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A CHARACTERIZATION OF L-DUAL FRAMES AND L-DUAL RIESZ BASES

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Abstract. This paper is an investigation of L-dual frames with respect to a function-valued inner product, the so called L-bracket product on $L^2(G)$, where G is a locally compact abelian group with a uniform lattice L. We show that several well known theorems for dual frames and dual Riesz bases in a Hilbert space remain valid for L-dual frames and L-dual Riesz bases in $L^2(G)$.

1. Introduction

In [2], the bracket product is defined as a function valued inner product on $L^2(\mathbb{R})$ and, in [9], the ϕ-bracket product is defined as its extension to $L^2(G)$, where G is a locally compact abelian group (LCA) and ϕ is a topological isomorphism on G. As a new inner product on $L^2(G)$, we define the $L-$bracket product which can be applied to extend several ideas and constructions from the theory of shift invariant spaces, factorable operators and Wely-Heisenberg frames on $\mathbb{R}^n$ to the setting of LCA groups. These extensions are, in a more general and different way, using various tools in abstract harmonic analysis.

Dual frames and Riesz Bases for Hilbert spaces are defined in [1, 3].

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The present paper deals with characterizing L-dual frames and L-dual Riesz bases on $L^2(G)$, and consists of four sections. In the first section some definitions and preliminaries related to locally compact abelian groups and L-bracket products are introduced. In Section 2, we state some definitions and notations related to L-frames. In Section 3, we define and characterize L-dual frames and, finally, in Section 4, we define L-dual Riesz basis.

2. **Preliminaries**

In this section we give a brief review of definitions and notations from LCA groups and L-bracket product. For more details on LCA groups we refer to the book [4] and an extensive study of the L-bracket product theory can be found in [7].

Definition 2.1. A subgroup L of G is called a uniform lattice, if it is discrete and co-compact; i.e., G/L is compact.

Definition 2.2. Let $f, g \in L^2(G)$. The L-bracket product of f and g is defined as the mapping $[.,.]_L \colon L^2(G) \times L^2(G) \to L^1(G/L)$ given by

$$[f,g]_L(\dot{x}) = \sum_{k \in L} f\bar{g}(xk^{-1}) \quad for \quad all \quad \dot{x} \in G/L.$$

We define the L-norm of `f` as $\|f\|_L(\dot{x}) = ([f,f]_L(\dot{x}))^{\frac{1}{2}}$.

The above definition appears in [7] with the following formula

$$[f,g]_\phi(\dot{x}) = \sum_{\phi(k) \in \phi(L)} f\bar{g}(x\phi(k^{-1})) \quad for \quad all \quad \dot{x} \in G/\phi(L),$$

where ϕ is any topological isomorphism on G.

Note 2.3. *If $\phi \colon G \to G$ is a topological automorphism and L is a uniform lattice in G, then $\phi(L)$ is also a uniform lattice in G [3]. Thus, we assume that G is a LCA group with uniform lattice L' and we set $L = \phi(L')$. In the present paper we always assume that G/L is normalized; i.e., $|G/L|=1$.*

Let L be a uniform lattice in G. Choosing the counting measure on L, a relation between the Haar measure dx on G and $d\dot{x}$ on $\frac{G}{L}$ is given by the following case of Weil's formula [4]. For $f \in L^1(G)$, we have

$\sum_{k\in L} f(xk^{-1}) \in L^1(G)$ and

$$\int_G f(x)dx = \int_{G/L} \sum_{k\in L} f(xk^{-1})d\dot{x}.$$

Example 2.4. *Examples of L-bracket product:*

(1) *Consider $G = \mathbb{R}^n$ with the uniform lattice $L' = \mathbb{Z}^n$. Let the topological automorphism $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, given by $\phi(x) = Ax$, where A is an invertible $n \times n$ matrix. The L-bracket product is the A-bracket product defined as $[f, g]_L(x) = \sum_{n\in\mathbb{Z}^n} f\overline{g}(x - An)$, for $f, g \in L^2(\mathbb{R}^n)$ (see [5]).*

In particular, let $n = 1$ and $G = \mathbb{R}$ with the uniform lattice $L' = \mathbb{Z}$. Fix $a \in \mathbb{R}^+$, we define: $\phi(x) = ax$, for $x \in \mathbb{R}$. The mapping $[.,.]_L : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \longrightarrow L^1([0, 1])$, defined by $[f, g]_L(x) = \sum_{n\in\mathbb{Z}} f\overline{g}(x - na)$ is the a-pointwise inner product of f and g (see [2]).

