Bulletin of the Iranian Mathematical Society Vol. 37 *No. 3* (*2011*)*, pp 33-37.*

ON RAINBOW 4-TERM ARITHMETIC PROGRESSIONS

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Communicated by Michel Waldschimidt

Abstract. Let $[n] = \{1, \ldots, n\}$ be colored in k colors. A rainbow AP(k) in $[n]$ is a k term arithmetic progression whose elements have different colors. Conlon, Jungić and Radoičić [3] prove that there exists an equinumerous 4-coloring of $[4n]$ which is rainbow AP(4) free, when n is even. Based on their construction, we show that such a coloring of $[4n]$ also exists for odd $n > 1$. We conclude that for nonnegative integers $k \geq 3$ and $n > 1$, every equinumerous k-coloring of $[kn]$ contains a rainbow AP(k) if and only if $k = 3$.

1. Introduction and Results

In his efforts to prove the Fermat's last theorem, Schur [7] proved that for each nonnegative integer k, every k-coloring of $[n] = \{1, \ldots, n\}$ contains a monochromatic solution of the equation $x + y = z$, provided that n is sufficiently large. Alekseev and Savchev [1] turn this problem to rainbow solutions of the equation $x + y = z$; i.e., solutions in which x, y and z are colored in different colors. Later on, in 2003, Jungić et al. [4] considered the rainbow arithmetic progressions arising in k-colorings of $[n]$. Jungić and Radoičić [6] proved the following theorem which is conjectured in [4].

MSC(2000): Primary: 05C55; Secondary: 11B25.
Keywords: Rainbow arithmetic progression, 4-term arithmetic progression, AP(4), AP(k).
Received: 26 Aug 2009, Accepted: 10 January 2010.
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Theorem 1.1. [6, Theorem 1] *For every equinumerous 3-coloring of* $[3n]$*, there exists a rainbow AP(3).*

What about more than 3 colors? Axenovich and Fon-Der-Flass [2] find equinumerous k-coloring of $[2mk]$ which contains no rainbow AP(k), for every $k \geq 5$. The most challenging case is $k = 4$ [5, Problem 1]. Sicherman [5, page 4], makes equinumerous 4-coloring of $[4n]$, for $1 < n \leq 15$, without rainbow AP(4). In 2007, Conlon, et al.[3] made a rainbow free equinumerous 4-coloring of $[4n]$, whenever n is even.

Theorem 1.2. *(*[3, Theorem 2]*) For every positive integer* m*, there exists an equinumerous 4-coloring of* $[8m]$ *with no rainbow AP(4).*

Based on their construction, we prove the following theorem. The proof appears in the next section.

Theorem 1.3. *For every positive integer* m*, there exists an equinumerous 4-coloring of* $[8m + 4]$ *with no rainbow AP(4).*

Hence, we have the following theorem, which in some sense finishes the story of the existence of rainbow AP(k) in equinumerous random k-coloring of $[kn]$, $n > 1$.

Theorem 1.4. *For nonnegative integers* $k \geq 3$ *and* $n > 1$*, every equinumerous* k*-coloring of* $[kn]$ *contains a rainbow AP*(k) *if and only if* $k = 3$*.*

Proof. If $k = 3$, see Theorem 1. For $k = 4$, by theorems 2 and 3, a rainbow AP(4) free 4-coloring of $[4n]$ is at hand for every $n > 1$. To construct 5-coloring of $[5n]$, we use easily a equinumerous 4-coloring of $[4n]$, which has no rainbow AP(4) and then color $\{4n + 1, \dots, 5n\}$ with the fifth color. Plainly, this equinumerous 5-coloring has no rainbow AP(5). One can inductively use this construction to provide equinumerous k-coloring of $[kn]$ for every $k > 5$, $n > 1$, with no rainbow AP(k). □

Note that the construction of equinumerous k-coloring of $[kn]$, $k \geq 5$, by Axenovich and Fan-Der-Flaass [2], is only for n even.

2. **Proof of Theorem 1.3**

The following equinumerous 4-coloring of $[4n]$ in the proof of Theorem 2 in [3] is rainbow AP(4) free, whenever $n = 2m$ is even.

Let W, X, Y and Z be our four colors and denote by $\mathcal{A}$ the block $WXYY$ and by $\mathcal{B}$ the block $ZZWX$. The coloring

$$(*) \qquad \underbrace{\mathcal{A}\ldots\mathcal{A}}_{m \text{ times}}\underbrace{\mathcal{B}\ldots\mathcal{B}}_{m \text{ times}}$$

is the desired coloring of $[4n] = [8m]$.

Our construction for $[4n]$, whenever $n = 2m+1 > 1$, is as follows:

$$(**) \qquad XWY\underbrace{\mathcal{A}\ldots\mathcal{A}}_{m \text{ times}}\underbrace{\mathcal{B}\ldots\mathcal{B}}_{m \text{ times}}Z$$

What remains is to check that this coloring of $[8m+4]$ is rainbow AP(4) free.

