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GENERAL HARDY-TYPE INEQUALITIES WITH NON-CONJUGATE EXPONENTS

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Abstract. We derive whole series of new integral inequalities of the Hardy-type, with non-conjugate exponents. First, we prove and discuss two equivalent general inequalities of such type, as well as their corresponding reverse inequalities. General results are then applied to special Hardy-type kernel and power weights. Also, some estimates of weight functions and constant factors are obtained. In particular, we obtain generalizations and improvements of some recent results, in the literature.

1. **Introduction**

Let p and q be real parameters satisfying

$$p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} \geq 1, \tag{1.1}$$

and let $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$, respectively, be their conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'}, \tag{1.2}$$

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and observe that $0 < \lambda \leq 1$ holds for all p and q as in (1.1). In particular, equality $\lambda = 1$ holds in (1.2) if and only if $q = p'$, that is, only if p and q are mutually conjugate. Otherwise, we have $0 < \lambda < 1$, and such parameters p and q will be referred to as non-conjugate exponents.

These exponents, in conjugate or non-conjugate form, appear in numerous classical inequalities. One of them is the Hardy inequality. In 1925, G. H. Hardy stated and proved in [5] the following integral inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \|f\|_{L^p\langle 0,\infty\rangle}, \tag{1.3}$$

where, $p > 1$, and $f \in L^p\langle 0, \infty\rangle$ is a non-negative function. This is the original form of the Hardy integral inequality, which later on has been extensively studied and used as a model example for investigations of more general integral inequalities.

During subsequent decades, the Hardy inequality was generalized in several different ways. Roughly speaking, the Hardy inequality was extended to what we call nowadays the general Hardy inequality, or the Hardy-type inequality:

$$\left[\int_a^b \left(\int_a^x f(y)\, dy \right)^{q'} u(x)\, dx \right]^{\frac{1}{q'}} \leq C_{p,q'} \left(\int_a^b f^p(x) v(x)\, dx \right)^{\frac{1}{p}},\ f \geq 0, \tag{1.4}$$

with parameters a, b, p, q', such that $-\infty \leq a < b \leq \infty$, $0 < q' \leq \infty$, $1 \leq p \leq \infty$, and with u and v as the given weight functions. The main problem in connection with the Hardy inequality is to determine conditions on the parameters p and q' and on the weight functions u and v under which the inequality holds for some classes of functions.

The Hardy inequality also plays an important role in various fields of mathematics, specially in functional and spectral analysis, where one investigates properties of the Hardy operator, like continuity and compactness, and also behavior in more general function spaces. For more details about the Hardy inequality, its history and related results, see [**?**, 10, 11] and [12].

Although classical, the Hardy inequality is still a field of interest for numerous mathematicians. In papers [2] and [3], Čizmešija and Pečarić investigated finite sections of the Hardy inequality, i.e., inequalities of the same type, where the integrals are taken over certain subsets of $\langle 0, \infty\rangle$. In such a way, they obtained some generalizations and refinements of inequality (1.3). For example, in [3], they proved the inequality

$$\int_0^b x^{-k}\left(\int_0^x f(y)\,dy\right)^p dx \le \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)\,dx, \tag{1.5}$$

where, $0<b<\infty$, $1<p,k<\infty$, $f\ge 0$, and $x^{1-\frac{k}{p}} f \in L^p\langle 0,b\rangle$.

The Hardy inequality is closely related to another important classical inequality, the so-called Hilbert inequality. That relation may be explained in a more general setting. Namely, in a recent work [7], Krnić and Pečarić provided an unified treatment of the Hardy and Hilbert inequalities in general form and extended them to cover the case when p and p' are conjugate exponents. More precisely, they obtained the following two inequalities:

$$\begin{aligned}\int_a^b \int_a^y (hg)(y) f(x)\,d\mu_1(x)d\mu_2(y) &\le \left[\int_a^b \varphi^p(x)\left(\int_x^b H(y)\,d\mu_2(y)\right) f^p(x)\,d\mu_1(x)\right]^{\frac{1}{p}} \cdot \\ &\quad \left[\int_a^b (\psi^{p'} h)(y)\left(\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right) g^{p'}(y)\,d\mu_2(y)\right]^{\frac{1}{p'}}\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}\int_a^b H(y)\left(\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right)^{1-p}\left(\int_a^y f(x)\,d\mu_1(x)\right)^p d\mu_2(y) \\ \le \int_a^b (\psi^{p'} h)(y)\left(\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right) g^{p'}(y)\,d\mu_2(y),\end{aligned} \tag{1.7}$$

where, $p>1$, μ_1,μ_2 are positive σ-finite measures, h,f,g,φ,ψ are measurable, positive functions a.e. on $\langle a,b\rangle$ and $H=h\psi^{-p}$. Inequalities (1.6) and (1.7) are equivalent in the sense that one is a consequence of another. Also, the same authors obtained inequalities (1.6) and (1.7) in a more general manner, with arbitrary measurable kernel instead of function h (see [7], Theorem 1). Such inequalities are generalizations of the classical Hilbert and Hardy inequalities. So, inequalities deduced from (1.6) will be referred to as the Hilbert-type inequalities and inequalities deduced from (1.7), as the Hardy-type inequalities. Hence, inequality (1.7) generalizes the inequality (1.5), as well as some other results in [2] and [3] and numerous other results, in the literature.

Very recently, Čižmešija et al. [4] developed a unified treatment of the general Hilbert-type inequalities extended to the case of non-conjugate exponents. Our aim here is also an extension of the general Hardy-type inequalities to the case of non-conjugate exponents. Similarly, as in [4], we use one Bonsall's idea, developed in [1], about reducing the case of non-conjugate parameters to the case of conjugate exponents. In particular, we obtain a series of integral Hardy-type inequalities for some special Hardy-type kernels. All results are given in two equivalent forms, analogous to (1.6) and (1.7).

The remainder of our work is organized in the following way: in Section 2 we state and prove a pair of new equivalent Hilbert and Hardy-type inequalities with non-conjugate exponents p and q, related to general measure spaces with positive σ-finite measures, and to the Hardy-type kernels. These relations are also discussed with respect to parameters p and q, in order to obtain the corresponding reverse inequalities. We also give necessary and sufficient conditions for equality in the inequalities obtained. In Section 3, we discuss duality in the Hardy-type inequalities. Moreover, we obtain dual analogues of the results in Section 2. Also, we prove the equivalence of these results in the case of the Lebesgue measure. Further, in Section 4, we apply our general results to the special Hardy-type kernels and power weights with integrals taken over intervals in $\mathbb{R}_+$. We obtain a whole series of inequalities with explicit constant factors on their right-hand sides. In Section 5, we further estimate the inequalities from the previous section. More precisely, we estimate some factors included in integrals of the inequalities from the previous section, depending on non-conjugate parameters and the exponents of power weights. Section 6 is dedicated to some uniform bounds for constant factors in the Hardy-type inequalities. We perform a detailed analysis for optimal constant factor, depending on non-conjugate factors p and q which give the Hardy-type or the reversed Hardy-type inequality. Finally, in the last section, we synthesize the methods developed in Sections 4, 5, and 6, and consider some special cases. In that manner, we obtain both generalizations and refinements of some known results.

Conventions. Throughout, let r' be the conjugate exponent to a positive real number $r \neq 1$, that is, $\frac{1}{r} + \frac{1}{r'} = 1$, or $r' = \frac{r}{r-1}$. All measures are assumed to be positive and σ-finite, and functions to be a non-negative and measurable. A weight function is assumed to be a non-negative measurable function. Further, $L^p(\mu)$ denotes measure space $L^p\left(\langle a,b\rangle;\mu\right)$ and $\|f\|_{L^p(\mu)} = \left(\int_a^b f^p(x)\,d\mu(x)\right)^{\frac{1}{p}}$ is the corresponding norm. Moreover, in the case of the Lebesgue measure, $L^p(\mu)$ will be simply denoted as L^p. Expressions of the form $0\cdot\infty$, $\frac{0}{\infty}$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero.