(2) *Consider the LCA group $G = \mathbb{R} \times \Delta_p$, where p is a prime and Δ_p denote the group of p-adic integers as defined in [6]. and let L be the subgroup $\{(n, nu)\}_{n\in\mathbb{Z}}$ of $\mathbb{R} \times \Delta_p$, where $u = (1, 0, 0, ...)$. Then, L is a uniform lattice in $\mathbb{R} \times \Delta_p$ (obviously L is discrete and by Theorem 10.13 in [6], $(\mathbb{R} \times \Delta_p)/L$ is compact). Let $a := (1/p, 0, 0, ...) \in \Delta_p$. Then, the mapping $\phi : \mathbb{R}\times\Delta_p \to \mathbb{R}\times\Delta_p$ defined by $(x, v) \in \mathbb{R} \times \Delta_p$, by $\phi(x, v) = (2x, av)$, is a topological isomorphism on $\mathbb{R} \times \Delta_p$, thus The L-bracket product is $[f, g]_L(x, v) = \sum_{n\in\mathbb{Z}} f\overline{g}(x - 2n, v - anu)$, for $f, g \in L^2(\mathbb{R} \times \Delta_p)$ (see [8]).*

Definition 2.5. The function $g \in L^2(G)$ is L-bounded, if there exists $M > 0$ such that $\|g\|_L (\dot{x}) \le M$; a.e.

For $f, g \in L^2(G)$ the function $[f, g]_L g$ need not generally be in $L^2(G)$. For example consider $f(x) = g(x) = \chi_{[0,a]}x^{-\frac{1}{3}}$, where $a \in \mathbb{R}^+$ and $\phi(x) = ax$, for $x \in \mathbb{R}$. But, if $f, g, h \in L^2(G)$ and g, h are L-bounded, then $[f, g]_L h \in L^2(G)$ (see [7]).

3. L-frame

In [9], ϕ-frames and its associated ϕ-analysis and ϕ-frame operators are defined. They obtained criteria for a sequence to be a ϕ-frame or a ϕ-Bessel sequence. In this section we state those concepts in $L-$bracket product sense.

Definition 3.1. The sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^2(G)$ is said to be a L-frame, if there exist positive constants $0 < A \leq B < \infty$ such that $for \ \ all \ \ f \in L^2(G)$

$$A \|f\|_L^2 (\dot{x}) \leq \sum_{n\in\mathbb{N}} |[f, f_n]_L(\dot{x})|^2 \leq B \|f\|_L^2 (\dot{x}) \quad for \quad \dot{x} \in G/L \ a.e..$$

Those sequences in $L^2(G)$, which satisfy only the right-hand inequality in the above formula are called L-Bessel sequences.

We now intend to define L-pre frame and L-analysis operators. We need to introduce a vector space which plays the role of $l^2(\mathbb{N})$ in the standard case. To this end, define $l_1^2(G/L)$ as the space all sequences in $L^\infty(G/L)$ such that convergent in $L^1(G/L)$; i.e.

$$l_1^2(G/L) = \{\{g_i\}_{i\in\mathbb{N}} \subset L^\infty(G/L); \int_{G/L} \sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2 \, d\dot{x} < \infty\},$$

$l_1^2(G/L)$ is an inner product space with respect to the following inner product:

$$[.,.]_{l_1^2(G/L)} \colon l_1^2(G/L) \times l_1^2(G/L) \to L^1(G/L), [\{g_i\}, \{h_i\}]_{l_1^2(G/L)} = \sum_{i\in\mathbb{N}} g_i \bar{h_i}.$$

