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GENERALIZED DISTANCE AND COMMON FIXED POINT THEOREMS IN MENGER PROBABILISTIC METRIC SPACES

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Abstract. We introduce the concept of r-distance on a Menger probabilistic metric space. Furthermore we prove some fixed point theorems in a complete Menger probabilistic metric space generalizing some famous fixed point theorems.

1. Introduction and preliminaries

Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [12]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of the distance. A probabilistic generalization of metric spaces appears to have interest in the investigation of

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physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. Probabilistic normed spaces were introduced by Šerstnev [14] in 1962 by means of a definition that was closely modelled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. In the sequel, we shall adopt the usual terminologies, notations and conventions of the theory of probabilistic normed spaces, as in [1-5, 8, 9, 11, 12].

Throughout this paper, the space of all probability distribution functions (briefly, d.f.s) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1] :$ F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here, $l^- f(x)$ denotes the left limit of the function f at the point x and is defined to be $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all, t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the d.f. given by:

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. [12] A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t–norm if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$, for all $a \in [0,1]$;
(d) $T(a,b) \leq T(c,d)$, whenever $a \leq c$ and $c \leq d$, and $a,b,c,d \in [0,1]$.

Two typical examples of continuous t–norms are $T(a,b) = ab$ and $T(a,b) = \min(a,b)$.

Now, t–norms are recursively defined by $T^1 = T$ and

$$T^n(x_1, \cdots, x_{n+1}) = T(T^{n-1}(x_1, \cdots, x_n), x_{n+1}),$$

for $n \geq 2$, $x_i \in [0,1]$, and $i \in \{1, 2, \ldots, n+1\}$. The t-norm T is *Hadžić type I* if for given $\varepsilon \in]0,1[$, there is $\delta \in]0,1[$ (which may depend on m) such that

$$T^m(1-\delta, ..., 1-\delta) > 1 - \varepsilon, \; m \in \mathbb{N}. \tag{1.1}$$

We assume that, here, all t–norms are Hadžić type I.

Definition 1.2. [12] A mapping $S : [0, 1]\times[0, 1] \longrightarrow [0, 1]$ is a continuous s–norm if S satisfies the following conditions:

(a) S is associative and commutative;
(b) S is continuous;
(c) $S(a, 0) = a$, for all $a \in [0, 1]$;
(d) $S(a, b) \leq S(c, d)$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Two typical examples of continuous s–norms are $S(a, b) = \min(a + b, 1)$ and $S(a, b) = \max(a, b)$.

Definition 1.3. A *Menger probabilistic metric space* (briefly, Menger PM-space) is a triplet $(X, \mathcal{F}, T)$, where X is a nonempty set, T is a continuous t–norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x, y), then the following conditions hold, for all x, y, z in X:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$, for all $t > 0$, if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$, for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 1.4. A *Menger probabilistic normed space* (briefly, Menger PN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t–norm, and μ is a mapping from X into D^+ such that the following conditions hold, for all x, y in X:

(PN1) $\mu_x(t) = \varepsilon_0(t)$, for all $t > 0$, if and only if $x = 0$;
(PN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$, for $\alpha \neq 0$;
(PN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$, for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 1.5. Let $(X, \mathcal{F}, T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$, whenever $n \geq N$.

(2) A sequence $\{x_n\}_n$ in X is called *Cauchy sequence* if for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$, whenever $n, m \geq N$.

(3) A Menger PM-space $(X, \mathcal{F}, T)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Definition 1.6. Let $(X, \mathcal{F}, T)$ be a Menger PM space. For each p in X and $\lambda > 0$, the strong $\lambda-$*neighborhood* of p is the set,

$$N_p(\lambda) \quad = \quad \{q \in X : F_{p,q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for X is the union $\bigcup_{p\in V} \mathcal{N}_p$, where $\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}$.

The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 1.7. [8, 13] *If $(X, \mathcal{F}, T)$ is a PM-space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$, then $\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$.*

Remark 1.8. In certain situations we assume the following. Suppose for every $\mu \in]0, 1[$ there exists a $\lambda \in]0, 1[$ (which does not depend on n) with

$$T^{n-1}(1 - \lambda, .., 1 - \lambda) > 1 - \mu, \text{ for each } n \in \{1, 2, ...\}. \tag{1.2}$$

2. R–distance

Recently, Kada, et al. [7] introduced the concept of w–distance on a metric space and proved some fixed point theorems. Here, using the concept of w–distance, we define the concept of r-distance on a Menger PM-space.

Definition 2.1. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then, the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ is called an r-distance on X if the followings are satisfied:

(r1) $f_{x,z}(t + s) \geq T(f_{x,y}(t), f_{y,z}(s))$, for all $x, y, z \in X$ and $t, s \geq 0$;
(r2) for any $x \in X$ and $t \geq 0$, $f_{x,.} : X \times [0, \infty] \longrightarrow [0, 1]$ is continuous;
(r3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f_{z,x}(t) \geq 1 - \delta$ and $f_{z,y}(s) \geq 1 - \delta$ imply $F_{x,y}(t + s) \geq 1 - \varepsilon$.

Let us give some examples of r-distances.

Example 2.2. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then, $f = F$ is an r-distance on X.

