Delta_p \), thus The \( L \)-bracket product is \( [f, g]_L(x,v) = \sum_{n \in \mathbb{Z}} f\overline{g}(x - 2n,v - anu) \), for \( f, g \in L^2(\mathbb{R} \times \Delta_p) \) (see [8]).

**Definition 2.5.** The function \( g \in L^2(G) \) is \( L \)-bounded, if there exists \( M > 0 \) such that \( \|g\|_L(\dot{x}) \leq M \); a.e.

For \( f, g \in L^2(G) \) the function \( [f, g]_Lg \) need not generally be in \( L^2(G) \). For example consider \( f(x) = g(x) = \chi_{[0,a]}x^{-\frac{1}{2}} \), where \( a \in \mathbb{R}^+ \) and \( \phi(x) = ax \), for \( x \in \mathbb{R} \). But, if \( f, g, h \in L^2(G) \) and \( g, h \) are \( L \)-bounded, then \( [f, g]_Lh \in L^2(G) \) (see [7]).

### 3. \( L \)-frame

In [9], \( \phi \)-frames and its associated \( \phi \)-analysis and \( \phi \)-frame operators are defined. They obtained criteria for a sequence to be a \( \phi \)-frame or a \( \phi \)-Bessel sequence. In this section we state those concepts in \( L \)-bracket product sense.
Definition 3.1. The sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( L^2(G) \) is said to be a \( L \)-frame, if there exist positive constants \( 0 < A \leq B < \infty \) such that for all \( f \in L^2(G) \)

\[
A \|f\|_L^2(\hat{x}) \leq \sum_{n \in \mathbb{N}} |[f, f_n]_L(\hat{x})|^2 \leq B \|f\|_L^2(\hat{x}) \quad \text{for} \quad \hat{x} \in G/L \ a.e..
\]

Those sequences in \( L^2(G) \), which satisfy only the right-hand inequality in the above formula are called \( L \)-Bessel sequences.

We now intend to define \( L \)-pre frame and \( L \)-analysis operators. We need to introduce a vector space which plays the role of \( l^2(\mathbb{N}) \) in the standard case. To this end, define \( l^2_1(G/L) \) as the space all sequences in \( L^\infty(G/L) \) such that convergent in \( L^1(G/L) \); i.e.

\[
l^2_1(G/L) = \{\{g_i\}_{i \in \mathbb{N}} \subset L^\infty(G/L) : \int_{G/L} \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 d\hat{x} < \infty\},
\]

\( l^2_1(G/L) \) is an inner product space with respect to the following inner product:

\[
[.,.]_{l^2_1(G/L)} : l^2_1(G/L) \times l^2_1(G/L) \rightarrow L^1(G/L), \{\{g_i\}, \{h_i\}\}_{l^2_1(G/L)} = \sum_{i \in \mathbb{N}} g_i h_i.
\]

For \( \{g_i\}_{i \in \mathbb{N}} \in l^2_1(G/L) \), The pointwise norm is defined by

\[
\|\{g_i\}\|_{l^2_1(G/L)}(\hat{x}) = \left( \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 \right)^{\frac{1}{2}},
\]

and the uniform norm by

\[
\|\{g_i\}\|_{l^2_1(G/L)} = \left( \int_{G/L} \sum_{i \in \mathbb{N}} |g_i(\hat{x})|^2 d\hat{x} \right)^{\frac{1}{2}}.
\]