For $\{g_i\}_{i\in\mathbb{N}} \in l_1^2(G/L)$, The pointwise norm is defined by

$$\|\{g_i\}\|_{l_1^2(G/L)} (\dot{x}) = (\sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2)^{\frac{1}{2}},$$

and the uniform norm by

$$\|\{g_i\}\|_{l_1^2(G/L)} = (\int_{G/L} \sum_{i\in\mathbb{N}} |g_i(\dot{x})|^2) d\dot{x})^{\frac{1}{2}}.$$

Let $\{f_n\}_{n\in\mathbb{N}} (= f)$ be a L-bounded sequence in $L^2(G)$. Define the L-analysis operator as the mapping $T_L^f \colon L^2(G) \to l_1^2(G/L)$ by

$$T_L^f g = \{[g, f_n]_L\}_{n\in\mathbb{N}}, \quad for \ \ all \ \ g \in L^2(G),$$

and the L-pre frame operator as the mapping $T_L^{*f} \colon l_1^2(G/L) \to L^2(G)$ by

$$T_L^{*f}(\{g_n\}) = \sum_{n\in\mathbb{N}} f_n g_n, \quad for \ \ all \ \ \{g_n\}_{n\in\mathbb{Z}} \in l_1^2(G/L).$$

Remark 3.2. *Our purpose is to consider a special l_1^2-orthonormal basis for $l_1^2(G/L)$. Consider the functions $g(\dot{x}) = 1$ and $h(\dot{x}) = 0$, for all $\dot{x} \in G/L$, and the sequence $\Delta_n = \{\Delta_n^k\}_{k=1}^{\infty}$ of functions $\Delta_n^k\colon G/L \to \mathbb{C}$ for $n = 1, 2, 3...$ in $L^\infty(G/L)$, defined by*

$$\Delta_n^k(\dot{x}) = \begin{cases} g(\dot{x}) & \text{if } k = n, \\ h(\dot{x}) & \text{if } k \neq n. \end{cases} \tag{3.1}$$

The vectors $\{\Delta_n\}_{n\in\mathbb{N}}$ defined by (3.1) constitute an l_1^2-orthonormal basis for $l_1^2(G/L)$ that, called canonical l_1^2-orthonormal basis.

In the following theorem we characterize L-Bessel sequence in terms of the L-pre frame operators.

Theorem 3.3. [9] Let $\{f_n\}_{n\in\mathbb{N}}$ be a L-bounded sequence in $L^2(G)$.

(1) $\{f_n\}_{n\in\mathbb{N}}$ is L-Bessel with bound B if and only if T_L^{*f} is a well defined, bounded operator from $l_1^2(G/L)$ into $L^2(G)$ and $\left\| T_L^{*f} \right\| \leq \sqrt{B}$.

(2) $\{f_n\}_{n\in\mathbb{N}}$ is a L-frame if and only if T_L^{*f} is a well defined, bounded operator from $l_1^2(G/L)$ onto $L^2(G)$.

Remark 3.4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a L-frame. Assume that each f_n, $n \in \mathbb{N}$, is L-bounded in $L^2(G)$. Then, the L-frame operator defined by $S_L := T_L^{*f} T_L^f$ is bounded. For all $g \in L^2(G)$ we have*

$$[S_L g, g]_L(\dot{x}) = [\textstyle\sum_{n\in\mathbb{N}} [g, f_n]_L(\dot{x}) f_n, g]_L(\dot{x}) = \textstyle\sum_{n\in\mathbb{N}} [g, f_n]_L \overline{[g, f_n]_L}(\dot{x})$$
$$= \sum_{n\in\mathbb{N}} \left|[g, f_n]_L(\dot{x})\right|^2, \quad a.e., \quad \dot{x} \in G/L.$$

So, we have: $A[g, g]_L(\dot{x}) \leq [S_\phi g, g]_L(\dot{x}) \leq B[g, g]_L(\dot{x})$ a.e. Therefore, $AI \leq S_L \leq BI$. By a standard argument as in the frame theory S_L is invertible (see [9]), and

$$B^{-1} I \leq S_L^{-1} \leq A^{-1} I. \tag{3.2}$$

Definition 3.5. A function $h \in L^\infty(G)$ is said to be L-periodic, if $h(xk) = h(x)\ for\ \ all\ \ k \in L, x \in G$. We will denote by $B_L(G)$ the set of all L-periodic functions in $L^\infty(G)$.