Proof. Properties (r1) and (r2) are obvious. We show (r3). Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, for $F_{z,x}(t) \geq 1-\delta$ and $F_{z,y}(s) \geq 1-\delta$, we have,

$$\begin{aligned} F_{x,y}(t+s) &\geq T(F_{z,x}(t), F_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

□

Example 2.3. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then, the function $f : X^2 \times [0, \infty) \longrightarrow [0, 1]$ defined by $f_{x,y}(t) = 1 - c$ for every $x, y \in X$ and $t > 0$ is an r-distance on X, where $c \in]0, 1[$.

Proof. Properties (r1) and (r2) are obvious. To show (r3), for any $\varepsilon > 0$, put $\delta = 1 - c/2$. Then, we have, that $f_{z,x}(t) \geq 1 - c/2$ and $f_{z,y}(s) \geq 1 - c/2$ imply $F_{x,y}(t+s) \geq 1-\varepsilon$. □

Example 2.4. Let (X, μ, T) be a Menger PN-space. Then, the function $f : X^2 \times [0, \infty) \longrightarrow [0, 1]$ defined by $f_{x,y}(t+s) = T(\mu_x(t), \mu_y(s))$ for every $x, y \in X$ and $t, s > 0$ is an r-distance on X.

Proof. Let $x, y, z \in X$ and $t, s > 0$. Then, we have,

$$\begin{aligned} f_{x,z}(t+s) &= T(\mu_x(t), \mu_z(s)) \\ &\geq T(T(\mu_x(t/2), \mu_y(t/2)), T(\mu_y(s/2), \mu_z(s/2))) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence, (r1) holds. Also, (r2) is obvious. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, for $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$, we have,

$$\begin{aligned} F_{x,y}(t+s) &= \mu_{x-y}(t+s) \geq T(\mu_x(t), \mu_y(s)) \\ &\geq T(T(\mu_x(t/2), \mu_z(t/2)), T(\mu_y(s/2), \mu_z(s/2))) \\ &= T(f_{z,x}(t), f_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

Hence, (r3) holds. □

Example 2.5. Let (X, μ, T) be a Menger PN-space. Then, the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ defined by $f_{x,y}(t) = \mu_x(t)$ for every $x, y \in X$ and $t > 0$ is an r-distance on X.

Proof. Let $x, y, z \in X$ and $t, s > 0$. Then, we have,

$$\begin{aligned} f_{x,z}(t+s) &= \mu_z(t+s) \\ &\geq T(\mu_y(t), \mu_z(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence, (r1) holds. Also, (r2) is obvious. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, for $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$, we have,

$$\begin{aligned} F_{x,y}(t+s) &= \mu_{x-y}(t+s) \\ &\geq T(\mu_x(t), \mu_y(s)) \\ &= T(f_{z,x}(t), f_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

Hence, (r3) holds. □

Example 2.6. Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let A be a continuous mapping from X into X. Then, the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ defined by

$$f_{x,y}(t) = \min(F_{Ax,y}(t), F_{Ax,Ay}(s))$$

for every $x, y \in X$ and $t, s > 0$ is an r-distance on X.

Proof. Let $x, y, z \in X$ and $t, s > 0$. If $F_{Ax,z}(t) \leq F_{Ax,Ay}(t)$, then we have,

$$\begin{aligned} f_{x,z}(t+s) &= F_{Ax,z}(t+s) \geq T(F_{Ax,Ay}(t), F_{Ay,z}(s)) \\ &\geq T(\min(F_{Ax,y}(t), F_{Ax,Ay}(t)), \min(F_{Ay,z}(s), F_{Ax,Ay}(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

With this inequality, we have,

$$
\begin{aligned}
f_{x,z}(t+s) &= F_{Ax,Az}(t+s) \geq T(F_{Ax,Ay}(t), F_{Ay,Az}(s)) \\
&\geq T(\min(F_{Ax,y}(t), F_{Ax,Ay}(t)), \min(F_{Ay,z}(s), F_{Ax,Ay}(s)) \\
&= T(f_{x,y}(t), f_{y,z}(s)).
\end{aligned}
$$

Hence, (r1) holds. Since A is continuous, then (r2) is obvious . Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Then, from $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$, we have $F_{Az,x}(t) \geq 1-\delta$ and $F_{Az,y}(s) \geq 1-\delta$. Therefore,

$$
\begin{aligned}
F_{x,y}(t+s) &\geq T(F_{Az,x}(t), F_{Az,y}(s)) \\
&\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon.
\end{aligned}
$$

Hence, (r3) holds. □

Next, we discuss some properties of r-distance.

