Bulletin of the Iranian Mathematical Society Vol. 37 *No. 3* (*2011*), *pp 39-75.*

SECOND SYMMETRIC POWERS OF CHAIN COMPLEXES

A. J. FRANKILD, S. SATHER-WAGSTAFF* AND A. TAYLOR

Communicated by Jürgen Herzog

ABSTRACT. We investigate Buchbaum and Eisenbud's construction of the second symmetric power $\mathrm{S}^2_R(X)$ of a chain complex X of modules over a commutative ring R. We state and prove a number of results from the folklore of the subject for which we know of no good direct references. We also provide several explicit computations and examples. We use this construction to prove the following version of a result of Avramov, Buchweitz, and Şega: let $R \to S$ be a module-finite ring homomorphism such that R is noetherian and local, and such that 2 is a unit in R. Let X be a complex of finite rank free S-modules such that $X_n = 0$ for each $n < 0$. If $\cup_n \mathrm{Ass}_R(\mathrm{H}_n(X \otimes_S X)) \subseteq \mathrm{Ass}(R)$ and if $X_\mathfrak{p} \simeq S_\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}(R)$, then $X \simeq S$.

1. **Introduction**

Multilinear constructions like tensor products and symmetric powers are important tools for studying modules over commutative rings. In recent years, these notions have been extended to the realm of chain complexes of R-modules. (Consult Section 2 for background information on complexes.) For instance, Buchsbaum and Eisenbud's description [5]

MSC(2000): Primary: 13C10; Secondary: 13D25.

Keywords: Chain complex, symmetric power, symmetric square.

Received: 30 March 2009, Accepted: 18 January 2010.

∗Corresponding author

of the minimal free resolutions of Gorenstein ideals of grade 3 uses the second symmetric power of a certain free resolution.

Here, we investigate Buchsbaum and Eisenbud's second symmetric power functor: for a chain complex X of modules over a commutative ring R, we set $\mathrm{S}^2_R(X) = (X \otimes_R X)/(Y + Z)$, where Y is the graded submodule generated by all elements of the form $x \otimes x' - (-1)^{|x||x'|} x' \otimes x$ and Z is the graded submodule generated by all elements of the form $x \otimes x$ with x having an odd degree.[1]

Our main result is the following version of a result of Avramov, et al. [2, (2.2)] for complexes. It is motivated by our work in [10] extending the results of [2]. Note that $\mathrm{S}^2_R(X)$ does not appear in the statement of Theorem below; however, it is the key tool for the proof given in 4.7.

Theorem A. Let $R \to S$ be a module-finite ring homomorphism such that R is noetherian and local, and such that 2 is a unit in R. Let X be a complex of finite rank free S-modules such that $X_n = 0$ for each $n < 0$. If $\cup_n \mathrm{Ass}_R(\mathrm{H}_n(X \otimes_S X)) \subseteq \mathrm{Ass}(R)$ and if $X_\mathfrak{p} \simeq S_\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}(R)$, then $X \simeq S$.

Most of our work is devoted to statements and proofs of results from the folklore in this subject. Section 3 contains basic properties of $\mathrm{S}^2_R(X)$, most of which are motivated by the behavior of tensor products of complexes and the properties of symmetric powers of modules. This section ends with an explicit description of the modules occuring in $\mathrm{S}^2_R(X)$; see Theorem 3.9. Section 4 examines the homological properties of $\mathrm{S}^2_R(X)$, and includes the proof of Thoerem A. We conclude with Section 5, which is devoted to explicit computations.

This work was completed after the untimely passing of Anders J. Frankild on 10 June 2007.

2. Complexes

Throughout this paper, R and S are commutative rings with identity. The term "module" is short for "unital module".

---

[1]Note that the definition of $\mathrm{S}^2_R(X)$ given in [5] does not yield the complex described in [5, p. 452], unless 2 is a unit in R. The corrected definition can be found, for instance, in [4, (3.4.3)].

This section consists of definitions, notation and background information for use in the remainder of the paper.

Definition 2.1. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n. A complex X is *degreewise-finite* if each X_n is finitely generated; it is *bounded-below* if $X_n = 0$ for $n \ll 0$.