Let \( \{f_n\}_{n \in \mathbb{N}}(= f) \) be a \( L \)-bounded sequence in \( L^2(G) \). Define the \( L \)-analysis operator as the mapping \( T^f_L : L^2(G) \rightarrow l^2_1(G/L) \) by

\[
T^f_Lg = \{[g, f_n]_L\}_{n \in \mathbb{N}}, \quad \text{for all} \quad g \in L^2(G),
\]

and the \( L \)-pre frame operator as the mapping \( T^{sf}_L : l^2_1(G/L) \rightarrow L^2(G) \) by

\[
T^{sf}_L(\{g_n\}) = \sum_{n \in \mathbb{N}} f_n g_n, \quad \text{for all} \quad \{g_n\}_{n \in \mathbb{Z}} \in l^2_1(G/L).
\]
Remark 3.2. Our purpose is to consider a special $l^2_1$-orthonormal basis for $l^2_1(G/L)$. Consider the functions $g(\hat{x}) = 1$ and $h(\hat{x}) = 0$, for all $\hat{x} \in G/L$, and the sequence $\Delta_n = \{\Delta_n^k\}_{k=1}^{\infty}$ of functions $\Delta_n^k: G/L \to \mathbb{C}$ for $n = 1, 2, 3...$ in $L^\infty(G/L)$, defined by

$$
\Delta_n^k(\hat{x}) = \begin{cases} 
g(\hat{x}) & \text{if } k = n, \\
h(\hat{x}) & \text{if } k \neq n.
\end{cases}
$$

(3.1)

The vectors $\{\Delta_n\}_{n \in \mathbb{N}}$ defined by (3.1) constitute an $l^2_1$-orthonormal basis for $l^2_1(G/L)$ that, called canonical $l^2_1$-orthonormal basis.

In the following theorem we characterize $L$-Bessel sequence in terms of the $L$-pre frame operators.

Theorem 3.3. [9] Let $\{f_n\}_{n \in \mathbb{N}}$ be a $L$-bounded sequence in $L^2(G)$.

1. $\{f_n\}_{n \in \mathbb{N}}$ is $L$-Bessel with bound $B$ if and only if $T_L^f$ is a well defined, bounded operator from $l^2_1(G/L)$ into $L^2(G)$ and $\|T_L^f\| \leq \sqrt{B}$.

2. $\{f_n\}_{n \in \mathbb{N}}$ is a $L$-frame if and only if $T_L^s$ is a well defined, bounded operator from $l^2_1(G/L)$ onto $L^2(G)$.

Remark 3.4. Let $\{f_n\}_{n \in \mathbb{N}}$ be a $L$-frame. Assume that each $f_n$, $n \in \mathbb{N}$, is $L$-bounded in $L^2(G)$. Then, the $L$-frame operator defined by $S_L := T_L^s T_L^f$ is bounded. For all $g \in L^2(G)$ we have

$$
[S_L g, g]_L(\hat{x}) = \sum_{n \in \mathbb{N}} \langle g, f_n \rangle L(\hat{x}) f_n, g \rangle L(\hat{x}) = \sum_{n \in \mathbb{N}} \langle g, f_n \rangle L(\hat{x}) \langle g, f_n \rangle L(\hat{x})
$$

$$
= \sum_{n \in \mathbb{N}} \|g, f_n \rangle L(\hat{x})\|^2, \ a.e., \ \hat{x} \in G/L.
$$

So, we have: $A[g, g]_L(\hat{x}) \leq [S_L g, g]_L(\hat{x}) \leq B[g, g]_L(\hat{x})$ a.e. Therefore, $AI \leq S_L \leq BI$. By a standard argument as in the frame theory $S_L$ is invertible (see [9]), and

$$
(3.2)
$$

$$
B^{-1}I \leq S_L^{-1} \leq A^{-1}I.
$$

Definition 3.5. A function $h \in L^\infty(G)$ is said to be $L$-periodic, if $h(xk) = h(x)$ for all $k \in L, x \in G$. We will denote by $B_L(G)$ the set of all $L$-periodic functions in $L^\infty(G)$.

Definition 3.6. Let $E$ be a subgroup of $G$ or $G/L$. An operator $U: L^2(G) \to L^p(E), 1 \leq p \leq \infty$, is said to be $L$-factorable, if $U(hf) = hU(f)$ for all $f \in L^2(G)$ and all $L$-periodic functions $h \in L^\infty(G)$. 