Lemma 2.7. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and f be a r-distance on it. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero, and $x, y, z \in X$ and $t, s > 0$. Then, the followings hold:*

(1) *if $f_{x_n,y}(t) \geq 1-\alpha_n$ and $f_{x_n,z}(s) \geq 1-\beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $f_{x,y}(t) = 1$ and $f_{x,z}(s) = 1$, then $y = z$;*
(2) *if $f_{x_n,y_n}(t) \geq 1-\alpha_n$ and $f_{x_n,z}(s) \geq 1-\beta_n$ for any $n \in \mathbb{N}$, then $F_{y_n,z}(t+s) \to 1$;*
(3) *if $f_{x_n,x_m}(t) \geq 1-\alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *if $f_{y,x_n}(t) \geq 1-\alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. We first prove (2). Let $\varepsilon > 0$ be given. From the definition of r-distance, there exists $\delta > 0$ such that $f_{u,v}(t) \geq 1-\delta$ and $f_{u,z}(s) \geq 1-\delta$ imply $F_{v,z}(t+s) \geq 1-\varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every $n \geq n_0$. Then, we have, for any $n \geq n_0$, $f_{x_n,y_n}(t) \geq 1-\alpha_n \geq 1-\delta$ and $f_{x_n,z}(t) \geq 1-\beta_n \geq 1-\delta$, and hence $F_{y_n,z}(t+s) \geq 1-\varepsilon$. This implies that $\{y_n\}$ converges to z. It follows from (2) that (1) holds. We prove (3). Let $\varepsilon > 0$ be given. As in the proof of (1), choose $\delta > 0$ and

$n_0 \in \mathbb{N}$. Then, for any $n, m \geq n_0 + 1$,

$$f_{x_{n_0},x_n}(t) \geq 1 - \alpha_{n_0} \geq 1 - \delta \text{ and } f_{x_{n_0},x_m}(s) \geq 1 - \alpha_{n_0} \geq 1 - \delta,$$

and hence $F_{x_n,x_m}(t+s) \geq 1 - \varepsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.8. *Let $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ be an r-distance on X. If we define $E_{\lambda,f} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,f}(x, y) = \inf\{t > 0 : f_{x,y}(t) > 1 - \lambda\},$$

for each $\lambda \in]0, 1[$ and $x, y \in X$, then we have the followings:

(1) *For any $\mu \in]0, 1[$, there exists $\lambda \in]0, 1[$ such that*

$$E_{\mu,f}(x_1, x_k) \leq E_{\lambda,f}(x_1, x_2) + E_{\lambda,f}(x_2, x_3) + \cdots + E_{\lambda,f}(x_{k-1}, x_k),$$

for any $x_1, ..., x_k \in X$;

(2) *For any sequence $\{x_n\}$ in X, we have, $f_{x_n,x}(t) \longrightarrow 1$ if and only if $E_{\lambda,f}(x_n, x) \to 0$. Also, the sequence $\{x_n\}$ is Cauchy w.r.t. f if and only if it is Cauchy with $E_{\lambda,f}$.*

Proof. The proof is the same as in Lemma 1.6 of [10]. □

Remark 2.9. If (1.2) holds, then the λ in Lemma 2.8(1) does not depend on k (see [10]).

3. Common fixed point theorems with r–distance

Now, we are in a position to prove some fixed point theorems in complete Menger PM-spaces.

Theorem 3.1. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, f be an r-distance on it and A be a mapping from X into itself. Suppose that there exists $k \in]0, 1[$ such that*

$$f_{Ax,A^2x}(t) \geq f_{x,Ax}(t/k),$$

for every $x \in X$, $t > 0$ and that

$$\sup\{T(f_{x,y}(t), f_{x,Ax}(t)) : x \in X\} < 1,$$

for every $y \in X$ with $y \neq Ay$.
Then, we have:

(a) *If (1.1) holds and there exists a $u \in X$ with*

$$E_f(u, Au) = \sup\{E_{\gamma,f}(u, Au) : \gamma \in]0,1[\} < \infty,$$

then there exists $z \in X$ such that $z = Az$.

(b) *If (1.2) holds, then there exists $z \in X$ such that $z = Az$. Moreover, if $v = Av$ and $f \in D^+$, then $f_{v,v} = \varepsilon_0$.*

Proof. (a). Let $u \in X$ be such that $E_f(u, Au) < \infty$. Define:

$$u_n = A^n u \text{ for any } n \in \mathbb{N}.$$

Then, we have, for any $n \in \mathbb{N}$,

$$f_{u_n,u_{n+1}}(t) \geq f_{u_{n-1},u_n}(\frac{t}{k}) \geq \cdots \geq f_{u,u_1}(\frac{t}{k^n}).$$

Therefore,

$$\begin{aligned} E_{\lambda,f}(u_n, u_{n+1}) &= \inf\{t > 0 : f_{u_n,u_{n+1}}(t) > 1-\lambda\} \\ &\leq \inf\{t > 0 : f_{u,u_1}(\frac{t}{k^n}) > 1-\lambda\} \\ &= k^n E_{\lambda,f}(u, u_1). \end{aligned}$$

Thus, for $m > n$ and $\lambda \in]0,1[$ there exists $\gamma \in]0,1[$ such that

$$\begin{aligned} E_{\lambda,f}(u_n, u_m) &\leq E_{\gamma,f}(u_n, u_{n+1}) + \cdots + E_f(u_{m-1}, u_m) \\ &\leq \frac{k^n}{1-k} E_{\gamma,f}(u, u_1). \end{aligned}$$

There exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have $E_{\lambda,f}(u_n, u_m) \longrightarrow 0$. By lemmas 2.7 and 2.8, $\{u_n\}$ is a Cauchy sequence. Therefore, by Lemma 2.8(2), there exist $n_1 \in \mathbb{N}$ and a sequence $\{\delta_n\}$ convergent to 0 such that for $n \geq \max\{n_0, n_1\}$ we have,

$$f_{u_n,u_m}(t) \geq 1 - \delta_n.$$

Since X is complete, then $\{u_n\}$ converges to some point $z \in X$. Hence, by $(r2)$, we have,

$$f_{u_n,z} = \lim_{m\to\infty} f_{u_n,u_m} \geq 1 - \delta_n,$$

and

$$f_{u_n,u_{n+1}} \geq 1 - \delta_n.$$

Assume $z \neq Az$. Then, by the hypothesis, we have,

$$\begin{aligned} 1 &> \sup\{T(f_{x,z}(t), f_{x,Ax}(t)) : x \in X\} \\ &\geq \sup\{T(f_{u_n,z}(t), f_{u_n,u_{n+1}}(t)) : n \in \mathbb{N}\} \\ &\geq \sup\{T(1-\delta_n, 1-\delta_n) : n \in \mathbb{N}\} = 1. \end{aligned}$$

This is a contradiction. Therefore, we have $z = Az$.

(b). The argument is as in case (a) except in this case we make use of Remark 2.9.

Now, if $v = Av$ and $f \in D^+$, then we have,

$$f_{v,v}(t) = f_{Av,A^2v}(t) \geq f_{v,Av}(\frac{t}{k}) = f_{v,v}(\frac{t}{k}).$$

Continuing this process, we have,

$$f_{v,v}(t) = f_{v,v}(\frac{t}{k^n}).$$

Also, we have $f_{v,v} = \varepsilon_0$. □

Corollary 3.2. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, f be an r-distance on it and A be a continuous mapping from X into itself. Suppose there exists $k \in]0, 1[$ such that*

$$f_{Ax,A^2x}(t) \geq f_{x,Ax}(t/k),$$

for every $x \in X$, $t > 0$. Then, we have:

(a) *If (1.1) holds and $u \in X$ with $E_F(u, Au) < \infty$, then there exists $z \in X$ such that $z = Az$.*

(b) *If (1.2) holds, then there exists $z \in X$ such that $z = Az$.*

Moreover, if $v = Av$ and $f \in D^+$, then $f_{v,v} = \varepsilon_0$.

Proof. (a). Assume that there exists $y \in X$ with $y \neq Ay$ and

$$\sup\{T(f_{x,y}(t), f_{x,Ax}(t)) : x \in X\} = 1.$$

Then, there exists $\{x_n\}$ such that

$$\lim_{n\to\infty} T(f_{x_n,y}(t), f_{x_n,Ax_n}(t)) = 1.$$

Since $f_{x_n,y}(t) \longrightarrow 1$ and $f_{x_n,Ax_n}(t) \longrightarrow 1$, then by Lemma 2.7, we have $\{Ax_n\}$ converging to y. We also have,

$$\begin{aligned} f_{x_n,A^2x_n}(t) &\geq T(f_{x_n,Ax_n}(\frac{t}{2}), f_{Ax_n,A^2x_n}(\frac{t}{2})) \\ &\geq T(f_{x_n,Ax_n}(\frac{t}{2}), f_{x_n,Ax_n}(\frac{t}{2k})) \\ &\longrightarrow 1, \end{aligned}$$

and hence $\{A^2x_n\}$ converges to y. Since A is continuous, then we have,

$$Ay = A(\lim_{n\to\infty} Ax_n) = \lim_{n\to\infty} A^2x_n = y.$$

This is a contradiction. Hence, if $y \neq Ay$, then

$$\sup\{T(f_{x,y}(t), f_{x,Ax}(t)) : x \in X\} < 1.$$

Thus, we have the desired result from Theorem 3.1.

(b). The argument is as in case (a) except in this case we make use of Remark 2.9. □

Theorem 3.3. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space and A be a mapping from X into itself. Suppose that there exists $\beta \in]0, 1[$ such that*

$$F_{Ax,Ay}(t) \geq S(F_{x,Ax}(\frac{t}{\beta}), F_{y,Ay}(\frac{t}{\beta})), \tag{3.1}$$

for every $x, y \in X$ and $t > 0$.

(a) *If (1.1) holds and there exists a $u \in X$ with $E_F(u, Au) < \infty$, then A has a unique fixed point.*
(b) *If (1.2) holds then A has a unique fixed point.*

Proof. (a). Let $x \in X$. From the inequality (3.1), we have,

$$F_{Ax,A^2x}(t) \geq S(F_{x,Ax}(\frac{t}{\beta}), F_{Ax,A^2x}(\frac{t}{\beta})),$$

and hence,

$$F_{Ax,A^2x}(t) \geq F_{x,Ax}(\frac{t}{\beta}).$$

Since the probabilistic metric F is an r-distance, assume that there exists $y \in X$ with $y \neq Ay$ and

$$\sup\{T(F_{x,y}(t), F_{x,Ax}(t)) : x \in X\} = 1.$$

Then, there exists $\{x_n\}$ such that

$$\lim_{n\to\infty} T(F_{x_n,y}(t), F_{x_n,Ax_n}(t)) = 1.$$

Since $F_{x_n,y}(t) \longrightarrow 1$ and $F_{x_n,Ax_n}(t) \longrightarrow 1$, then by Lemma 2.7, we have $\{Ax_n\}$ converging to y. On the other hand, since A satisfies the condition (3.1), then we have,

$$F_{Ax_n,Ay}(t) \geq S(F_{x_n,Ax_n}(\frac{t}{\beta}), F_{y,Ay}(\frac{t}{\beta})) \longrightarrow 1,$$

as $n \longrightarrow \infty$, i.e., $y = Ay$. This is a contradiction. Hence, if $y \neq Ay$, then

$$\sup\{T(F_{x,y}(t), F_{x,Ax}(t)) : x \in X\} < 1.$$

By Theorem 3.1, there exists $z \in X$ such that $z = Az$. Since $F \in D^+$, then the uniqueness is trivial. This completes the proof for (a).