2. General Inequalities of the Hardy-Type

Here we prove our main result that extends relations (1.6) and (1.7) to the case of non-conjugate exponents.

Let $\langle a,b\rangle$ be an interval in $\mathbb{R}$, let $T=\{(x,y)\in\mathbb{R}^2 : a<x\le y<b\}$, and let μ_1 and μ_2 be positive $\sigma-$finite measures on $\langle a,b\rangle$. We define the Hardy-type kernel $K:\langle a,b\rangle\times\langle a,b\rangle\to\mathbb{R}$ as:

$$K(x,y)=h(y)\chi_T(x,y)=\begin{cases} h(y) & ,\ x\le y,\\ 0 & ,\ x>y,\end{cases} \tag{2.1}$$

where, h is a measurable a.e. positive function on $\langle a,b\rangle$. Further, we define the functions $F:\langle a,b\rangle\to\mathbb{R}$ and $G:\langle a,b\rangle\to\mathbb{R}$ as:

$$\begin{aligned} F(x) &= \left[\int_x^b h(y)\psi^{-q'}(y)\,d\mu_2(y)\right]^{\frac{1}{q'}}, \quad x\in\langle a,b\rangle,\\ G(y) &= \left[h(y)\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}}, \quad y\in\langle a,b\rangle, \end{aligned} \tag{2.2}$$

where, ψ and φ are measurable a.e. positive functions on $\langle a,b\rangle$.

We also introduce the related Hardy-type operator by the formula

$$\left(Hf\right)(y)=\int_a^y f(x)\,d\mu_1(x), \qquad y\in\langle a,b\rangle. \tag{2.3}$$

Now, we are ready to state and prove our main result.

Theorem 2.1. *Let real parameters p, q, and λ be as in (1.1) and (1.2), and let μ_1 and μ_2 be σ-finite measures on $\langle a,b\rangle$, $-\infty\le a<b\le\infty$. Let h, φ, ψ be measurable, a.e. positive functions on $\langle a,b\rangle$ and let H be the operator defined by (2.3). If the functions F and G are defined by (2.2), then the inequalities*

$$\int_a^b (h^\lambda g)(y)\left(Hf\right)(y)\,d\mu_2(y)\le\|\varphi F f\|_{L^p(\mu_1)}\|\psi G g\|_{L^q(\mu_2)} \tag{2.4}$$

and

$$\left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} (Hf)^{q'}(y)\, d\mu_2(y) \right\}^{\frac{1}{q'}} \tag{2.5}$$
$$\leq \|\varphi F f\|_{L^p(\mu_1)}$$

hold for all non-negative functions f and g on $\langle a, b\rangle$, such that $\varphi F f \in L^p(\mu_1)$ and $\psi G g \in L^q(\mu_2)$, and are equivalent.

Proof. We prove the inequality (2.4) first. Let h, φ, and ψ be as given in the statement of the theorem and let f and g be arbitrary non-negative measurable functions on $\langle a, b\rangle$. Let L denote the left-hand side of inequality (2.4). Since $\frac{1}{q'} + \frac{1}{p'} + (1-\lambda) = 1$, the left-hand side of (2.4) can be rewritten as follows:

$$\begin{aligned} L &= \int_a^b \int_a^y (h^\lambda g)(y) f(x)\, d\mu_1(x) d\mu_2(y) \\ &= \int_a^b \int_a^y \left[(h\psi^{-q'})(y) (\varphi^p F^{p-q'} f^p)(x) \right]^{\frac{1}{q'}} \cdot \\ &\qquad \left[\varphi^{-p'}(x) (h\psi^q G^{q-p'} g^q)(y) \right]^{\frac{1}{p'}} \cdot \\ &\qquad \left[(\varphi F f)^p (x) \, (\psi G g)^q (y) \right]^{1-\lambda} d\mu_1(x) d\mu_2(y). \end{aligned} \tag{2.6}$$

Further, by using the Hölder inequality, either with the parameters $q', p', \frac{1}{1-\lambda} > 1$ in the case of non-conjugate exponents p and q, or with the parameters p and p' when $q' = p$ (that is, when $\lambda = 1$), and then applying the Fubini theorem, we obtain that L is not greater than

$$\begin{aligned} R = &\left\{ \int_a^b \left[\int_x^b (h\psi^{-q'})(y)\, d\mu_2(y) \right] (\varphi^p F^{p-q'} f^p)(x)\, d\mu_1(x) \right\}^{\frac{1}{q'}} \cdot \\ &\left\{ \int_a^b \left[h(y) \int_a^y \varphi^{-p'}(x)\, d\mu_1(x) \right] (\psi^q G^{q-p'} g^q)(y)\, d\mu_2(y) \right\}^{\frac{1}{p'}} \cdot \\ &\left\{ \int_a^b (\psi G g)^q (y) \int_a^y (\varphi F f)^p (x)\, d\mu_1(x) d\mu_2(y) \right\}^{1-\lambda} . \end{aligned}$$

Now, by considering definitions (2.2) of functions F and G, the above expression can be rewritten as

$$R = \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \|\psi G g\|_{L^q(\mu_2)}^{\frac{q}{p'}} \cdot \left\{ \|\varphi F f\|_{L^p(\mu_1)}^{p} \|\psi G g\|_{L^q(\mu_2)}^{q} - \right.$$
$$\left. - \int_a^b \int_y^b (\varphi F f)^p(x) (\psi G g)^q(y)\, d\mu_1(x) d\mu_2(y) \right\}^{1-\lambda}. \tag{2.7}$$

Of course, from relations (2.6) and (2.7) we obtain the inequality

$$L \leq R. \tag{2.8}$$

Finally, considering (2.7), we easily have

$$R \leq \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}+p(1-\lambda)} \|\psi G g\|_{L^q(\mu_2)}^{\frac{q}{p'}+q(1-\lambda)} = \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)},$$

and so (2.4) is proved. The further step is to prove that (2.4) implies (2.5) to hold for all non-negative measurable functions f on $\langle a, b\rangle$. In particular, for any such f and the function g defined by

$$g(y) = (\psi^{-q'} h^{1-\lambda})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} (Hf)^{q'-1}(y), \quad y \in \langle a, b\rangle, \tag{2.9}$$

applying the Fubini theorem, the left-hand side of (2.4) becomes

$$L_f = \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} (Hf)^{q'}(y)\, d\mu_2(y),$$

that is, we get the integral on the left-hand side of (2.5), while on the right-hand side of (2.4) we have

$$R_f = \|\varphi F f\|_{L^p(\mu_1)} \left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} \cdot \right.$$
$$\left. (Hf)^{q'}(y)\, d\mu_2(y) \right\}^{\frac{1}{q}} = \|\varphi F f\|_{L^p(\mu_1)} L_f^{\frac{1}{q}}.$$

Hence,

$$L_f \leq \|\varphi F f\|_{L^p(\mu_1)} L_f^{\frac{1}{q}},$$

which directly yields (2.5), and so the implication (2.4) $\Rightarrow$ (2.5) is proved. Conversely, by using the Hölder inequality for the conjugate exponents q and q', together with relation (2.5) and definitions (2.2), for arbitrary $f, g \geq 0$, we have

$$
\begin{aligned}
&\int_a^b (h^\lambda g)(y)(Hf)(y)\,d\mu_2(y) \\
&\quad = \int_a^b (\psi G g)(y)\left[(\psi G)^{-1}(y) h^\lambda(y)(Hf)(y)\right] d\mu_2(y) \\
&\quad \leq \|\psi G g\|_{L^q(\mu_2)} \left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} \cdot \right. \\
&\qquad \left. (Hf)^{q'}(y)\,d\mu_2(y) \right\}^{\frac{1}{q'}} \\
&\quad \leq \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)}.
\end{aligned}
$$

Thus, (2.5) implies (2.4), so these inequalities are equivalent. The proof of Theorem 2.1 is now complete.