The nth *homology module* of X is $\mathrm{H}_n(X) := \mathrm{Ker}(\partial^X_n)/\mathrm{Im}(\partial^X_{n+1})$. The *infimum* of X is $\inf(X) := \inf\{i \in \mathbb{Z} \mid \mathrm{H}_n(X) \neq 0\}$, and the *large support* of X is

$$\mathrm{Supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid X_{\mathfrak{p}} \not\simeq 0\} = \cup_n \mathrm{Supp}_R(\mathrm{H}_n(X)).$$

For each $x \in X_n$, we set $|x| := n$. An R-complex X is *homologically degreewise-finite* if $\mathrm{H}_n(X)$ is finitely generated for each n; it is *homologically finite* if the R-module $\oplus_{n\in\mathbb{Z}} \mathrm{H}_n(X)$ is finitely generated.

For each integer i, the ith *suspension* (or *shift*) of X, denoted by $\Sigma^i X$, is the complex with $(\Sigma^i X)_n = X_{n-i}$ and $\partial^{\Sigma^i X}_n = (-1)^i \partial^X_{n-i}$. The notation ΣX is short for $\Sigma^1 X$.

Definition 2.2. Let X and Y be R-complexes. A *morphism* from X to Y is a sequence of R-module homomorphisms $\{f_n \colon X_n \to Y_n\}$ such that $f_{n-1}\partial^X_n = \partial^Y_n f_n$, for each n. A morphism of complexes $\alpha\colon X \to Y$ induces homomorphisms on homology modules $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$, and α is a *quasiisomorphism* when each $\mathrm{H}_n(\alpha)$ is bijective. Quasiisomorphisms are designated by the symbol "$\simeq$".

Definition 2.3. Let X, Y be R-complexes. Two morphisms $f, g\colon X \to Y$ are *homotopic* if there exists a sequence of homomorphisms $s = \{s_n \colon X_n \to Y_{n+1}\}$ such that $f_n = g_n + \partial^Y_{n+1}s_n + s_{n-1}\partial^X_n$, for each n; here, we say that s is a *homotopy* from f to g. The morphism f is a *homotopy equivalence* if there is a morphism $h\colon Y \to X$ such that the compositions fh and hf are homotopic to the respective identity morphisms id_Y and id_X, and then f and h are *homotopy inverses*.

Definition 2.4. Given two bounded-below complexes P and Q of projective R-modules, we write $P \simeq Q$, when there is a quasiisomorphism $P \xrightarrow{\simeq} Q$.

Fact 2.5. The relation $\simeq$ from Definition 2.4 is an equivalence relation; see [3, (2.8.8.2.2')] or [8, (6.6.ii)] or [9, (6.21)].

Let P and Q be bounded-below complexes of projective R-modules. Then, any quasiisomorphism $P \xrightarrow{\simeq} Q$ is a homotopy equivalence; see [3, (1.8.5.3)] or [8, (6.4.iii)]. Conversely, it is straightforward to show that any homotopy equivalence between R-complexes is a quasiisomorphism.

Definition 2.6. Let X be a homologically bounded-below R-complex. A *projective (or free) resolution* of X is a quasiisomorphism $P \xrightarrow{\simeq} X$ such that each P_n is projective (or free) and P is bounded-below; the resolution $P \xrightarrow{\simeq} X$ is *degreewise-finite* if P is degreewise-finite. We say that X has *finite projective dimension* when it admits a projective resolution $P \xrightarrow{\simeq} X$ such that $P_n = 0$ for $n \gg 0$.

Fact 2.7. Let X be a homologically bounded-below R-complex. Then, X has a free resolution $P \xrightarrow{\simeq} X$ such that $P_n = 0$, for all $n < \inf(X)$; see [3, (2.11.3.4)] or [8, (6.6.i)] or [9, (2.6.P)]. It follows that $P_{\inf(X)} \neq 0$. If $P \xrightarrow{\simeq} X$ and $Q \xrightarrow{\simeq} X$ are projective resolutions of X, then there is a homotopy equivalence $P \xrightarrow{\simeq} Q$; see [8, (6.6.ii)] or [9, (6.21)]. If R is noetherian and X is homologically degreewise-finite, then P may be chosen degreewise-finite; see [3, (2.11.3.3)] or [9, (2.6.L)].

Definition 2.8. Let X be an R-complex that is homologically both bounded-below and degreewise-finite. Assume that R is noetherian and local with maximal ideal $\mathfrak{m}$. A projective resolution $P \xrightarrow{\simeq} X$ is *minimal* if the complex P is minimal, that is, if $\operatorname{Im}(\partial_n^P) \subseteq \mathfrak{m}P_{n-1}$, for each n.