Definition 3.6. Let E be a subgroup of G or G/L. An operator $U\colon L^2(G) \to L^p(E)$, $1 \leq p \leq \infty$, is said to be L -factorable, if $U(hf) = hU(f)\ for\ \ all\ \ f \in L^2(G)$ and all L-periodic functions $h \in L^\infty(G)$.

Lemma 3.7. [8]

(1) Let U be a bounded L-factorable operator on $L^2(G)$. Then, for every $f \in L^2(G)$ we have $\|Uf\|_L(\dot{x}) \leq \|U\| \, \|f\|_L(\dot{x})$ *a.e.*,

(2) Let $f, g \in L^2(G)$. Then, for all periodic functions h,

$$[fh, g]_L = h[f, g]_L, \quad [f, \bar{h}g]_L = h[f, g]_L.$$

The following proposition shows that every bounded L-factorable operator on $L^2(G)$ is adjointable.

Proposition 3.8. [8] Let $U\colon L^2(G) \to L^2(G)$ be a bounded L-factorable operator and U^* be its adjoint. Then, U^* is L-factorable. Moreover, for all $f, g \in L^2(G)$, $[U(f), g]_L(\dot{x}) = [f, U^*(g)]_L(\dot{x})$ *a.e.*, $\dot{x} \in G/L$.

Lemma 3.9. Let $\{f_n\}_{n\in\mathbb{N}}$ be a L-frame that is L-bounded and S_L is L-frame operator for $\{f_n\}_{n\in\mathbb{N}}$ then S_L is L-factorable.

Proof. Let h be a L-periodic function we show that

$$S_L(hf) = hS_L(f) \quad for \;\; all \;\; f \in L^2(G).$$

$S_L(hf) = \sum_{n\in\mathbb{N}} [hf, f_n]_L f_n$, on the other hand, by Lemma 3.7(2) we have $S_L(hf) = \sum_{n\in\mathbb{N}} h[f, f_n]_L f_n = hS_L(f)$. Thus, S_L is a L-factorable.

4. L -dual frame

Our goal in this section is to define and characterize L-dual frames for L-frames in $L^2(G)$.

Definition 4.1. Let $\{f_n\}_{n\in\mathbb{N}}$ be a L-bounded, L-frame, then the L-bounded, L-frame $\{g_n\}_{n\in\mathbb{N}}$ is called a L -dual frame for $\{f_n\}_{n\in\mathbb{N}}$ if

$$g = \sum_{n\in\mathbb{N}} [g, g_n]_L f_n \quad for \;\; all \;\; g \in L^2(G) \tag{4.1}$$

Remark 4.2. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a L-dual frame for $\{f_n\}_{n\in\mathbb{N}}$, thus they are L-Bessel sequences and we denote the L-pre frame operator for $\{f_n\}_{n\in\mathbb{N}}$ by T_L^{*f}, and the L-pre frame operator for $\{g_n\}_{n\in\mathbb{N}}$ by T_L^{*g}. In terms of these operators (4.1) means*

$$T_L^{*f} T_L^{g} = I. \tag{4.2}$$

Remark 4.3. *By the equation (3.2), $\{S_L^{-1}f_n\}_{n\in\mathbb{N}}$ is a L-frame and by Lemma 3.9, S_L^{-1} is L-factorable and then by Lemma 3.7(1):*