(b). The argument is as in case (a) except in this case we make use of Remark 2.9. □

Definition 3.4. [6] We say that the r-distance f has property (C'), if it satisfies the following condition:

$$f_{x,y}(t) = C', \quad \text{for all } t > 0, \quad \text{implies } C' = 1.$$

Theorem 3.5. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space, f be an r-distance on it and $A, B : X \longrightarrow X$ be maps that satisfy the following conditions:*

(a) *$B(X) \subseteq A(X)$;*
(b) *A and B are continuous;*
(c) *$f_{B(x),B(y)}(t) \geq f_{A(x),A(y)}(\frac{t}{k})$, for all $x, y \in X$, $t > 0$ and $0 < k < 1$.*

Assume that for each $x \in X$,

$$E_f(A(x), B(x)) + E_f(A(x), z) + E_f(B(x), z) + E_f(B(x), B(B(x))) < \infty,$$

for all $z \in X$ with $B(z) \neq B(B(z))$, where,

$$E_f(w, u) = \sup\{E_{\gamma,f}(w, u) : \gamma \in (0, 1)\}.$$

Also, suppose if $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} A(x_n) = y \in X$, then for every $\mu \in]0, 1[$, we have,

$$E_{\mu,f}(A(x_n), y) \leq \lim_{p\to\infty} E_{\mu,f}(A(x_n), A(x_p)).$$

In addition,

(i) If (1.1) holds and there exists a $x_0 \in X$ *with*

$$E_f(A(x_0), B(x_0)) = \sup\{E_{\gamma,f}(A(x_0), B(x_0)) : \gamma \in (0,1)\} < \infty,$$

and

$$E_f(B(x_0), B(B(x_0))) = \sup\{E_{\gamma,f}(B(x_0), B(B(x_0))) : \gamma \in (0,1)\} < \infty,$$

then A *and* B *have a common fixed point provided that* A *and* B *commute.*

(ii) If (1.2) holds, then A *and* B *have a common fixed point provided that* A *and* B *commute.*

Moreover, if f *has property* (C'), $f(.)$ *is non-decreasing and* $B(v) = B(B(v))$, *for all* $v \in X$, *then* $f_{B(v),B(v)}(t) = 1$.

Proof. *(i).* We claim that for every $x \in X$,

$$\inf\{E_f(A(x), B(x)) + E_f(A(x), z) + E_f(B(x), z)$$

$$+E_f(B(x), B(B(x)))\} > 0,$$

for every $z \in X$ with $B(z) \neq B(B(z))$. For the moment, suppose that the claim is true. Let x_0 in X with $E_f(A(x_0), B(x_0)) < \infty$ and $E_f(B(x_0), B(B(x_0))) < \infty$. By (a), we can find x_1 such that $A(x_1) = B(x_0)$. By induction, we can define a sequence $\{x_n\}_n$ such that $A(x_n) = B(x_{n-1})$. By induction again,

$$\begin{aligned} f_{A(x_n),B(x_{n+1})}(t) &= f_{B(x_{n-1}),B(x_n)}(t) \\ &\geq f_{A(x_{n-1}),A(x_n)}(\frac{t}{k}) \\ &\geq \cdots \geq f_{A(x_0),A(x_1)}(\frac{t}{k^n}), \end{aligned}$$

and therefore,

$$E_{\lambda,f}(A(x_n), A(x_{n+1})) \leq k^n E_{\lambda,f}(A(x_0), A(x_1)),$$

for $n = 1, 2, \cdots$, which implies that, for $p > n$ and for any $\mu \in]0, 1[$, there exists $\lambda \in]0, 1[$ such that

$$\begin{aligned}
& E_{\mu,f}(A(x_n), A(x_p)) \\
\leq \ & E_{\lambda,f}(A(x_{p-1}), A(x_p)) + E_{\lambda,f}(A(x_{p-2}), A(x_{p-1})) + \\
& \cdots + E_{\lambda,f}(A(x_n), A(x_{n+1})) \\
\leq \ & E_f(A(x_0), A(x_1)) \sum_{j=n}^{p-1} k^j \leq \frac{k^n}{1-k} E_f(A(x_0), A(x_1)).
\end{aligned}$$