Remark 2.2. *Note that equality in (2.4) holds if and only if it holds in the Hölder inequality and in (2.8). Equality in the Hölder inequality holds if and only if the functions $h\psi^{-q'}\varphi^p F^{p-q'} f^p$, $h\varphi^{-p'}\psi^q G^{q-p'} g^q$, and $(\varphi F f)^p (\psi G g)^q$ are effectively proportional on the set T. Clearly, this trivially happens if at least one of the functions involved in the left-hand side of (2.4) is a zero-function. To discuss other non-trivial cases of equality in (2.4), we can, without loss of generality, assume that the functions f and g are positive. Otherwise, instead of T, we consider the set $S = \{(x,y) \in T : f(x)g(y) > 0\}$, which has a positive measure. Under such assumptions, equality in (2.4) occurs if and only if there exist positive real constants α_1, β_1, and γ_1, such that the relations*

$$
\begin{aligned}
\alpha_1 (h\psi^{-q'})(y)(\varphi^p F^{p-q'} f^p)(x) &= \beta_1 \varphi^{-p'}(x)(h\psi^q G^{q-p'} g^q)(y) \\
&= \gamma_1 (\varphi F f)^p(x)\,(\psi G g)^q(y)
\end{aligned}
$$

hold for a.e. $a < x \leq y < b$. Further, these relations can be rewritten in a more suitable, equivalent form, as:

(2.10)

$$\alpha_1(\varphi^{p+p'} F^{p-q'} f^p)(x) = \beta_1(\psi^{q+q'} G^{q-p'} g^q)(y), \quad \textit{for a.e. } a < x \leq y < b,$$

and

$$\alpha_1 F^{-q'}(x) = \gamma_1 (h^{-1}\psi^{q+q'} G^q g^q)(y), \quad \text{for a.e. } a < x \le y < b. \tag{2.11}$$

Since the left-hand side of (2.10) depends only on $x \in \langle a, b\rangle$, while the right-hand side of this relation is a single-variable function in $y \in \langle a, b\rangle$, (2.10) holds only if

$$\varphi^{p+p'} F^{p-q'} f^p = \alpha^p = const. \quad a.e. \text{ on } \langle a, b\rangle$$

and

$$\psi^{q+q'} G^{q-p'} g^q = \beta^p = const. \quad a.e. \text{ on } \langle a, b\rangle,$$

for some positive real constants α and β. Considering $1 + \frac{p'}{p} = p'$ and $1 + \frac{q'}{q} = q'$, these identities can be finally transformed to:

$$f = \alpha\varphi^{-p'} F^{\frac{q'}{p}-1} \quad \text{and} \quad g = \beta\psi^{-q'} G^{\frac{p'}{q}-1} \quad a.e. \text{ on } \langle a, b\rangle. \tag{2.12}$$

By using the same argument, we obtain from (2.11) that $F = const.$ Further, by inserting (2.12) in (2.11), we also get that $h^{-1}G^{p'} = const.$ Hence, considering definitions (2.2), we arrive at a contradiction since the functions h, φ, and ψ are positive a.e. on $\langle a, b\rangle$. So, equality in the Hölder inequality holds only if $f = 0$ or $g = 0$ a.e. on $\langle a, b\rangle$. Clearly, in that case, equality holds in (2.8) and consequently in (2.4). Moreover, it is clear from the proof of Theorem 2.1 that equality in (2.5) holds only if it holds in (2.4), that is, if $f = 0$ a.e. on $\langle a, b\rangle$. □

Remark 2.3. *Note that the sign of inequality in (2.4) depends only on the parameters p', q', and λ, since the crucial step in proving this relation was an application of the Hölder inequality. Therefore, besides $p', q' > 1$ and $\lambda \in \langle 0, 1]$, as in (1.1) and (1.2), we can consider exponents which provide the reversed sign of inequality in (2.4). Specially, if the parameters p and q from Theorem 2.1 are such that*

$$p < 0, \ q \in \langle 0, 1\rangle, \ \frac{1}{p} + \frac{1}{q} \le 1, \tag{2.13}$$

and λ is defined by (1.2), we have $p' \in \langle 0, 1\rangle$, $q' < 0$, and $1-\lambda \le 0$, and so the sign of inequality in (2.4) is reversed as a direct consequence of the so-called reversed Hölder inequality (for details, see e.g., [12, Chapter V]*). The same result is also obtained with the parameters p and q satisfying*

$$p \in \langle 0, 1\rangle, \ q < 0, \ \frac{1}{p} + \frac{1}{q} \le 1, \tag{2.14}$$

since from (2.14) *one obtains* $p' < 0$, $q' \in \langle 0, 1\rangle$, *and* $1 - \lambda \leq 0$. *Further, it is obvious that for all cases of the parameters* p', q', *and* λ, *the relations* (2.4) *and* (2.5) *hold with the same sign of inequality since they are equivalent.* $\square$

Remark 2.4. *In the case of conjugate exponents, that is, when* $q = p'$ *and* $\lambda = 1$, *inequalities* (2.4) *and* (2.5) *reduce to relations* (1.6) *and* (1.7) *from the Introduction. Thus, our Theorem* 2.1 *may be regarded as an extension of the corresponding results from* [7] *to the case of non-conjugate exponents. Clearly, reversed inequalities in* (1.6) *and* (1.7) *hold if* $0 \neq p < 1$. $\square$

Remark 2.5. *In the proof of Theorem* 2.1, *we have obtained the inequality* (2.8), *which is a refinement of inequality* (2.4). *Let us write that inequality once again:*

$$\int_a^b (h^\lambda g)(y)\,(Hf)\,(y)\,d\mu_2(y) \leq \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \|\psi G g\|_{L^q(\mu_2)}^{\frac{q}{p'}} \cdot$$

$$\left\{\int_a^b \int_a^y (\varphi F f)^p\,(x)\,(\psi G g)^q\,(y)\,d\mu_1(x) d\mu_2(y)\right\}^{1-\lambda}. \tag{2.15}$$

Clearly, by substituting the function g, *defined by* (2.9), *in the preceding inequality, we obtain its equivalent Hardy-type form*

$$\left\{\int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} (Hf)^{q'}\,(y)\,d\mu_2(y)\right\}^{\frac{1}{p}}$$

$$\leq \|\varphi F f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \cdot \left\{\int_a^b (h\psi^{-q'})(y) \left[\int_a^y \varphi^{-p'}(x)\,d\mu_1(x)\right]^{-\frac{q'}{p'}} \cdot\right.$$

$$\left.(Hf)^{q'}\,(y) \int_a^y (\varphi F f)^p\,d\mu_1(x) d\mu_2(y)\right\}^{1-\lambda}. \tag{2.16}$$

Inequality (2.16) *is also a slight refinement of* (2.5). *Note that these refinements hold only in the non-conjugate case. Further, the reversed sign of inequality in* (2.15) *and* (2.16) *holds as in Remark 2 (conditions* (2.13) *or* (2.14)*) and also in the case when* $p, q \in \langle 0, 1\rangle$. $\square$

3. General Inequalities with Dual Hardy-Type Kernel

One important property of the Hardy inequality is duality. Here, we obtain dual analogues of the relations in the previous section.