To get a contradiction, let $t_1 < t_2 < t_3 < t_4$ denote the terms of a rainbow AP(4) in $(**)$ with common difference d. Obviously, $d > 1$. Since $(*)$ is rainbow AP(4) free, we must have either $t_1 \in \{1, 2, 3\}$ or $t_4 = 8m + 4$ or both. Since the left side (the first $4m + 3$ numbers) of $(**)$ is colored only by W, X and Y, therefore $t_4 > 4m + 3$. Similarly, $t_1 \leq 4m + 3$. Now, five cases occur.

Case 1. $t_1 = 1$ and $t_4 \neq 8m + 4$.

subcase 1a. $t_1 < t_2 \leq 4m + 3 < t_3 < t_4$. If $d \equiv 0 \pmod 4$, then t_1 and t_2 are colored X. If $d \equiv 1 \pmod 4$, then t_1 and t_3 are colored X. If $d \equiv 2 \pmod 4$, then t_1 and t_4 are colored X. If $d \equiv 3 \pmod 4$, then t_1 and t_3 are colored X.

subcase 1b. $t_1 < t_2 < t_3 \leq 4m + 3 < t_4$. If $d \equiv 0 \pmod 4$, then t_1 and t_2 are colored X. If $d \equiv 1 \pmod 4$, then t_2 and t_3 are colored Y. If $d \equiv 2 \pmod 4$, then t_1 and t_3 are colored X. If $d \equiv 3 \pmod 4$, then t_2 and t_4 are colored W.

Case 2. $t_1 = 2$ and $t_4 \neq 8m + 4$.

subcase 2a. $t_1 < t_2 \leq 4m + 3 < t_3 < t_4$. If $d \equiv 0 \pmod 4$, then t_1 and t_3 are colored W. If $d \equiv 1 \pmod 4$, then t_3 and t_4 are colored Z. If $d \equiv 2 \pmod 4$, then t_1 and t_2 are colored W. If $d \equiv 3 \pmod 4$, then t_2 and t_4 are colored X.

subcase 2b. $t_1 < t_2 < t_3 \leq 4m+3 < t_4$. If $d \equiv 0 \pmod 4$, then t_2 and t_3 are colored Y. If $d \equiv 1 \pmod 4$, then t_1 and t_3 are colored W. If $d \equiv 2 \pmod 4$, then t_1 and t_2 are colored W. If $d \equiv 3 \pmod 4$, then t_1 and t_3 are colored W.

Case 3. $t_1 = 3$ and $t_4 \neq 8m+4$.

subcase 3a. $t_1 < t_2 \leq 4m+3 < t_3 < t_4$. If $d \equiv 0 \pmod 4$, then t_1 and t_2 are colored Y. If $d \equiv 1 \pmod 4$, then t_2 and t_4 are colored W. If $d \equiv 2 \pmod 4$, then t_2 and t_3 are colored X. If $d \equiv 3 \pmod 4$, then t_1 and t_2 are colored Y.

subcase 3b. $t_1 < t_2 < t_3 \leq 4m+3 < t_4$. If $d \equiv 0 \pmod 4$, then t_1 and t_2 are colored Y. If $d \equiv 1 \pmod 4$, then t_2 and t_4 are colored W. If $d \equiv 2 \pmod 4$, then t_1 and t_3 are colored Y. If $d \equiv 3 \pmod 4$, then t_1 and t_2 are colored Y.

Case 4. $t_1 > 3$ and $t_4 = 8m+4$.

subcase 4a. $t_1 < t_2 \leq 4m+3 < t_3 < t_4$. If $d \equiv 0 \pmod 4$, then t_3 and t_4 are colored Z. If $d \equiv 1 \pmod 4$, then t_1 and t_3 are colored X. If $d \equiv 2 \pmod 4$, then t_2 and t_3 are colored W. If $d \equiv 3 \pmod 4$, then t_3 and t_4 are colored Z.

subcase 4b. $t_1 \leq 4m+3 < t_2 < t_3 < t_4$. If $d \equiv 0 \pmod 4$, then t_3 and t_4 are colored Z. If $d \equiv 1 \pmod 4$, then t_1 and t_3 are colored X. If $d \equiv 2 \pmod 4$, then t_2 and t_4 are colored Z. If $d \equiv 3 \pmod 4$, then t_3 and t_4 are colored Z.

Case 5. $t_1 \in \{1, 2, 3\}$ and $t_4 = 8m+4$. In this case, since $t_4 \equiv 0 \pmod 4$ and $t_4 - t_1 = 3d$, it follows that $d \equiv t_1 \pmod 4$. Also, $t_2 < 4m+3 < t_3$ and t_4 is colored Z.

subcase 5a. $1 = t_1 < t_2 < 4m+3 < t_3 < t_4 = 8m+4$. Here, by our construction, t_1 and t_3 are colored X because $d \equiv t_1 \equiv 1 \pmod 4$.

subcase 5b. $2 = t_1 < t_2 < 4m+3 < t_3 < t_4 = 8m+4$. In this subcase, we have $d \equiv t_1 \equiv 2 \pmod 4$. Therefore, t_1 and t_2 are colored W.

subcase 5c. $3 = t_1 < t_2 < 4m+3 < t_3 < t_4 = 8m+4$. In this subcase, since $d \equiv t_1 \equiv 3 \pmod 4$, t_1 and t_2 are colored Y. $\square$

Acknowledgments

The authors thank the referee for constructive comments and shortening the proof.

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