Thus, $\{A(x_n)\}$ is a Cauchy sequence. Since X is complete, then there exists $y \in X$ such that $\lim_{n\to\infty} A(x_n) = y$. As a result, $B(x_{n-1}) = A(x_n)$ tends to y, and so $\{B(A(x_n))\}_n$ converges to $B(y)$. However, $B(A(x_n)) = A(B(x_n))$ by the commutativity of A and B, and so $A(B(x_n))$ converges to $A(y)$. Because limits are unique, $A(y) = B(y)$, and so $A(A(y)) = A(B(y))$. On the other hand, we have,

$$E_{\mu,f}(A(x_n), y) \leq \lim_{p\to\infty} E_{\mu,f}(A(x_n), A(x_p)) \leq \frac{k^n}{1-k} E_f(A(x_0), A(x_1)).$$

So since this holds for all $\mu \in]0, 1[$, then we have,

$$E_f(A(x_n), y) \leq \frac{k^n}{1-k} E_f(A(x_0), A(x_1)).$$

Similarly, since $B(x_n) = A(x_{n+1})$, then we have,

$$E_f(B(x_n), y) \leq \frac{k^{n+1}}{1-k} E_f(A(x_0), A(x_1)),$$

and

$$\begin{aligned}
f_{B(x_n),B(B(x_n))}(t) \ & \geq \ f_{A(x_n),A(B(x_n))}(\frac{t}{k}) \\
& = \ f_{B(x_{n-1}),B(B(x_{n-1}))}(\frac{t}{k}) \\
& \geq \ f_{A(x_{n-1}),A(g(x_{n-1}))}(\frac{t}{k^2}) \\
& = \ f_{B(x_{n-2}),B(B(x_{n-2}))}(\frac{t}{k^2}) \\
& \geq \ \cdots \geq f_{A(x_1),B(A(x_1))}(\frac{t}{k^n}),
\end{aligned}$$

which imply:

$$\begin{aligned}
E_{\mu,f}(B(x_n), B(B(x_n))) \ & \leq \ k^n E_{\mu,f}(A(x_1), B(A(x_1))) \\
& \leq \ k^n E_f(A(x_1), B(A(x_1))),
\end{aligned}$$

and so,

$$E_f(B(x_n), B(B(x_n))) \le k^n E_f(A(x_1), B(A(x_1)).$$

Now, we show $B(y) = B(B(y))$. Suppose $B(y) \neq B(B(y))$. By the claim above, we have,

$$\begin{aligned}
0 &< \inf\{E_f(A(x), B(x)) + E_f(A(x), y) + E_f(B(x), y) + \\
&\quad E_f(A(x), B(B(x))) : x \in X\} \\
&\le \inf\{E_f(A(x_n), B(x_n)) + E_f(A(x_n), y) + E_f(B(x_n), y) + \\
&\quad E_f(B(x_n), B(B(x_n))) : n \in \mathbf{N}\} \\
&= \inf\{E_f(A(x_n), A(x_{n+1})) + E_f(A(x_n), y) + E_f(B(x_n), y) + \\
&\quad E_f(B(x_n), B(B(x_n))) : n \in \mathbf{N}\} \\
&\le \inf\{k^n E_f(A(x_0), A(x_1)) + \frac{k^n}{1-k} E_f(A(x_0), A(x_1)) + \\
&\quad \frac{k^{n+1}}{1-k} E_f(A(x_0), A(x_1)) + k^n E_f(A(x_1), B(A(x_1))) : n \in \mathbf{N}\} \\
&= 0.
\end{aligned}$$

This is a contradiction. Therefore, $B(y) = B(B(y))$. Thus, $B(y) = B(B(y)) = A(B(y))$, and so $B(y)$ is a common fixed point of A and B.

Furthermore, if $B(y)$ is a common fixed point of A and B and $B(v) = B(B(v))$, for all $v \in X$, then we have,

$$\begin{aligned}
f_{B(y),B(y)}(t) &= f_{B(B(y)),B(B(y))}(t) \\
&\ge f_{A(B(y)),A(B(y))}(t/k) \\
&= f_{B(y),B(y)}(t/k).
\end{aligned}$$

On the other hand, since f is non-decreasing, then we have,

$$f_{B(y),B(y)}(t) \le f_{B(y),B(y)}(t/k).$$

Hence,

$$f_{B(y),B(y)}(t) = f_{B(y),B(y)}(t/k),$$

which implies $f_{B(y),B(y)}(t) = C'$, for every $t > 0$. Thus, by property (C'), we have $f_{B(y),B(y)}(t) = 1$.

Now, it remains to prove the claim. Assume that there exists $y \in X$ with $B(y) \neq B(B(y))$ and

$$\begin{aligned}
\inf\{E_f(A(x), B(x)) &+ E_f(A(x), y) + E_f(B(x), y) \\
&\quad + E_f(B(x), B(B(x))) : x \in X\} = 0.
\end{aligned}$$

Then, there exists $\{x_n\}$ such that

$$\lim_{n\to\infty}\{E_f(A(x_n),B(x_n)) + E_f(A(x_n),y)+E_f(B(x_n),y) + E_f(B(x_n),B(B(x_n)))\}=0.$$