Let $\langle a,b\rangle$ be an interval in $\mathbb{R}$, let $\widetilde{T}=\{(x,y)\in\mathbb{R}^2 : a<y\leq x<b\}$, and let μ_1 and μ_2 be positive $\sigma-$finite measures on $\langle a,b\rangle$. We define the dual Hardy-type kernel $\widetilde{K}:\langle a,b\rangle\times\langle a,b\rangle\to\mathbb{R}$ as:

$$K(x,y)=h(y)\chi_{\widetilde{T}}(x,y)=\begin{cases} h(y), & x\geq y,\\ 0, & x<y,\end{cases} \tag{3.1}$$

where, h is a measurable a.e. positive function on $\langle a,b\rangle$. Moreover, we define the functions $\widetilde{F}:\langle a,b\rangle\to\mathbb{R}$ and $\widetilde{G}:\langle a,b\rangle\to\mathbb{R}$ as:

$$\begin{aligned} \widetilde{F}(x) &= \left[\int_a^x (h\psi^{-q'})(y)\,d\mu_2(y)\right]^{\frac{1}{q'}}, \quad x\in\langle a,b\rangle,\\ \widetilde{G}(y) &= \left[h(y)\int_y^b \varphi^{-p'}(x)\,d\mu_1(x)\right]^{\frac{1}{p'}}, \quad y\in\langle a,b\rangle, \end{aligned} \tag{3.2}$$

where, ψ and φ are measurable a.e. positive functions on $\langle a,b\rangle$ with respect to the corresponding σ-finite measures.

Further, the dual Hardy-type operator to the operator H in (2.3) is defined in the following way:

$$(\widetilde{H}f)(y)=\int_y^b f(x)\,d\mu_1(x), \qquad y\in\langle a,b\rangle. \tag{3.3}$$

In this setting, we obtain a dual analogue of Theorem 2.1.

Theorem 3.1. *Let real parameters p, q, and λ be as in (1.1) and (1.2), and let μ_1 and μ_2 be σ-finite measures on $\langle a,b\rangle$, $-\infty\leq a<b\leq\infty$. Let h, φ, ψ be measurable a.e. positive functions on $\langle a,b\rangle$ and let $\widetilde{H}$ be the operator defined by (3.3). If the functions $\widetilde{F}$ and $\widetilde{G}$ are defined by (3.2), then the inequalities*

$$\int_a^b (h^\lambda g)(y)(\widetilde{H}f)(y)\,d\mu_2(y)\leq \|\varphi\widetilde{F}f\|_{L^p(\mu_1)}\|\psi\widetilde{G}g\|_{L^q(\mu_2)} \tag{3.4}$$

and

$$\left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_y^b \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} (\widetilde{H}f)^{q'}(y)\, d\mu_2(y) \right\}^{\frac{1}{q'}} \tag{3.5}$$
$$\leq \|\varphi \widetilde{F} f\|_{L^p(\mu_1)}$$

hold for all non-negative functions f and g on $\langle a, b\rangle$, such that $\varphi\widetilde{F}f \in L^p(\mu_1)$ and $\psi\widetilde{G}g \in L^q(\mu_2)$, and are equivalent.

Proof. Follows the same lines as the proof of Theorem 2.1. □

Remark 3.2. *Similarly as in Remark 2.2, one obtains that equality in (3.4) holds if and only if $f = 0$ or $g = 0$ a.e. on $\langle a, b\rangle$. Equality in (3.5) holds if and only if $f = 0$ a.e. on $\langle a, b\rangle$. Finally, the reversed inequalities of (3.4) and (3.5) hold under the same assumptions as in Remark 2.3.*□

Remark 3.3. *In the case of conjugate exponents, that is, when $q = p'$ and $\lambda = 1$, Theorem 3.1 reduces to Theorem in [7], and so our Theorem 3.1 can be regarded as an extension of the mentioned results to non-conjugate exponents.* □

Remark 3.4. *Similarly as in Remark 2.5, one easily obtains a refinement of (3.4) in the non-conjugate case:*

$$\int_a^b (h^\lambda g)(y)(\widetilde{H}f)(y)\, d\mu_2(y) \leq \|\varphi\widetilde{F}f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \|\psi\widetilde{G}g\|_{L^q(\mu_2)}^{\frac{q}{p'}} \cdot$$
$$\left\{ \int_a^b \int_y^b (\varphi\widetilde{F}f)^p(x)(\psi\widetilde{G}g)^q(y)\, d\mu_1(x) d\mu_2(y) \right\}^{1-\lambda}, \tag{3.6}$$

with sharp inequality for $f, g \neq 0$ a.e. on $\langle a, b\rangle$. Furthermore, by substituting the function g, defined by

$$g(y) = (\psi^{-q'} h^{1-\lambda})(y) \left[\int_y^b \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} (\widetilde{H}f)^{q'-1}(y), \quad y \in \langle a, b\rangle,$$

in (3.6)*, we obtain the inequality*

$$\left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_y^b \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} (\widetilde{H}f)^{q'}(y)\, d\mu_2(y) \right\}^{\frac{1}{p}}$$

$$\leq \|\varphi \widetilde{F} f\|_{L^p(\mu_1)}^{\frac{p}{q'}} \left\{ \int_a^b (h\psi^{-q'})(y) \left[\int_y^b \varphi^{-p'}(x)\, d\mu_1(x) \right]^{-\frac{q'}{p'}} \cdot \right.$$

$$\left. (\widetilde{H}f)^{q'}(y) \int_y^b (\varphi \widetilde{F} f)^p\, d\mu_1(x) d\mu_2(y) \right\}^{1-\lambda}, \tag{3.7}$$

which can be regarded as a refinement of (3.5)*. Of course, the reversed sign of inequality in* (3.6) *and* (3.7) *holds as in Remark* 2.5*.* ☐

The most interesting case in our work appears when the interval $\langle a, b\rangle$ is a subset of the set of non-negative real numbers, that is, $0 \leq a < b \leq \infty$. Namely, we show that Theorems 2.1 and 3.1 are equivalent in the case of Lebesgue measures.

Theorem 3.5. *Let* $0 \leq a < b \leq \infty$ *and under conditions of Theorem* 2.1 *and Theorem* 3.1*, let* $d\mu_1(x) = dx$*,* $d\mu_2(y) = dy$*. Then, the inequalities* (2.4) *and* (3.4) *are equivalent. Moreover, the inequalities* (2.5) *and* (3.5) *are equivalent as well.*

Proof. Suppose that the inequality (2.4) holds for an arbitrary interval $\langle a, b\rangle \subseteq \mathbb{R}_+$ and arbitrary non-negative measurable functions φ, ψ, h, f, g on $\langle a, b\rangle$. We define $\widetilde{a} = \frac{1}{b}$ and $\widetilde{b} = \frac{1}{a}$, with conventions $\widetilde{a} = 0$ for $b = \infty$ and $\widetilde{b} = \infty$ for $a = 0$. We also define the functions $\widetilde{h}, \widetilde{\varphi}, \widetilde{\psi}, \widetilde{f}$, and $\widetilde{g}$ on $\langle \widetilde{a}, \widetilde{b}\rangle$ by $\widetilde{h}(t) = h\left(\frac{1}{t}\right)$, $\widetilde{\varphi}(t) = t^{\frac{2}{p'}}\varphi\left(\frac{1}{t}\right)$, $\widetilde{\psi}(t) = t^{\frac{2}{q'}}\psi\left(\frac{1}{t}\right)$, $\widetilde{f}(t) = t^{-2}f\left(\frac{1}{t}\right)$, and $\widetilde{g}(t) = t^{-2}g\left(\frac{1}{t}\right)$.

Rewrite (2.4) with these new parameters. More precisely, by using the substitutions $x = \frac{1}{u}$ and $y = \frac{1}{v}$, the left-hand side of (2.4) becomes:

$$\int_{\widetilde{a}}^{\widetilde{b}} (\widetilde{h}^{\lambda}\widetilde{g})(y) t(H\widetilde{f})(y)\, dy = \int_a^b (h^{\lambda}g)(v) \int_v^b f(u)\, du\, dv,$$

that is, the left hand side of inequality (3.4). Analogously, for the first factor on the right-hand side of (2.4), we have

$$\|\widetilde{\varphi} F\widetilde{f}\|_{L^p(\widetilde{a}, \widetilde{b})}^p = \int_a^b (\varphi f)^p(u) \left[\int_a^u (h\psi^{-q'})(v) dv \right]^{\frac{p}{q'}} du,$$

which obviously represents the first factor on the right-hand side of inequality (3.4). The same argument holds for the second factor on the right-hand side of (2.4). Thus, inequality (2.4) implies (3.4). In the same manner, one obtains the reverse implication, and so inequalities (2.4) and (3.4) are equivalent.