Lemma 3.7. [8]

(1) Let $U$ be a bounded $L$-factorable operator on $L^2(G)$. Then, for every $f \in L^2(G)$ we have $\|Uf\|_L(\hat{x}) \leq \|U\| \|f\|_L(\hat{x})$ a.e.,

(2) Let $f, g \in L^2(G)$. Then, for all periodic functions $h$,

$$[fh, g]_L = h[f, g]_L, \quad [f, \bar{h}g]_L = h[f, g]_L.$$  

The following proposition shows that every bounded $L$-factorable operator on $L^2(G)$ is adjointable.

Proposition 3.8. [8] Let $U : L^2(G) \to L^2(G)$ be a bounded $L$-factorable operator and $U^*$ be its adjoint. Then, $U^*$ is $L$-factorable. Moreover, for all $f, g \in L^2(G)$, $[U(f), g]_L(\hat{x}) = [f, U^*(g)]_L(\hat{x})$ a.e., $\hat{x} \in G/L$.

Lemma 3.9. Let $\{f_n\}_{n \in \mathbb{N}}$ be a $L$-frame that is $L$-bounded and $S_L$ is $L$-frame operator for $\{f_n\}_{n \in \mathbb{N}}$ then $S_L$ is $L$-factorable.

Proof. Let $h$ be a $L$-periodic function we show that

$$S_L(hf) = hS_L(f) \text{ for all } f \in L^2(G).$$

$$S_L(hf) = \sum_{n \in \mathbb{N}} [hf, f_n]_L f_n, \text{ on the other hand, by Lemma 3.7 (2) we have }$$

$$S_L(hf) = \sum_{n \in \mathbb{N}} h[f, f_n]_L f_n = hS_L(f). \text{ Thus, } S_L \text{ is a } L \text{-factorable.}$$

4. $L$-dual frame

Our goal in this section is to define and characterize $L$-dual frames for $L$-frames in $L^2(G)$.

Definition 4.1. Let $\{f_n\}_{n \in \mathbb{N}}$ be a $L$-bounded, $L$-frame, then the $L$-bounded, $L$-frame $\{g_n\}_{n \in \mathbb{N}}$ is called a $L$-dual frame for $\{f_n\}_{n \in \mathbb{N}}$ if

$$g = \sum_{n \in \mathbb{N}} [g, g_n]_L f_n \text{ for all } g \in L^2(G).$$

(4.1)

Remark 4.2. Let $\{g_n\}_{n \in \mathbb{N}}$ be a $L$-dual frame for $\{f_n\}_{n \in \mathbb{N}}$, thus they are $L$-Bessel sequences and we denote the $L$-pre frame operator for $\{f_n\}_{n \in \mathbb{N}}$ by $T_L^{*f}$, and the $L$-pre frame operator for $\{g_n\}_{n \in \mathbb{N}}$ by $T_L^{*g}$. In terms of these operators (4.1) means

$$T_L^{*f}T_L^{*g} = I.$$
Remark 4.3. By the equation (3.2), \( \{S_L^{-1}f_n\}_{n \in \mathbb{N}} \) is a L-frame and by Lemma 3.9, \( S_L^{-1} \) is L-factorable and then by Lemma 3.7(1):

\[
\|S_L^{-1}(f_n)\|_L(\hat{x}) \leq \|S_L^{-1}\| \|f_n\|_L(\hat{x}) \text{ a.e., for all } n \in \mathbb{N}
\]

thus \( \{S_L^{-1}f_n\}_{n \in \mathbb{N}} \) is L-bounded and also we have

\[
g = S_LS_L^{-1}g = \sum_{i \in \mathbb{N}} [g, S^{-1}f_n]_L f_n,
\]

and \( \{S_L^{-1}f_n\}_{n \in \mathbb{N}} \) is a L-dual frame for \( \{f_n\}_{n \in \mathbb{N}} \), that is called the canonical or standard L-dual frame.