By Lemma 2.8(2), $f_{A(x_n),B(x_n)}(t)\longrightarrow 1$ and $f_{A(x_n),y}(t)\longrightarrow 1$, and therefore by Lemma 2.7, we have,

$$\lim_{n\to\infty} B(x_n)=y. \tag{3.2}$$

Also, by Lemma 2.8(2), $f_{B(x_n),y}(t)\longrightarrow 1$ and $f_{B(x_n),B(B(x_n))}(t)\longrightarrow 1$, and therefore by Lemma 2.7, we have,

$$\lim_{n\to\infty} B(B(x_n))=y. \tag{3.3}$$

By (3.2), (3.3) and the continuity of B we have,

$$B(y)=B(\lim_n B(x_n))=\lim_n B(B(x_n))=y.$$

Therefore, $B(y)=B(B(y))$, which is a contradiction. Hence, if $B(y)\neq B(B(y))$, then

$$\inf\{E_f(A(x),B(x)) + E_f(A(x),y)+E_f(B(x),y) + E_f(B(x),B(B(x))) : x\in X\}>0.$$

(ii). The argument is as in case (i) except in this case we use Remark 2.9. □

If we work with the usual distance function we can get a more general result improving Theorem 2.3 in [10] (we do not need to assume $\sum_{n=1}^{\infty}\phi^n(t)<\infty$, for $t>0$).

Definition 3.6. Let f and g be maps from a Menger PM-space $(X,\mathcal{F},T)$ into itself. The maps f and g are said to be weakly commuting if

$$F_{fgx,gfx}(t)\geq F_{fx,gx}(t),$$

for each x in X and $t>0$.

For the remainder of the paper, let Φ be the set of all onto and strictly increasing functions,

$$\phi:[0,\infty)\longrightarrow[0,\infty),$$

which satisfy $\lim_{n\to\infty}\phi^n(t)=0$, for $t>0$. Here, $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$.

Remark 3.7. First notice that if $\phi \in \Phi$, then $\phi(t) < t$, for $t > 0$. To see this, suppose there exists $t_0 > 0$ with $t_0 \leq \phi(t_0)$. Then, since ϕ is nondecreasing, we have $t_0 \leq \phi^n(t_0)$, for each $n \in \{1, 2, ...\}$, which is a contradiction. Note also that $\phi(0) = 0$.

Lemma 3.8. [10] *Suppose a Menger PM-space $(X, \mathcal{F}, T)$ satisfies the following condition:*

$$F_{x,y}(t) = C, \quad \textit{for all } t > 0.$$

Then, we have $C = \varepsilon_0(t)$ and $x = y$.

Theorem 3.9. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space and f and g be weakly commuting self-mappings of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*
(b) *f or g is continuous;*
(c) *$F_{fx,fy}(\phi(t)) \geq F_{gx,gy,}(t)$, where $\phi \in \Phi$.*

(i) If (1.1) holds and there exists $x_0 \in X$ with

$$E_F(gx_0, fx_0) = \sup\{E_{\gamma,F}(gx_0, fx_0) : \gamma \in (0,1)\} < \infty,$$

then f and g have a unique common fixed point.

(ii) If (1.2) holds, then f and g have a unique common fixed point.

Proof. *(i).* Choose $x_0 \in X$ with $E_F(gx_0, fx_0) < \infty$. Choose $x_1 \in X$ with $fx_0 = gx_1$. In general, choose x_{n+1} such that $fx_n = gx_{n+1}$. Now, $F_{fx_n,fx_{n+1}}(\phi^{n+1}(t)) \geq F_{gx_n,gx_{n+1}}(\phi^n(t)) = F_{fx_{n-1},fx_n}(\phi^n(t)) \geq ... \geq F_{gx_0,gx_1}(t)$.

Note that for each $\lambda \in (0,1)$ (see Lemma 1.9. of [10]),

$$\begin{aligned} E_{\lambda,F}(fx_n, fx_{n+1}) &= \inf\{\phi^{n+1}(t) > 0 : F_{fx_n,fx_{n+1}}(\phi^{n+1}(t)) > 1 - \lambda\} \\ &\leq \inf\{\phi^{n+1}(t) > 0 : F_{gx_0,fx_0}(t) > 1 - \lambda\} \\ &\leq \phi^{n+1}(\inf\{t > 0 : F_{gx_0,fx_0}(t) > 1 - \lambda\}) \\ &= \phi^{n+1}(E_{\lambda,F}(gx_0, fx_0)) \\ &\leq \phi^{n+1}(E_F(gx_0, fx_0)). \end{aligned}$$

Thus, $E_{\lambda,F}(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_F(gx_0, fx_0))$, for each $\lambda \in (0,1)$, and so,

$$E_F(fx_n, fx_{n+1}) \leq \phi^{n+1}(E_F(gx_0, fx_0)).$$

Let $\epsilon > 0$. Choose $n \in \{1, 2, ...\}$ so that $E_F(fx_n, fx_{n+1}) < \epsilon - \phi(\epsilon)$. For $\lambda \in (0, 1)$, there exists $\mu \in (0, 1)$ with

$$\begin{aligned} E_{\lambda,F}(fx_n, fx_{n+2}) &\leq E_{\mu,F}(fx_n, fx_{n+1}) + E_{\mu,F}(fx_{n+1}, fx_{n+2}) \\ &\leq E_{\mu,F}(fx_n, fx_{n+1}) + \phi(E_{\mu,F}(fx_n, fx_{n+1})) \\ &\leq E_F(fx_n, fx_{n+1}) + \phi(E_F(fx_n, fx_{n+1})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon - \phi(\epsilon)) \\ &\leq \epsilon. \end{aligned}$$