Finally, the pairs of inequalities (2.4) and (2.5) as well as (3.4) and (3.5) are equivalent (see Theorems 2.1 and 3.1), which implies the equivalence of (2.5) and (3.5). □

4. **Some Special Hardy-Type Kernels and Weight Functions**

Here, we consider the case with Lebesgue measure for some particular Hardy-type kernels and weight functions. Namely, let $0 \le a < b \le \infty$ and $h, \varphi, \psi : \langle a, b\rangle \to \mathbb{R}$ be, respectively, defined by $h(y) = \frac{1}{y}$, $\varphi(x) = x^{A_1}$, $\psi(y) = y^{A_2}$, $A_1, A_2 \in \mathbb{R}$. As shown in the previous section, it is sufficient to consider only the Hardy-type inequalities in Theorem 2.1, since their duals are equivalent with them.

In particular, we have to distinguish the cases

$$0 < a < b < \infty, \tag{4.1}$$

$$0 = a < b < \infty, \tag{4.2}$$

$$0 < a < b = \infty, \tag{4.3}$$

$$0 = a < b = \infty, \tag{4.4}$$

since one obtains different integration formulas for the functions F and G, defined by (2.2).

Specially, if $0 < a < b < \infty$, then

$$F(x) = \begin{cases} |q'A_2|^{-\frac{1}{q'}} x^{-A_2} \left|1 - \left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{1}{q'}}, & A_2 \neq 0, \\ \left(\ln \frac{b}{x}\right)^{\frac{1}{q'}}, & A_2 = 0, \end{cases} \tag{4.5}$$

and

$$G(y) = \begin{cases} |1 - p'A_1|^{-\frac{1}{p'}} y^{-A_1} \left|1 - \left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{1}{p'}}, & A_1 \neq \frac{1}{p'}, \\ y^{-\frac{1}{p'}} \left(\ln \frac{y}{a}\right)^{\frac{1}{p'}}, & A_1 = \frac{1}{p'}. \end{cases} \tag{4.6}$$

Note that we have, in previous two relations, included the cases $A_1 > \frac{1}{p'}$ and $A_2 < 0$, by means of the modulus function. In this setting, we obtain four corollaries arising from Theorem 2.1.

If $A_1 \neq \frac{1}{p'}$ and $A_2 \neq 0$, then we have the following result.

Corollary 4.1. *Let $-\infty < a < b < \infty$, let real parameters p, q, and λ be as in (1.1) and (1.2), let A_1, A_2 be real parameters such that $A_1 \neq \frac{1}{p'}$, $A_2 \neq 0$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \leq \frac{|1-p'A_1|^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}} \left[\int_a^b x^{(A_1-A_2)p} \left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b y^{(A_2-A_1)q} \left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.7}$$

and

$$\left[\int_a^b y^{(A_1-A_2-\lambda)q'} \left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{-\frac{q'}{p'}} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq \frac{|1-p'A_1|^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}} \left[\int_a^b x^{(A_1-A_2)p} \left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.8}$$

hold for all non-negative measurable functions f and g on $\langle a, b\rangle$, and are equivalent.

In the case $A_1 \neq \frac{1}{p'}$, $A_2 = 0$ we have the following result.

Corollary 4.2. *Let $-\infty < a < b < \infty$, let real parameters p, q, and λ be as in (1.1) and (1.2), let A_1 be real parameter such that $A_1 \neq \frac{1}{p'}$, and let H be the operator defined by (2.3). Then the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \leq |1-p'A_1|^{-\frac{1}{p'}} \left[\int_a^b x^{A_1 p} \left(\ln\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b y^{-A_1 q} \left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.9}$$

and

$$\left[\int_a^b y^{(A_1-\lambda)q'}\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq |1-p'A_1|^{-\frac{1}{p'}}\left[\int_a^b x^{A_1 p}\left(\ln\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.10}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$, and are equivalent.

If $A_1=\frac{1}{p'}$ and $A_2\neq 0$, we obtain the following corollary

Corollary 4.3. *Let $-\infty<a<b<\infty$, let real parameters p, q, and λ be as in (1.1) and (1.2), let A_2 be real parameter such that $A_2\neq 0$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy$$

$$\leq |q'A_2|^{-\frac{1}{q'}}\left[\int_a^b x^{(1-A_2)p-1}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}}\cdot$$

$$\left[\int_a^b y^{(A_2-\frac{1}{p'})q}\left(\ln\frac{y}{a}\right)^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.11}$$

and

$$\left[\int_a^b y^{-q'A_2-1}\left(\ln\frac{y}{a}\right)^{-\frac{q'}{p'}}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq |q'A_2|^{-\frac{1}{q'}}\left[\int_a^b x^{(1-A_2)p-1}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.12}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$, and are equivalent.

Finally, if $A_1=\frac{1}{p'}$ and $A_2=0$, then we have the following result.

Corollary 4.4. *Let $-\infty < a < b < \infty$, let real parameters p, q, and λ be as in (1.1) and (1.2), and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \le \left[\int_a^b x^{p-1}\left(\ln\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot$$

$$\left[\int_a^b y^{-\frac{q}{p'}}\left(\ln\frac{y}{a}\right)^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.13}$$

and

$$\left[\int_a^b y^{-1}\left(\ln\frac{y}{a}\right)^{-\frac{q'}{p'}} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \le \left[\int_a^b x^{p-1}\left(\ln\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.14}$$

hold for all non-negative measurable functions f and g on $\langle a, b\rangle$, and are equivalent.

Now, we consider the second case, (4.2), where, $a = 0$. Then, F is defined by (4.5), and

$$G(y) = (1 - p'A_1)^{-\frac{1}{p'}} y^{-A_1}, \qquad y \in \langle 0, b\rangle, \tag{4.15}$$

where, $1 - p'A_1 > 0$. In this setting, we obtain two results, depending on value of the parameter A_2 ($A_2 \neq 0$ or $A_2 = 0$).

Corollary 4.5. *Let real parameters p, q, and λ be as in (1.1) and (1.2), let $0 < b < \infty$, let A_1, A_2 be real parameters such that $p'A_1 < 1$, $A_2 \neq 0$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_0^b y^{-\lambda} g(y)(Hf)(y)\,dy$$

$$\le \frac{(1 - p'A_1)^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}} \left[\int_0^b x^{(A_1 - A_2)p}\left|1 - \left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}}$$

$$\left[\int_0^b y^{(A_2 - A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.16}$$

and

$$\left[\int_0^b y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{|q'A_2|^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right|^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.17}$$

hold for all non-negative measurable functions f and g on $\langle 0,b\rangle$, and are equivalent.

Corollary 4.6. *Let real parameters p, q, and λ be as in (1.1) and (1.2), let $0<b<\infty$, let A_1 be real parameter such that $p'A_1<1$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_0^b y^{-\lambda}g(y)(Hf)(y)\,dy \leq (1-p'A_1)^{-\frac{1}{p'}}\left[\int_0^b x^{A_1p}\left(\ln\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}}\cdot$$

$$\left[\int_0^b y^{-A_1q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.18}$$

and

$$\left[\int_0^b y^{(A_1-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq (1-p'A_1)^{-\frac{1}{p'}}\left[\int_0^b x^{A_1p}\left(\ln\frac{b}{x}\right)^{\frac{p}{q'}} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.19}$$

hold for all non-negative measurable functions f and g on $\langle 0,b\rangle$, and are equivalent.