We begin with a lemma, which shows the roles of \( \{f_n\}_{n \in \mathbb{N}} \) and \( \{g_n\}_{n \in \mathbb{N}} \) can be interchanged:

Lemma 4.4. Assume that \( \{f_n\}_{n \in \mathbb{N}} \) and \( \{g_n\}_{n \in \mathbb{N}} \) are L-bounded, L-Bessel sequences in \( L^2(G) \). Then, the following are equivalent:

(i) \( g = \sum_{n \in \mathbb{N}} [g, g_n]_L f_n \) for all \( g \in L^2(G) \)
(ii) \( g = \sum_{n \in \mathbb{N}} [f_n, g_n]_L g_n \) for all \( g \in L^2(G) \)

Proof. In terms of the L-pre frame operators (i) means that \( T^sfL^q = I \)

\[
(T^sfL^q = I)^* = T^{s^q}T^fL = I
\]

which is identical to the statement in (ii). In a similar way (ii) implies (i).

When (4.3) is satisfied, we say that \( T^{s^q} \) is a left inverse of \( T^fL \).

Lemma 4.5. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a L-bounded, L-frame for \( L^2(G) \) and \( \{\Delta_k\}_{k \in \mathbb{N}} \) be the canonical \( l^2 \)-orthonormal basis for \( l^2(G/L) \). The L-dual frames for \( \{f_n\}_{n \in \mathbb{N}} \) are precisely the family \( \{g_n\}_{n \in \mathbb{N}} = \{V(\Delta_n)\} \), where \( V: l^2(G/L) \to L^2(G) \) is a bounded left inverse of \( T^fL \).

Proof. If \( V \) is a bounded, left inverse of \( T^fL \), then \( V \) is surjective and \( \{g_n\}_{n \in \mathbb{N}} = \{V(\Delta_n)\} \) is L-frame by Theorem 3.3(2). Since

\[
\|g_k\|_{L^2(G)} = \|V(\Delta_k)\|_{L^2(G)} \leq \|V\| \|\Delta_k\|_{l^2(G/L)},
\]

\( \|\Delta_k\|_{l^2(G/L)} = 1 \) and \( \|V\| \leq M \), this implies that \( \|g_k\|_{L^2(G)} \leq M \) and by Weil’s formula we have

\[
\int_{G/L} \|g_k\|_L(\hat{x}) d\hat{x} = \int_{G/L} \sum_{l \in L} |g_k(xl)|^2 d\hat{x} = \int_G |g_k(x)|^2 dx = \|g_k\|_{L^2(G)} \leq M.
\]
Therefore, we have \( \|g_k\| \leq M \) a.e., for all \( k \in \mathbb{N} \). Thus, \( \{g_k\}_{k \in \mathbb{N}} \) is a \( L \)-bounded sequence. Also, we have,

\[
T_L^f g = \{[g, f_k]_L\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} [g, f_k]_L \Delta_k, \quad \text{for all} \quad g \in L^2(G),
\]

thus \( g = VT_L^f g = \sum_{k=1}^{\infty} [g, f_k]_L g_k \) for all \( g \in L^2(G) \); i.e., \( \{g_k\}_{k \in \mathbb{N}} \) is \( L \)-dual frame of \( \{f_k\}_{k \in \mathbb{N}} \). Assume that \( \{g_k\}_{k \in \mathbb{N}} \) is the \( L \)-dual frame of \( \{f_k\}_{k \in \mathbb{N}} \), then, by Theorem 3.3(2), the \( L \)-pre frame \( T_L^{sf} \) of \( \{g_k\}_{k \in \mathbb{N}} \) is bounded. In fact, \( \{g_k\}_{k \in \mathbb{N}} = \{T_L^{sg}(\Delta_k)\}_{k \in \mathbb{N}} \), by Lemma 4.4, \( T_L^{sg}T_L^f = I \) and the proof is complete.

**Lemma 4.6.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a \( L \)-bounded, \( L \)-frame with \( L \)-pre frame operator \( T_L^{sf} \). Then, the bounded left inverses of \( T_L^{sf} \) are precisely the operators having the form \( S_L^{-1}T_L^{sf} + W(I - T_L^{sf}S_L^{-1}T_L^{sf}) \), where \( W: I_1^2(G/L) \to L^2(G) \) and \( I \) denote the identity operator on \( I_1^2(G/L) \).