Fact 2.9. Let X be an R-complex that is homologically both bounded-below and degreewise-finite. Assume that R is noetherian and local with maximal ideal $\mathfrak{m}$. Then, X has a minimal free resolution $P \xrightarrow{\simeq} X$ such that $P_n = 0$, for all $n < \inf(X)$; see [1, Prop. 2] or [3, (2.12.5.2.1)]. Let $P \xrightarrow{\simeq} X$ and $Q \xrightarrow{\simeq} X$ be projective resolutions of X. If P is minimal, then there is a bounded-below exact complex P' of projective R-modules such that $Q \cong P \oplus P'$; see [3, (2.12.5.2.3)]. It follows that X has finite projective dimension if and only if every minimal projective resolution of X is bounded. It also follows that if P and Q are both minimal, then $P \cong Q$; see [3, (2.12.5.2.2)].

Definition 2.10. Let X and Y be R-complexes. The R-complex $X \otimes_R Y$ is given by

$$(X \otimes_R Y)_n = \bigoplus_p X_p \otimes_R Y_{n-p}$$

with nth differential $\partial_n^{X \otimes_R Y}$ given on generators by

$$x \otimes y \mapsto \partial_{|x|}^X(x) \otimes y + (-1)^{|x|} x \otimes \partial_{|y|}^Y(y).$$

Fix two more R-complexes X', Y' and morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$. Define the tensor product $f \otimes_R g\colon X \otimes_R Y \to X' \otimes_R Y'$ on generators as

$$x \otimes y \mapsto f_{|x|}(x) \otimes g_{|y|}(y).$$

One checks readily that $f \otimes_R g$ is a morphism.

Fact 2.11. Let P and Q be bounded-below complexes of projective R-modules. If $f\colon X \xrightarrow{\simeq} Y$ is a quasiisomorphism, then so are the induced morphisms $f \otimes_R Q\colon X \otimes_R Q \to Y \otimes_R Q$ and $P \otimes_R f\colon P \otimes_R X \to P \otimes_R Y$; see [3, (1.10.4.2.2')] or [8, (6.10)] or [9, (7.8)]. In particular, if $g\colon P \xrightarrow{\simeq} Q$ is a quasiisomorphism, then so is $g \otimes g\colon P \otimes_R P \to Q \otimes_R Q$; see [8, (6.10)]. This can be used to show the following facts from [9, (7.28)]:

$$\inf(P \otimes_R Q) \geqslant \inf(P) + \inf(Q)$$
$$\mathrm{H}^R_{\inf(P)+\inf(Q)}(P \otimes_R Q) \cong \mathrm{H}_{\inf(P)}(P) \otimes_R \mathrm{H}_{\inf(Q)}(Q).$$

Assume that R is noetherian and that P and Q are homologically degreewise-finite. One can use degreewise-finite projective resolutions of P and Q in order to show that each R-module $\mathrm{H}_n(P \otimes_R Q)$ is finitely generated; see [9, (7.31)]. In particular, if R is local, then Nakayama's Lemma conspires with the previous display to imply that $\inf(P \otimes_R Q) = \inf(P) + \inf(Q)$; see [9, (7.28)].

The following technical lemma about power series is used in the proofs of Theorem 4.8 and Corollary 4.10.

Lemma 2.12. *Let $Q(t) = \sum_{i=0}^{\infty} r_i t^i$ be a power series with nonnegative integer coefficients, and assume $r_0 > 0$. If either $Q(t)^2 + Q(-t^2)$ or $Q(t)^2 - Q(-t^2)$ is a non-negative integer, then $r_i = 0$, for all $i > 0$. Furthermore,*

(a) $Q(t)^2 + Q(-t^2) \neq 0$;
(b) *if* $Q(t)^2 - Q(-t^2) = 0$, *then* $Q(t) = 1$;
(c) *if* $Q(t)^2 + Q(-t^2) = 2$, *then* $Q(t) = 1$; *and*
(d) *if* $Q(t)^2 - Q(-t^2) = 2$, *then* $Q(t) = 2$.

Proof. We begin by showing that $r_n = 0$, for each $n \geqslant 1$, by induction on n. The coefficients of $Q(-t^2)$ in odd degree are all 0. Hence, the degree 1 coefficient of $Q(t)^2 \pm Q(-t^2)$ is

$$0 = r_1 r_0 + r_0 r_1 = 2 r_1 r_0.$$

It follows that $r_1 = 0$, since $r_0 > 0$. Inductively, assume that $n \geqslant 1$ and that $r_i = 0$, for each $i = 1, \ldots, n$. Since the degree $n+1$ coefficient of $Q_X^R(-t^2)$ is either $\pm r_{\frac{n+1}{2}}$ (when $n+1$ is even) or 0 (when $n+1$ is odd), the induction hypothesis implies that this coefficient is 0. The degree $n+1$ coefficient of $Q(t)^2 \pm Q(-t^2)$ is

$$0 = r_{n+1} r_0 + \underbrace{r_n r_1 + \cdots + r_1 r_n}_{=0} + r_0 r_{n+1} = 2 r_{n+1} r_0$$

and so $r_{n+1} = 0$.