The next case (4.3) includes $b=\infty$. Then, G is defined by (4.6), and

$$F(x)=(q'A_2)^{-\frac{1}{q'}}x^{-A_2}, \qquad x\in\langle a,\infty\rangle, \tag{4.20}$$

where, $q'A_2>0$. In this setting, we get two results, depending on value of the parameter A_1 ($A_1\neq\frac{1}{p'}$ or $A_1=\frac{1}{p'}$).

Corollary 4.7. *Let $0<a<\infty$, let real parameters p, q, and λ be as in (1.1) and (1.2), let A_1, A_2 be real parameters such that $A_1\neq\frac{1}{p'}$, $q'A_2>$*

0, and let H be the operator defined by (2.3). Then, the inequalities

$$\int_a^\infty y^{-\lambda} g(y)(Hf)(y)\,dy$$

$$\leq \frac{|1-p'A_1|^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot$$

$$\left[\int_a^\infty y^{(A_2-A_1)q} \left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.21}$$

and

$$\left[\int_a^\infty y^{(A_1-A_2-\lambda)q'} \left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right|^{-\frac{q'}{p'}} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq \frac{|1-p'A_1|^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.22}$$

hold for all non-negative measurable functions f and g on $\langle a,\infty\rangle$, and are equivalent.

Corollary 4.8. *Let real parameters p, q, and λ be as in (1.1) and (1.2), let $0<a<\infty$, let A_2 be real parameter such that $q'A_2>0$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^\infty y^{-\lambda} g(y)(Hf)(y)\,dy \leq (q'A_2)^{-\frac{1}{q'}} \left[\int_a^\infty x^{(1-A_2)p-1} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot$$

$$\left[\int_a^\infty y^{q(A_2-\frac{1}{p'})} \left(\ln\frac{y}{a}\right)^{\frac{q}{p'}} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.23}$$

and

$$\left[\int_a^\infty y^{-q'A_2-1} \left(\ln\frac{y}{a}\right)^{-\frac{q'}{p'}} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq (q'A_2)^{-\frac{1}{q'}} \left[\int_a^\infty x^{(1-A_2)p-1} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.24}$$

hold for all non-negative measurable functions f and g on $\langle a,\infty\rangle$, and are equivalent.

Finally, we consider the case (4.4), that is, with $a = 0$ and $b = \infty$. In that case, the functions F and G are respectively defined by (4.20) and (4.15), where, $1 - p'A_1 > 0$ and $q'A_2 > 0$. Hence, we have only one possibility, given in the following corollary.

Corollary 4.9. *Suppose p, q, and λ are as in (1.1) and (1.2), A_1 and A_2 are real parameters such that $p'A_1 < 1$, $q'A_2 > 0$, and H is the operator defined by (2.3). Then, the inequalities*

$$\int_0^\infty y^{-\lambda} g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_0^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_0^\infty y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{4.25}$$

and

$$\left[\int_0^\infty y^{(A_1-A_2-\lambda)q'} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_0^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{4.26}$$

hold for all non-negative measurable functions f and g on $\mathbb{R}_+$ and are equivalent.

Remark 4.10. *Some results from this section, specialized to the case of conjugate exponents, can be found in [7] (see Corollaries 1–3). Hence, our results generalize the mentioned results in [7]. Note that the inequalities in this section hold with the reversed signs of inequality if the exponents p and q satisfy conditions (2.13) or (2.14).* □

Remark 4.11. *As already mentioned, we can easily obtain related results which correspond to the dual Hardy-type kernel. Here, they are omitted. Also, in [8] (see Section 4), one can find the results that are similar to the ones in this section.* □

5. **Further Analysis of Parameters**

We proceed with estimates of some factors included in integrals in the inequalities of the previous section, depending on non-conjugate exponents and the real parameters A_1 and A_2. Applying these estimates, we shall get closer to the classical Hardy inequality. More precisely, by using the same notation as in the previous section, the estimates

$$\left|1-\left(\frac{a}{y}\right)^{1-p'A_1}\right| \leq \left|1-\left(\frac{a}{b}\right)^{1-p'A_1}\right|, \quad y \in \langle a,b\rangle,\ A_1 \neq \frac{1}{p'}, \tag{5.1}$$

and

$$\left|1-\left(\frac{x}{b}\right)^{q'A_2}\right| \leq \left|1-\left(\frac{a}{b}\right)^{q'A_2}\right|, \quad x \in \langle a,b\rangle,\ A_2 \neq 0, \tag{5.2}$$

hold, where $0 < a < b < \infty$. Further, estimates

$$\ln\frac{b}{x} \leq \ln\frac{b}{a}, \quad x \in \langle a,b\rangle, \tag{5.3}$$

and

$$\ln\frac{y}{a} \leq \ln\frac{b}{a}, \quad y \in \langle a,b\rangle, \tag{5.4}$$

are obviously valid for the natural logarithm function, where, $0 < a < b < \infty$.

Our aim is an application of these estimates to the results obtained in Section 4. In such a way, we shall simplify these inequalities and obtain new constant factors included in the right-hand sides of appropriate inequalities as well. Constant factors will be expressed in terms of function $k : \mathbb{R} \to \mathbb{R}$, defined by the formula

$$k(\alpha) = \begin{cases} \frac{1-\left(\frac{a}{b}\right)^{\alpha}}{\alpha}, & \alpha \neq 0, \\ \ln\frac{b}{a}, & \alpha = 0, \end{cases} \tag{5.5}$$

with $0 < a < b < \infty$. Obviously, k is a continuous function, since $\lim_{\alpha\to 0} k(\alpha) = k(0)$.

By combining Corollary 4.1 and estimates (5.1) and (5.2), we obtain the following result.

Corollary 5.1. *Suppose p, q, and λ are as in (1.1) and (1.2), $-\infty < a < b < \infty$, A_1 and A_2 are real parameters such that $A_1 \neq \frac{1}{p'}$, $A_2 \neq 0$,* *and H is the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \le k^{\frac{1}{p'}}(1-p'A_1)k^{\frac{1}{q'}}(q'A_2)\cdot$$

$$\left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{5.6}$$

and

$$\left[\int_a^b y^{(A_1-A_2-\lambda)q'} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\le k^{\frac{1}{p'}}(1-p'A_1)k^{\frac{1}{q'}}(q'A_2)\left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{5.7}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$, and are equivalent.

Note that inequality (5.7) has a form of the classical Hardy inequality (1.3). Now, let us compare inequalities (4.8) and (5.7). The left-hand side of inequality (4.8) is not less than the corresponding side of (5.7), while the right-hand side of (4.8) is not greater than the corresponding side of (5.7). Thus, we can regard (4.8) as both a generalization and a refinement of the classical Hardy inequality. The same argument will be valid for the remaining results of the Hardy-type in Section 4.

Of course, in a similar way, we obtain the results that correspond to Corollaries 4.2, 4.3 and 4.4.

Corollary 5.2. *Let $-\infty < a < b < \infty$, let p, q, and λ be as in (1.1) and (1.2), let A_1 be real parameter such that $A_1 \neq \frac{1}{p'}$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \le k^{\frac{1}{p'}}(1-p'A_1)k^{\frac{1}{q'}}(0)\left[\int_a^b x^{A_1 p} f^p(x)\,dx\right]^{\frac{1}{p}}\cdot$$

$$\left[\int_a^b y^{-A_1 q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{5.8}$$

and

$$\left[\int_a^b y^{(A_1-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq k^{\frac{1}{p'}}(1-p'A_1)k^{\frac{1}{q'}}(0)\left[\int_a^b x^{A_1p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{5.9}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$, and are equivalent.

Corollary 5.3. *Suppose p, q, and λ are as in (1.1) and (1.2), $0<a<b<\infty$, A_2 is real parameter such that $A_2\neq 0$, and H is the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy \leq k^{\frac{1}{p'}}(0)k^{\frac{1}{q'}}(q'A_2)\left[\int_a^b x^{(1-A_2)p-1}f^p(x)\,dx\right]^{\frac{1}{p}}\cdot$$

$$\left[\int_a^b y^{(A_2-\frac{1}{p'})q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{5.10}$$

and

$$\left[\int_a^b y^{-q'A_2-1}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}}$$

$$\leq k^{\frac{1}{p'}}(0)k^{\frac{1}{q'}}(q'A_2)\left[\int_a^b x^{(1-A_2)p-1}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{5.11}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$, and are equivalent.