We can do this argument for each $\lambda \in (0, 1)$, and so,

$$E_F(fx_n, fx_{n+2}) \leq \epsilon.$$

For $\lambda \in (0, 1)$, there exists $\mu \in (0, 1)$ with

$$\begin{aligned} E_{\lambda,F}(fx_n, x_{n+3}) &\leq E_{\mu,F}(fx_n, fx_{n+1}) + E_{\mu,F}(fx_{n+1}, fx_{n+3}) \\ &\leq E_{\mu,F}(fx_n, fx_{n+1}) + \phi(E_{\mu,F}(fx_n, fx_{n+2})) \\ &\leq E_F(fx_n, fx_{n+1}) + \phi(E_F(fx_n, fx_{n+2})) \\ &\leq \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon. \end{aligned}$$

Note here that we used the fact that $F_{fx_{n+1},fx_{n+3}}(\phi(t)) \geq F_{gx_{n+1},gx_{n+3}}(t) = F_{fx_n,fx_{n+2}}(t)$ so $E_{\lambda,F}(fx_{n+1}, fx_{n+3}) \leq \phi(E_{\mu,F}(fx_n, fx_{n+2}))$. Thus,

$$E_F(fx_n, fx_{n+3}) \leq \epsilon.$$

By induction,

$$E_F(fx_n, fx_{n+k}) \leq \epsilon \text{ for } k \in \{1, 2, ...\}.$$

Thus, $\{fx_n\}_n$ is Cauchy and by the completeness of X , $\{fx_n\}_n$ converges to say z in X. Also $\{gx_n\}_n$ converges to z. Let us now suppose that the mapping f is continuous. Then, $\lim_n ffx_n = fz$ and $\lim_n fgx_n = fz$. Furthermore, since f and g are weakly commuting, we have,

$$F_{fgx_n,gfx_n}(t) \geq F_{fx_n,gx_n}(t).$$

Let $n \to \infty$ in the above inequality and get $\lim_n gfx_n = fz$, by continuity of $\mathcal{F}$. We now prove $z = fz$. Suppose $z \neq fz$. By (c), for any $t > 0$, we have,

$$F_{fx_n,ffx_n}(\phi^{k+1}(t)) \geq F_{gx_n,gfx_n}(\phi^k(t)), \quad k \in \mathbb{N}.$$

Let $n \to \infty$ in the above inequality and get

$$F_{z,fz}(\phi^{k+1}(t)) \geq F_{z,fz}\phi^k(t)).$$

Also, we have,

$$F_{z,fz}(\phi^k(t)) \geq F_{z,fz}(\phi^{k-1}(t)),$$

and

$$F_{z,fz}(\phi(t)) \geq F_{z,fz}(t).$$

As a result, we have,

$$F_{z,fz}(\phi^{k+1}(t)) \geq F_{z,fz}(t).$$

On the other hand, we have (see Remark 3.7),

$$F_{z,fz}(\phi^{k+1}(t)) \leq F_{z,fz}(t).$$

Then, $F_{z,fz}(t) = C$, and by Lemma 3.8, $z = fz$. Since $f(X) \subseteq g(X)$, then we can find z_1 in X such that $z = fz = gz_1$. Now,

$$F_{ffx_n,fz_1}(t) \geq F_{gfx_n,gz_1}(\phi^{-1}(t)).$$

Taking the limit as $n \to \infty$, we get,

$$F_{fz,fz_1}(t) \geq F_{fz,gz_1}(\phi^{-1}(t)) = \varepsilon_0(t)$$

which implies $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. Also, for any $t > 0$, since f and g are weakly commuting, we have,

$$F_{fz,gz}(t) = F_{fgz_1,gfz_1}(t) \geq F_{fz_1,gz_1}(t) = \varepsilon_0(t),$$

which again implies $fz = gz$. Thus z, is a common fixed point of f and g.

Now, to prove uniqueness, suppose $z' \neq z$ is another common fixed point of f and g. Then, for any $t > 0$ and $n \in \mathbb{N}$, we have,

$$F_{z,z'}(\phi^{n+1}(t)) \;\; = \;\; F_{fz,fz'}(\phi^{n+1}(t)) \geq F_{gz,gz'}(\phi^n(t)) = F_{z,z'}(\phi^n(t)).$$

Also, we have,

$$F_{z,z'}(\phi^n(t)) \geq F_{z,z'}(\phi^{n-1}(t)),$$

and

$$F_{z,z'}(\phi(t)) \geq F_{z,z'}(t).$$

As a result, we have,

$$F_{z,z'}(\phi^{n+1}(t)) \;\; \geq \;\; F_{z,z'}(t).$$

On the other hand, we have,

$$F_{z,z'}(t) \leq F_{z,z'}(\phi^{n+1}(t)).$$

Then, $F_{z,z'}(t) = C$, and by Lemma 3.8, $z = z'$, which is contradiction. Therefore, z is the unique common fixed point of f and g.

(ii). The argument is as in case (i) except in this case we make use of Remark 1.11 in [10]. □

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