$$\left\|S_L^{-1}(f_n)\right\|_L(\dot{x}) \leq \left\|S_L^{-1}\right\| \|f_n\|_L(\dot{x}) \quad a.e., \quad for \quad all \quad n\in\mathbb{N}$$

thus $\{S_L^{-1}f_n\}_{n\in\mathbb{N}}$ is L-bounded and also we have

$$g = S_L S_L^{-1} g = \sum_{i\in\mathbb{N}} [g, S^{-1}f_n]_L f_n,$$

and $\{S_L^{-1}f_n\}_{n\in\mathbb{N}}$ is a L-dual frame for $\{f_n\}_{n\in\mathbb{N}}$, that is called the canonical or standard L-dual frame.

We begin with a lemma, which shows the roles of $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ can be interchanged:

Lemma 4.4. Assume that $\{f_n\}_{n\in\mathbb{N}}$ and$\{g_n\}_{n\in\mathbb{N}}$ are L-bounded, L-Bessel sequences in $L^2(G)$. Then, the following are equivalent:

(i) $g = \Sigma_{n\in\mathbb{N}}[g, g_n]_L f_n \quad for \quad all \quad g \in L^2(G)$
(ii) $g = \Sigma_{n\in\mathbb{N}}[g, f_n]_L g_n \quad for \quad all \quad g \in L^2(G)$

Proof. In terms of the L-pre frame operators (i) means that $T_L^{*f}T_L^g = I$

$$(T_L^{*f}T_L^g = I)^* = T_L^{*g}T_L^f = I \tag{4.3}$$

which is identical to the statement in (ii). In a similar way (ii) implies (i).

When (4.3) is satisfied, we say that T_L^{*g} is a left inverse of T_L^f.

Lemma 4.5. Let$\{f_n\}_{n\in\mathbb{N}}$ be a L-bounded, L-frame for $L^2(G)$ and $\{\Delta_k\}_{k\in\mathbb{N}}$ be the canonical l^2-orthonormal basis for $l^2(G/L)$. The L-dual frames for $\{f_n\}_{n\in\mathbb{N}}$ are precisely the family $\{g_n\}_{n\in\mathbb{N}} = \{V(\Delta_n)\}$, where $V : l^2(G/L) \to L^2(G)$ is a bounded left inverse of T_L^f.

Proof. If V is a bounded, left inverse of T_L^f, then V is surjective and $\{g_n\}_{n\in\mathbb{N}} = \{V(\Delta_n)\}$ is L-frame by Theorem 3.3(2). Since

$$\|g_k\|_{L^2(G)} = \|V(\Delta_k)\|_{L^2(G)} \leq \|V\| \|\Delta_k\|_{l^2(G/L)},$$

$\|\Delta_k\|_{l^2(G/L)} = 1$ and $\|V\| \leq M$, this implies that $\|g_k\|_{L^2(G)} \leq M$ and by Weil's formula we have

$$\int_{G/L} \|g_k\|_L(\dot{x})d\dot{x} = \int_{G/L}\sum_{l\in L}|g_k(xl)|^2\, d\dot{x} = \int_G |g_k(x)|^2\, dx = \|g_k\|_{L^2(G)} \leq M.$$

Therefore, we have $\|g_k\|_L(\dot{x}) \leq M$ a.e., for all $k \in \mathbb{N}$. Thus, $\{g_k\}_{k\in\mathbb{N}}$ is a L-bounded sequence. Also, we have,

$$T_L^f g = \{[g, f_k]_L\}_{k\in\mathbb{N}} = \sum_{k=1}^{\infty} [g, f_k]_L \Delta_k, \quad for \quad all \quad g \in L^2(G),$$

thus $g = VT_L^f g = \sum_{k=1}^{\infty}[g, f_k]_L g_k \ for \ all \ g \in L^2(G)$; i.e., $\{g_k\}_{k\in\mathbb{N}}$ is L-dual frame of $\{f_k\}_{k\in\mathbb{N}}$. Assume that $\{g_k\}_{k\in\mathbb{N}}$ is the L-dual frame of $\{f_k\}_{k\in\mathbb{N}}$, then, by Theorem 3.3(2), the L-pre frame T_L^{*g} of $\{g_k\}_{k\in\mathbb{N}}$ is bounded, In fact, $\{g_k\}_{k\in\mathbb{N}} = \{T_L^{*g}(\Delta_k)\}_{k\in\mathbb{N}}$, by Lemma 4.4, $T_L^{*g} T_L^f = I$ and the proof is complete.