Corollary 5.4. *Let $0<a<b<\infty$. Further, suppose p, q, and λ are as in (1.1) and (1.2) and H is the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda}g(y)(Hf)(y)\,dy$$

$$\leq k^{\lambda}(0)\left[\int_a^b x^{p-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_a^b y^{-\frac{q}{p'}}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{5.12}$$

and

$$\left[\int_a^b y^{-1}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \le k^{\lambda}(0)\left[\int_a^b x^{p-1}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{5.13}$$

hold for all non-negative measurable functions f and g on $\langle a,b\rangle$, and are equivalent.

Remark 5.5. *It is easy to see that inequalities with the reversed inequality sign do not hold in Corollaries 5.1–5.4, since inequalities with opposite inequality signs are combined.* □

Finally, by using the established estimates, we can also obtain results that correspond to Corollaries 4.5 and 4.7. Since $a = 0$ or $b = \infty$, we do not need to express the constant factors in terms of the function k.

Corollary 5.6. *Let $0 < b < \infty$, let p, q, and λ be as in (1.1) and (1.2), let A_1, A_2 be real parameters such that $p'A_1 < 1$, $q'A_2 > 0$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_0^b y^{-\lambda}g(y)(Hf)(y)\,dy \le \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_0^b y^{(A_2-A_1)q}g^q(y)\,dy\right]^{\frac{1}{q}} \tag{5.14}$$

and

$$\left[\int_0^b y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \le \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_0^b x^{(A_1-A_2)p}f^p(x)\,dx\right]^{\frac{1}{p}} \tag{5.15}$$

hold for all non-negative measurable functions f and g on $\langle 0,b\rangle$, and are equivalent.

Remark 5.7. *If the parameters p and q satisfy conditions (2.13), then the sign of inequality in (5.14) and (5.15) is reversed.* □

Corollary 5.8. *Suppose p, q, and λ are as in (1.1) and (1.2), $0 < a < \infty$, A_1 and A_2 are real parameters such that $p'A_1 < 1$, $q'A_2 > 0$, and H is the operator defined by (2.3). Then, the inequalities*

$$\int_a^\infty y^{-\lambda} g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_a^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_a^\infty y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{5.16}$$

and

$$\left[\int_a^\infty y^{(A_1-A_2-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}}\left[\int_a^\infty x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{5.17}$$

hold for all non-negative measurable functions f and g on $\langle a, \infty\rangle$, and are equivalent.

Remark 5.9. *If non-conjugate exponents p and q satisfy conditions (2.14), then the sign of inequality in (5.16) and (5.17) is reversed.* □

6. Uniform Bounds for Constant Factors

We investigate here some further estimates for the Hardy-type kernels. First, recall that Corollary 5.1 was obtained from Corollary 4.1 by means of the estimates (5.1) and (5.2). On the other hand, we may apply the uniform upper bound $1 - t^x \leq 1$, $t \in \langle 0, 1\rangle$, $x \geq 0$, to Corollary 4.1. A corresponding result, under some stronger conditions, is contained in the following corollary.

Corollary 6.1. *Let $0 < a < b < \infty$, let p, q, and λ be as in (1.1) and (1.2), let A_1, A_2 be real parameters such that $p'A_1 < 1$, $q'A_2 > 0$, and* *let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b y^{(A_2-A_1)q} g^q(y)\,dy\right]^{\frac{1}{q}} \tag{6.1}$$

and

$$\left[\int_a^b y^{(A_1-A_2-\lambda)q'} (Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \leq \frac{(1-p'A_1)^{-\frac{1}{p'}}}{(q'A_2)^{\frac{1}{q'}}} \left[\int_a^b x^{(A_1-A_2)p} f^p(x)\,dx\right]^{\frac{1}{p}} \tag{6.2}$$

hold for all non-negative measurable functions f and g on $\langle a, b\rangle$, and are equivalent.

Remark 6.2. *If we compare Corollaries 4.9, 5.6 and 5.8 with Corollary 6.1, we easily conclude that Corollary 6.1 holds also if $a = 0$ or $b = \infty$.* □

The constant factor involved in the right-hand sides of the inequalities (6.1) and (6.2) depends on the parameters A_1 and A_2, while the integrals are dependent only on the parameter $A = A_1 - A_2$. Hence, it is interesting to consider that constant factor for a fixed value of A. Then, $A_2 = A_1 - A$ and the constant factor can be regarded as the function,

$$C(A_1) = (1-p'A_1)^{-\frac{1}{p'}} (q'A_1 - q'A)^{-\frac{1}{q'}}. \tag{6.3}$$

It is interesting to find an optimal value for the constant factor (6.3). More precisely, depending on sign of inequality, we find maximal or minimal values for that factor. Having Remark 2 in mind, we have to consider three cases:

1. $p, q > 1$, $\lambda \geq 1$.
   In this case, we have $A < \frac{1}{p'}$ and $A_1 \in \left\langle A, \frac{1}{p'}\right\rangle$, and so we have to find
   $$\inf_{A<x<\frac{1}{p'}} C(x) = \inf_{A<x<\frac{1}{p'}} (1-p'x)^{-\frac{1}{p'}} (q'x - q'A)^{-\frac{1}{q'}}.$$

One easily obtains that $C'(x) = 0$ if and only if

$$x_0 = \frac{1 + q'A}{p' + q'}. \tag{6.4}$$

Further, since $x_0 \in \left\langle A, \frac{1}{p'} \right\rangle$ and $C''(x_0) > 0$, we conclude that $C(x)$ attains its minimal value on the interval $\left\langle A, \frac{1}{p'} \right\rangle$ at the point x_0. Hence, a straightforward computation gives the form of the minimal constant factor:

$$\inf_{A < x < \frac{1}{p'}} C(x) = C(x_0) = \left(\frac{p'\lambda}{1 - p'A} \right)^{\lambda}.$$

2. $p < 0$, $q \in \langle 0, 1 \rangle$, $\lambda \geq 1$
   It is easy to see that $A_1 \in \left\langle -\infty, \min\left\{ \frac{1}{p'}, A \right\} \right\rangle$, and so we distinguish two cases, depending on the relationship between the parameters A and $\frac{1}{p'}$.
   If $A < \frac{1}{p'}$, then we conclude, by a similar reasoning as in the previous case, that the function $C(x)$ attains its maximal value on the interval $\langle -\infty, A \rangle$, at the point x_0 defined by (6.4). Finally, since $C(A) = 0$, we have

$$\sup_{x \leq A} C(x) = C(x_0) = \left(\frac{p'\lambda}{1 - p'A} \right)^{\lambda}.$$

   If $A \geq \frac{1}{p'}$, then the stationary point (6.4) does not belong to the interval $\left\langle -\infty, \frac{1}{p'} \right\rangle$ and $C(x)$ is strictly increasing on that interval. Further, since $\lim_{x \to \frac{1}{p'}-} C(x) = \infty$, there is no upper bound for the constant factor $C(x)$ in that setting.
3. $p \in \langle 0, 1 \rangle$, $q < 0$, $\lambda \geq 1$.
   Here, we have to find the optimal value on the interval $\left\langle \max\left\{ \frac{1}{p'}, A \right\}, \infty \right\rangle$. Similarly, as above, we have to consider two cases.
   For $A < \frac{1}{p'}$, we easily obtain that the function $C(x)$ attains its maximal value on the the interval $\left\langle \frac{1}{p'}, \infty \right\rangle$ at the point defined

by (6.4). Moreover, since $C\left(\frac{1}{p'}\right) = 0$, we have

$$\sup_{x \geq \frac{1}{p'}} C(x) = C(x_0) = \left(\frac{p'\lambda}{1 - p'A}\right)^{\lambda}.$$

If $A \geq \frac{1}{p'}$, then the stationary point (6.4) is not contained in the interval $\langle A, \infty\rangle$ and $C(x)$ is strictly decreasing on that interval. Since $\lim_{x \to A+} C(x) = \infty$, there is no upper bound for the constant factor $C(x)$ in that case.