**Proof.** We show that \( S_L^{-1}T_L^{sf} + W(I - T_L^{sf}S_L^{-1}T_L^{sf}) \) is bounded and a left inverse of \( T_L^{sf} \). By Theorem 3.3.2, \( T_L^{sf} \), \( S_L \) and \( S_L^{-1} \) are bounded, so, \( S_L^{-1}T_L^{sf} + W(I - T_L^{sf}S_L^{-1}T_L^{sf}) \) is bounded too, and

\[
(S_L^{-1}T_L^{sf} + W(I - T_L^{sf}S_L^{-1}T_L^{sf}))T_L^{sf} = I.
\]

For implication, if \( U \) is a given left inverse of \( T_L^{sf} \), then by taking \( W = U \) we have

\[
S_L^{-1}T_L^{sf} + W(I - T_L^{sf}S_L^{-1}T_L^{sf}) = S_L^{-1}T_L^{sf} + U - UT_L^{sf}S_L^{-1}T_L^{sf} = U.
\]

Now, we are ready to characterize all \( L \)-dual frames associated to a given \( L \)-frame.

**Theorem 4.7.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a \( L \)-bounded, \( L \)-frame for \( L^2(G) \), then the \( L \)-dual frames of \( \{f_n\}_{n \in \mathbb{N}} \) are precisely the families

\[
\{g_n\}_{n \in \mathbb{N}} = \{S_L^{-1}f_n + h_n - \sum_{k=1}^{\infty} [S_L^{-1}f_n, f_k]_L h_k\}_{n \in \mathbb{N}},
\]

where \( \{h_n\}_{n \in \mathbb{N}} \) is a \( L \)-bounded, \( L \)-Bessel sequence in \( L^2(G) \).

**Proof.** By Lemmas 4.5 and 4.6 we can characterize the \( L \)-dual frames as all families of the form

\[
\{g_k\}_{k \in \mathbb{N}} = \{S_L^{-1}T_L^{sf}(\Delta_k) + W(I - T_L^{sf}S_L^{-1}T_L^{sf})\Delta_k\},
\]
where, \( W: L^2(G/L) \to L^2(G) \) is an operator of the form \( W\{k_j\}_{j \in \mathbb{N}} = \sum_{j=1}^{\infty} k_j h_j \), such that \( \{h_n\}_{n \in \mathbb{N}} \) is a \( L \)-bounded \( L \)-Bessel sequence in \( L^2(G) \). Thus, \( W\{\Delta_k\}_{k \in \mathbb{N}} = \{h_n\}_{n \in \mathbb{N}} \),

\[
\{g_k\}_{k \in \mathbb{N}} = \{S^{-1}_L T_s f(\Delta_k) + W(I - T_s^f S^{-1}_L T_s f)(\Delta_k)\} = \{S^{-1}_L f_n + h_n - \sum_{k=1}^{\infty} [S^{-1}_L f_n, f_k] L h_k\}_{n \in \mathbb{N}},
\]

that completes the proof.

5. A characterization of \( L \)-dual Riesz bases

In this section we show that every \( L \)-Riesz basis is a \( L \)-bounded, \( L \)-frame sequence in \( L^2(G) \). Thus, it has a \( L \)-dual frame which is a \( L \)-Riesz basis. Similar to the usual inner product, we define \( L \)-orthogonality.

**Definition 5.1.** Let \( f, g \in L^2(G) \). We say that \( f \) and \( g \) are \( L \)-orthogonal if \( [f, g]_L = 0 \).