Lemma 4.6. Let $\{f_n\}_{n\in\mathbb{N}}$ be a L-bounded, L-frame with L-pre frame operator T_L^{*f}. Then, the bounded left inverses of T_L^{*f} are precisely the operators having the form $S_L^{-1}T_L^{*f} + W(I - T_L^f S_L^{-1} T_L^{*f})$, where $W : l_1^2(G/L) \to L^2(G)$ and I denote the identity operator on $l_1^2(G/L)$.

Proof. We show that $S_L^{-1}T_L^{*f} + W(I - T_L^f S_L^{-1} T_L^{*f})$ is bounded and a left inverse of T_L^f. By Theorem 3.3(2) T_L^{*f}, S_L and S_L^{-1} are bounded, So, $S_L^{-1}T_L^{*f} + W(I - T_L^f S_L^{-1} T_L^{*f})$ is bounded too, and

$$(S_L^{-1}T_L^{*f} + W(I - T_L^f S_L^{-1} T_L^{*f}))T_L^f = I.$$

For implication, if U is a given left inverse of T_L^f, then by taking $W = U$ we have

$$S_L^{-1}T_L^{*f} + W(I - T_L^f S_L^{-1} T_L^{*f}) = S_L^{-1}T_L^{*f} + U - UT_L^f S_L^{-1} T_L^{*f} = U.$$

Now, we are ready to characterize all L-dual frames associated to a given L-frame.

Theorem 4.7. Let$\{f_n\}_{n\in\mathbb{N}}$ be a L-bounded, L-frame for $L^2(G)$, then the L-dual frames of $\{f_n\}_{n\in\mathbb{N}}$ are precisely the families

$$\{g_n\}_{n\in\mathbb{N}} = \{S_L^{-1} f_n + h_n - \sum_{k=1}^{\infty} [s_L^{-1} f_n, f_k]_L h_k\}_{n\in\mathbb{N}},$$

where $\{h_n\}_{n\in\mathbb{N}}$ is a L-bounded, L-Bessel sequence in $L^2(G)$.

Proof. By Lemmas 4.5 and 4.6 we can characterize the L-dual frames as all families of the form

$$\{g_k\}_{k\in\mathbb{N}} = \{S_L^{-1}T_L^{*f}(\Delta_k) + W(I - T_L^f S_L^{-1} T_L^{*f})\Delta_k\},$$

where, $W\colon l_1^2(G/L) \to L^2(G)$ is an operator of the form $W\{k_j\}_{j\in\mathbb{N}} = \sum_{j=1}^{\infty} k_j h_j$, such that $\{h_n\}_{n\in\mathbb{N}}$ is a L-bounded L-Bessel sequence in $L^2(G)$. Thus, $W\{\Delta_k\}_{k\in\mathbb{N}} = \{h_n\}_{n\in\mathbb{N}}$,

$$\begin{aligned}\{g_k\}_{k\in\mathbb{N}} &= \{S_L^{-1}T_L^{*f}(\Delta_k) + W(I - T_L^f S_L^{-1} T_L^{*f})(\Delta_k)\}\\ &= \{S_L^{-1} f_n + h_n - \sum_{k=1}^{\infty}[S_L^{-1} f_n, f_k]_L h_k\}_{n\in\mathbb{N}},\end{aligned}$$

that completes the proof.

5. **A characterization of L - dual Riesz bases**

In this section we show that every L-Riesz basis is a L-bounded, L-frame sequence in $L^2(G)$. Thus, it has a L-dual frame which is a L-Riesz basis. Similar to the usual inner product, we define L-orthogonality.