According to the previous analysis, we have just proved the following result.

Theorem 6.3. *Let $0 < a < b < \infty$, let p, q, and λ be as in (1.1) and (1.2), let A be real parameter such that $A < \frac{1}{p'}$, and let H be the operator defined by (2.3). Then, the inequalities*

$$\int_a^b y^{-\lambda} g(y)(Hf)(y)\, dy \leq \left(\frac{p'\lambda}{1 - p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\, dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b y^{-qA} g^q(y)\, dy\right]^{\frac{1}{q}} \tag{6.5}$$

and

$$\left[\int_a^b y^{(A-\lambda)q'} (Hf)^{q'}(y)\, dy\right]^{\frac{1}{q'}} \leq \left(\frac{p'\lambda}{1 - p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\, dx\right]^{\frac{1}{p}} \tag{6.6}$$

hold for all non-negative measurable functions f and g on $\langle a, b\rangle$, and are equivalent. Moreover, if $a = 0$ and p, q, λ are as in (2.13), then the signs of inequality in both relations are reversed. Further, if $b = \infty$ and p, q, λ are as in (2.14), the the signs of inequality are reversed as well.

7. Applications

Finally,we consider some interesting special cases involving the optimal constant factor in the Hardy-type inequality established in the previous section. Namely, we shall synthesize the methods developed in sections 4, 5 and 6 for such cases. By means of the established estimates, we obtain the whole series of interpolating inequalities which will provide both generalizations and refinements of recent results, known in the literature.

We can gather the previous discussion in the following two series of inequalities:

$$\begin{aligned}
&\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \\
&\qquad \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left\{\int_a^b x^{pA}\left[1-\left(\frac{x}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \cdot \\
&\qquad\quad \left\{\int_a^b y^{-qA}\left[1-\left(\frac{a}{y}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{q}{p'}} g^q(y)\,dy\right\}^{\frac{1}{q}} \\
&\qquad \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[1-\left(\frac{a}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\lambda} \cdot \\
&\qquad\quad \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b y^{-qA} g^q(y)\,dy\right]^{\frac{1}{q}} \\
&\qquad \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_a^b y^{-qA} g^q(y)\,dy\right]^{\frac{1}{q}}
\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}
&\left[\int_a^b y^{(A-\lambda)q'}(Hf)^{q'}(y)\,dy\right]^{\frac{1}{q'}} \\
&\qquad \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[1-\left(\frac{a}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{1}{p'}} \cdot \\
&\qquad\quad \left\{\int_a^b x^{pA}\left[1-\left(\frac{x}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx\right\}^{\frac{1}{p}} \\
&\qquad \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[1-\left(\frac{a}{b}\right)^{\frac{1-p'A}{\lambda p'}}\right]^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}} \\
&\qquad \le \left(\frac{p'\lambda}{1-p'A}\right)^{\lambda} \left[\int_a^b x^{pA} f^p(x)\,dx\right]^{\frac{1}{p}},
\end{aligned} \tag{7.2}$$

which hold under assumptions of Theorem 6.3. Of course, these set of inequalities are equivalent and reversed signs of inequalities hold as described in Theorem 6.3. For $A = 0$, the above set of inequalities (7.1) and (7.2) reduce, respectively, to:

$$\begin{aligned}
&\int_a^b y^{-\lambda} g(y)(Hf)(y)\,dy \\
&\qquad \leq (p'\lambda)^{\lambda} \left\{ \int_a^b \left[1 - \left(\frac{x}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx \right\}^{\frac{1}{p}} \cdot \\
&\qquad\quad \left\{ \int_a^b \left[1 - \left(\frac{a}{y}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{q}{p'}} g^q(y)\,dy \right\}^{\frac{1}{q}} \\
&\qquad \leq (p'\lambda)^{\lambda} \left[1 - \left(\frac{a}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\lambda} \|f\|_{L^p} \|g\|_{L^q} \leq (p'\lambda)^{\lambda} \|f\|_{L^p} \|g\|_{L^q}
\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}
&\left[\int_a^b y^{-\lambda q'} (Hf)^{q'}(y)\,dy \right]^{\frac{1}{q'}} \\
&\qquad \leq (p'\lambda)^{\lambda} \left[1 - \left(\frac{a}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{1}{p'}} \left\{ \int_a^b \left[1 - \left(\frac{x}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\frac{p}{q'}} f^p(x)\,dx \right\}^{\frac{1}{p}} \\
&\qquad \leq (p'\lambda)^{\lambda} \left[1 - \left(\frac{a}{b}\right)^{\frac{1}{\lambda p'}}\right]^{\lambda} \|f\|_{L^p} \leq (p'\lambda)^{\lambda} \|f\|_{L^p}.
\end{aligned} \tag{7.4}$$

Remark 7.1. *We can easily show that inequalities (7.3) and (7.4), with $A = 0$, are equivalent to inequalities (7.1) and (7.2), with condition $A < \frac{1}{p'}$. Namely, if we put*

$$a^{1-p'A}, \quad b^{1-p'A}, \quad x^{\frac{1}{1-p'A}-1} f\big(x^{\frac{1}{1-p'A}}\big), \quad y^{(1-\lambda)\frac{p'A}{1-p'A}} g\big(y^{\frac{1}{1-p'A}}\big)$$

in (7.3), respectively instead of $a, b, f(x), g(y)$, and then apply the variable substitution theorem, the set of inequalities (7.3) become (7.1). So, the case with condition $A < \frac{1}{p'}$ is equivalent to the case with condition $A = 0$. Thus, it is enough to observe the cases with $A = 0$, since all others follow by suitable substitutions. □

Finally, to conclude our work, we compare the results obtained here with some recent results in the literature and observe the corresponding generalizations and refinements.

Remark 7.2. *If we substitute* $a = 0$ *and* $b = \infty$ *in* (7.4) *and isolate outer expressions, then we obtain the inequality*

$$\left[\int_0^\infty y^{-\lambda q'} (Hf)^{q'}(y)\, dy\right]^{\frac{1}{q'}} \leq \left(p'\lambda\right)^{\lambda} \|f\|_{L^p},$$

which coincide with Opic's estimate (see [10]*). Clearly, for* $\lambda = 1$*, we obtain the Hardy inequality* (1.3) *in the original form. That inequality can also be found, however, in the conjugate case, in Kufner's work* [9]*. So, our results may be regarded as both generalizations and refinements of the mentioned results.* □

Remark 7.3. *Clearly, if* $A = \lambda - \frac{k}{q'}$*,* $k > 1$*, then* $A < \lambda - \frac{1}{q'} = \frac{1}{p'}$*, and so by setting* $A = \lambda - \frac{k}{q'}$ *in inequalities* (7.1) *and* (7.2)*, the optimum constant factor established in Theorem* 6.3 *takes the form:*

$$C = \left(\frac{\lambda q'}{k-1}\right)^{\lambda}. \tag{7.5}$$

In this setting, we see that inequalities (7.1) *and* (7.2) *provide an extension to non-conjugate case of the corresponding results in* [7] *(see* [7]*, Remark* 3,4*). Further, relation* (7.2) *can be seen as both a refinement and an extension of the results from the* [2] *and* [3]*. Namely, in the conjugate case, that is, when* $p = q'$ *and* $\lambda = 1$*, with* C *defined by* (7.5)*, from relation* (7.2) *we obtain related results from the mentioned works (for example, see* [2]*, Theorem* 2*,* [3]*, relation* (13) *and also relation* (1.5) *from the Introduction).* □

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