A sequence \( \{g_n\}_{n \in \mathbb{N}} \subseteq L^2(G) \) is called \( L \)-orthonormal, if \( [g_n, g_m]_L = 0 \) for all \( n \neq m \in \mathbb{N} \) and \( \|g_n\|_L = 1 \) for all \( n \in \mathbb{N} \). For \( \Lambda \subseteq L^2(G) \), the \( L \)-orthogonal complement of \( \Lambda \) is

\[
\Lambda^\perp_L = \{g \in L^2(G); [f, g]_L = 0, \text{ a.e., for all } f \in \Lambda\}.
\]

**Remark 5.2.** If \( \{g_n\}_{n \in \mathbb{N}} \) is a \( L \)-orthonormal basis in \( L^2(G) \), in [9] it is proved that the following are equivalent:
(a) For each \( f \in L^2(G) \), \( f(x) = \sum_{n \in \mathbb{N}} ([f, g_n]_L(\hat{x})) g_n(x) \), a.e., \( x \in G \).
(b) For all \( f \in L^2(G) \), \( \|f\|^2_L(\hat{x}) = \sum_{n \in \mathbb{N}} \|f, g_n\|^2_L(\hat{x}) \), a.e., (Parseval Identity).

**Definition 5.3.** A sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( L^2(G) \) is said to be \( L \)-Riesz basis, if there exists a \( L \)-orthonormal basis \( \{g_n\}_{n \in \mathbb{N}} \) and \( L \)-factorable operator \( U: L^2(G) \to L^2(G) \), which is a topological automorphism such that \( U(g_n) = f_n \), for every \( n \in \mathbb{N} \).

**Remark 5.4.** (i) If \( U: L^2(G) \to L^2(G) \) is invertible, then \( U \) is \( L \)-factorable. Indeed, for \( h \in B_L(G) \), we have

\[
UU^{-1}(h f) = h f = hUU^{-1} f = U(hU^{-1} f).
\]

Therefore, \( U^{-1}(h f) = hU^{-1} f \) for all \( f \in L^2(G) \), and so \( U^{-1} \) is \( L \)-factorable.

(ii) If \( \{g_n\}_{n \in \mathbb{N}} \) is a \( L \)-Riesz basis for \( L^2(G) \). According to the definition
we can write \( \{g_n\}_{n \in \mathbb{N}} = \{U(f_n)\}_{n \in \mathbb{N}} \), where \( U \) is a \( L \)-factorable operator which is a topological automorphism on \( L^2(G) \) and \( \{f_n\}_{n \in \mathbb{N}} \) is a \( L \)-orthonormal basis for \( L^2(G) \). Since \( U \) is a topological automorphism then \( U \) is bounded, thus by Lemma 3.7(1) we have

\[
\|g_n\|_L(\hat{x}) = \|U(f_n)\|_L(\hat{x}) \leq \|U\| \|f_n\|_L(\hat{x}) \text{ a.e.}
\]

Since \( \|f_n\|_L(\hat{x}) = 1 \text{ a.e. so } \|g_n\|_L(\hat{x}) \leq \|U\| \text{ a.e., that is, } g_n \text{ is a } L \text{-bounded for all } n \in \mathbb{N}.

**Proposition 5.5.** Let \( \{g_n\}_{n \in \mathbb{N}} \) be a \( L \)-Riesz basis for \( L^2(G) \). Then, \( \{g_n\}_{n \in \mathbb{N}} \) is a \( L \)-frame for \( L^2(G) \).

**Proof.** According to the definition we can write \( \{g_n\}_{n \in \mathbb{N}} = \{U(f_n)\}_{n \in \mathbb{N}} \), where \( U \) is a \( L \)-factorable operator which is a topological automorphism on \( L^2(G) \) and \( \{f_n\}_{n \in \mathbb{N}} \) is \( L \)-orthonormal basis for \( L^2(G) \). By Remark 5.2(b) we have for \( g \in L^2(G) \),

\[
\sum_{n=1}^{\infty} |\langle g, g_n \rangle_{L(\hat{x})}|^2 = \sum_{n=1}^{\infty} |\langle g, U f_n \rangle_{L(\hat{x})}|^2 = \|U^* g\|_{L(\hat{x})}^2 \text{ a.e.,}
\]