Definition 5.1. Let $f, g \in L^2(G)$. We say that f and g are L-orthogonal if $[f, g]_L = 0$.
A sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq L^2(G)$ is called L-orthonormal, if $[g_n, g_m]_L = 0$ for all $n \neq m \in \mathbb{N}$ and $\|g_n\|_L = 1$ for all $n \in \mathbb{N}$. For $\Lambda \subseteq L^2(G)$, the L-orthogonal complement of Λ is

$$\Lambda^{\perp_L} = \{g \in L^2(G); [f, g]_L = 0, \ \ a.e., \ \ for \ \ all \ \ f \in \Lambda\}.$$

Remark 5.2. If $\{g_n\}_{n\in\mathbb{N}}$ is a L-orthonormal basis in $L^2(G)$, in [9] it is proved that the following are equivalent:
(a) For each $f \in L^2(G)$, $f(x) = \sum_{n\in\mathbb{N}}([f, g_n]_L(\dot{x}))g_n(x)$, a.e., $x \in G$.
(b) For all $f \in L^2(G)$, $\|f\|_L^2(\dot{x}) = \sum_{n\in\mathbb{N}} |[f, g_n]_L(\dot{x})|^2$, *a.e., (Parseval Identity).*

Definition 5.3. A sequence $\{f_n\}_{n\in\mathbb{N}}$ in $L^2(G)$ is said to be L-Riesz basis, if there exists a L-orthonormal basis $\{g_n\}_{n\in\mathbb{N}}$ and L-factorable operator $U\colon L^2(G) \to L^2(G)$, which is a topological automorphism such that $U(g_n) = f_n$, for every $n \in \mathbb{N}$.

Remark 5.4. *(i) If* $U\colon L^2(G) \to L^2(G)$ *is invertible, then* U *is* L-*factorable. Indeed, for* $h \in B_L(G)$, *we have*

$$UU^{-1}(hf) = hf = hUU^{-1}f = U(hU^{-1}f).$$

Therefore, $U^{-1}(hf) = hU^{-1}f$ *for all* $f \in L^2(G)$, *and so* U^{-1} *is* L-*factorable.*
(ii) If $\{g_n\}_{n\in\mathbb{N}}$ *is a* L-*Riesz basis for* $L^2(G)$. *According to the definition*

we can write $\{g_n\}_{n\in\mathbb{N}} = \{U(f_n)\}_{n\in\mathbb{N}}$, *where* U *is a* L*-factorable operator which is a topological automorphism on* $L^2(G)$ *and* $\{f_n\}_{n\in\mathbb{N}}$ *is a* L*-orthonormal basis for* $L^2(G)$. *Since* U *is a topological automorphism then* U *is bounded, thus by Lemma 3.7(1) we have*

$$\|g_n\|_L(\dot{x}) = \|U(f_n)\|_L(\dot{x}) \leq \|U\| \, \|f_n\|_L(\dot{x}) \quad a.e.$$

Since $\|f_n\|_L(\dot{x}) = 1 \quad a.e.$ *so* $\|g_n\|_L(\dot{x}) \leq \|U\| \quad a.e.$, *that is,* g_n *is a* L*-bounded for all* $n \in \mathbb{N}$.

Proposition 5.5. *Let* $\{g_n\}_{n\in\mathbb{N}}$ *be a* L*-Riesz basis for* $L^2(G)$. *Then,* $\{g_n\}_{n\in\mathbb{N}}$ *is a* L*-frame for* $L^2(G)$.