by Lemma 3.7(1) we have \( \sum_{n=1}^{\infty} |\langle g, g_n \rangle_{L(\hat{x})}|^2 \leq \|U^*\|^2 \|g\|_{L(\hat{x})}^2 \text{ a.e., this implies that a } L \text{-Riesz basis is a } L \text{-Bessel sequence. Thus the lower bound property follows from}

\[
\|g\|_L(\hat{x}) = \|(U^*)^{-1} U^* g\|_L(\hat{x}) \leq \\
\|(U^*)^{-1}\| \|(U^*) g\|_L(\hat{x}) = \|U^{-1}\| \|(U^*) g\|_L(\hat{x}) \text{ a.e.}
\]

Using the above proposition, Remarks 5.4(i) and 5.4(ii) and the following theorem we have the characterization of \( L \)-dual frames for \( L \)-Riesz basis.

**Theorem 5.6.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a \( L \)-Riesz basis for \( L^2(G) \). Then, there exists a \( L \)-Riesz basis \( \{g_n\}_{n \in \mathbb{N}} \) in \( L^2(G) \) such that

\[
g = \sum_{n=1}^{\infty} \langle g, g_n \rangle_{L(\hat{x})} f_n \text{ for all } g \in L^2(G).
\]

**Proof.** By definition we have \( \{f_n\}_{n \in \mathbb{N}} = \{U(e_n)\}_{n \in \mathbb{N}} \), where \( U \) is a \( L \)-factorable operator which is a topological automorphism on \( L^2(G) \) and \( \{e_n\}_{n \in \mathbb{N}} \) is a \( L \)-orthonormal basis for \( L^2(G) \).
Let $g \in L^2(G)$. By expanding $U^{-1}g$ in the $L$-orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$, we have

$$U^{-1}g = \sum_{n=1}^{\infty} [U^{-1}g, e_n]_L e_n = \sum_{n=1}^{\infty} [g, (U^{-1})^* e_n]_L e_n.$$ 

Setting $g_n := (U^{-1})^* e_n$, we have $g = UU^{-1}g = U \sum_{n=1}^{\infty} [g, g_n]_L e_n$. Then, $[g, (U^{-1})^* e_n] = [U^{-1}g, e_n]_L$ and $[U^{-1}g, e_n]_L \in L^\infty(G/L)$ for every $n \in \mathbb{N}$ and by Bessel’s Inequality

$$\sum_{n=1}^{\infty} \| [g, (U^{-1})^* e_n]_L (\dot{x}) \|^2 \leq \| g \|_L (\dot{x}) < \infty \text{ for a.e., } \dot{x} \in G/L.$$ 

Also,

$$g = U \sum_{n=1}^{\infty} [g, (U^{-1})^* e_n]_L e_n = \sum_{n=1}^{\infty} [g, (U^{-1})^* e_n]_L U e_n = \sum_{n=1}^{\infty} [g, g_n]_L f_n.$$ 

This completes the proof.

The sequence $\{g_n\}_{n \in \mathbb{N}}$ in the above proof is called the $L$-dual Riesz basis, and so a $L$-dual frame, for $\{f_n\}_{n \in \mathbb{N}}$.

**Example 5.7.** Let $G$ be a LCA group with a uniform lattice $L$. In [9], it is proved that $L^2(G)$ admits a $L$-orthonormal basis. Let $\{E_n\}_{n \in \mathbb{N}}$ be a $L$-orthonormal basis for $L^2(G)$ then $\mathcal{F} = \{E_1, E_1, E_2, E_2, \ldots, E_k, E_k, \ldots\}$ is a $L$-frame for $L^2(G)$. Also, $\{\frac{1}{2}E_1, \frac{1}{2}E_1, \ldots, \frac{1}{2}E_k, \frac{1}{2}E_k, \ldots\}$ and $\{E_1, 0, \ldots, E_k, 0, \ldots\}$ are $L$-dual frames for $\mathcal{F}$.

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