Proof. According to the definition we can write $\{g_n\}_{n\in\mathbb{N}} = \{U(f_n)\}_{n\in\mathbb{N}}$, where U is a L-factorable operator which is a topological automorphism on $L^2(G)$ and $\{f_n\}_{n\in\mathbb{N}}$ is L-orthonormal basis for $L^2(G)$. By Remark 5.2(b) we have for $g \in L^2(G)$,

$$\sum_{n=1}^{\infty} |[g, g_n]_L(\dot{x})|^2 = \sum_{n=1}^{\infty} |[g, Uf_n]_L(\dot{x})|^2 = \|U^*g\|_L^2(\dot{x}) \quad a.e.,$$

by Lemma 3.7(1) we have $\sum_{n=1}^{\infty} |[g, g_n]_L(\dot{x})|^2 \leq \|U^*\|^2 \|g\|_L^2(\dot{x}) \; a.e.$, this implies that a L-Riesz basis is a L-Bessel sequence. The lower bound property follows from

$$\|g\|_L(\dot{x}) = \left\|(U^*)^{-1}U^*g\right\|_L(\dot{x}) \leq$$
$$\left\|(U^*)^{-1}\right\| \|(U^*)g\|_L(\dot{x}) = \left\|U^{-1}\right\| \|(U^*)g\|_L(\dot{x}) \quad a.e..$$

Using the above proposition, Remarks 5.4(i) and 5.4(ii) and the following theorem we have the characterization of L-dual frames for L-Riesz basis.

Theorem 5.6. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a* L*-Riesz basis for* $L^2(G)$. *Then, there exists a* L*-Riesz basis* $\{g_n\}_{n\in\mathbb{N}}$ *in* $L^2(G)$ *such that*

$$g = \sum_{n=1}^{\infty} [g, g_n]_L f_n \quad for \quad all \quad g \in L^2(G).$$

Proof. By definition we have $\{f_n\}_{n\in\mathbb{N}} = \{U(e_n)\}_{n\in\mathbb{N}}$, where U is a L-factorable operator which is a topological automorphism on $L^2(G)$ and $\{e_n\}_{n\in\mathbb{N}}$ is a L-orthonormal basis for $L^2(G)$.

Let $g \in L^2(G)$. By expanding $U^{-1}g$ in the L-orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, we have

$$U^{-1}g = \sum_{n=1}^{\infty}[U^{-1}g, e_n]_L e_n = \sum_{n=1}^{\infty}[g, (U^{-1})^* e_n]_L e_n.$$

Setting $g_n := (U^{-1})^* e_n$, we have $g = UU^{-1}g = U\sum_{n=1}^{\infty}[g, g_n]_L e_n$. Then, $[g, (U^{-1})^* e_n] = [U^{-1}g, e_n]_L$ and $[U^{-1}g, e_n]_L \in L^\infty(G/L)$ for every $n \in \mathbb{N}$ and by Bessel's Inequality

$$\sum_{n=1}^{\infty}\left|[g, (U^{-1})^* e_n]_L(\dot{x})\right|^2 \leq \|g\|_L(\dot{x}) < \infty \quad for \quad a.e., \quad \dot{x} \in G/L.$$

Also,

$$g = U\sum_{n=1}^{\infty}[g, (U^{-1})^* e_n]_L e_n = \sum_{n=1}^{\infty}[g, (U^{-1})^* e_n]_L U e_n = \sum_{n=1}^{\infty}[g, g_n]_L f_n.$$

This completes the proof.

The sequence $\{g_n\}_{n\in\mathbb{N}}$ in the above proof is called the L-dual Riesz basis, and so a L-dual frame, for $\{f_n\}_{n\in\mathbb{N}}$.

Example 5.7. *Let G be a LCA group with a uniform lattice L. In [9], it is proved that $L^2(G)$ admits a L-orthonormal basis. Let $\{E_n\}_{n\in\mathbb{N}}$ be a L-orthonormal basis for $L^2(G)$ then $\mathcal{F} = \{E_1, E_1, E_2, E_2, \ldots E_k, E_k, \ldots\}$ is a L-frame for $L^2(G)$. Also, $\{\frac{1}{2}E_1, \frac{1}{2}E_1, \ldots, \frac{1}{2}E_k, \frac{1}{2}E_k, \ldots\}$ and $\{E_1, 0, \ldots, E_k, 0, \ldots\}$ are L-dual frames for $\mathcal{F}$.*

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