The previous paragraph shows that $Q(t) = r_0$, so $Q(t)^2 \pm Q(-t^2) = r_0^2 \mp r_0$. The conclusions (a)–(d) follow readily, using the assumption $r_0 > 0$. □

3. Definition and Basic Properties of $\mathrm{S}_R^2(X)$

We begin this section with the definition of the second symmetric power of a complex. It is constructed based on the definition for modules.

Definition 3.1. Let X be an R-complex and let $\alpha^X \colon X \otimes_R X \to X \otimes_R X$ be the morphism described on generators by the formula

$$x \otimes x' \mapsto x \otimes x' - (-1)^{|x||x'|} x' \otimes x.$$

The *weak second symmetric power* of X is $\mathrm{s}_R^2(X) := \operatorname{Coker}(\alpha^X)$. The *second symmetric power* of X is $\mathrm{S}_R^2(X) := \mathrm{s}_R^2(X) / \langle \overline{x \otimes x} \mid |x| \text{ is odd} \rangle$. For each $i \in \mathbb{Z}$, let $\omega_i^X \colon \mathrm{s}_R^2(X)_i \to \mathrm{S}_R^2(X)_i$ be the natural surjection.

Remark 3.2. Let X be an R-complex. Since $\mathrm{s}_R^2(X)$ is defined as a cokernel of a morphism, it is an R-complex. Also, for each $n \in \mathbb{Z}$ and $x \in X_{2n+1}$, one has

$$\partial_{4n+2}^{X \otimes_R X}(x \otimes x) = \alpha_{4n+1}^X(\partial_{2n+1}^X(x) \otimes x).$$

It follows that $\mathrm{S}_R^2(X)$ is an R-complex, and that the sequence $\{\omega_i^X\}$ describes a morphism $\omega^X : \mathrm{s}_R^2(X) \to \mathrm{S}_R^2(X)$.

Here are computations for later use. Section 5 contains more involved examples.

Example 3.3. *If M is an R-module, then computing $\mathrm{S}_R^2(M)$ and $\mathrm{s}_R^2(M)$ as complexes (considering M as a complex concentrated in degree 0) and computing $\mathrm{S}_R^2(M)$ as a module give the same result. In particular, we have $\mathrm{S}_R^2(0) = 0 = \mathrm{s}_R^2(0)$ and $\mathrm{S}_R^2(R) \cong R \cong \mathrm{s}_R^2(R)$.*

Example 3.4. *For $0 \neq x, y \in \Sigma R$, we have $\alpha^{\Sigma R}(x \otimes y) = x \otimes y + y \otimes x$. Hence, the natural tensor-cancellation isomorphism $R \otimes_R R \xrightarrow{\cong} R$ yields the vertical isomorphisms in the following commutative diagram:*

$$\begin{array}{ccccc}
(\Sigma R) \otimes_R (\Sigma R) & \xrightarrow{\alpha^{\Sigma R}} & (\Sigma R) \otimes_R (\Sigma R) & \xrightarrow{p^{\Sigma R}} & \mathrm{s}_R^2(\Sigma R) \\
\cong \big\downarrow \beta & & \cong \big\downarrow \beta & & \cong \big\downarrow \overline{\beta} \\
\Sigma^2 R & \xrightarrow{(2)} & \Sigma^2 R & \longrightarrow & \Sigma^2 R/(2)
\end{array}$$

It follows that $\mathrm{s}_R^2(\Sigma R) \cong \Sigma^2 R/(2)$.

By definition, the kernel of the natural map $\omega^X : \mathrm{s}_R^2(X) \to \mathrm{S}_R^2(X)$ is generated by $\overline{1 \otimes 1} \in \mathrm{s}_R^2(\Sigma R)_2$. Since we have $\overline{\beta}_2\left(\overline{1 \otimes 1}\right) = \overline{1}$, it follows that $\mathrm{S}_R^2(\Sigma R) = 0$.

More generally, for each integer n we have $\mathrm{s}_R^2(\Sigma^{2n+1} R) \cong \Sigma^{4n+2} R/(2)$ and $\mathrm{S}_R^2(\Sigma^{2n+1} R) = 0$. In particular, if $2R \neq 0$, then

$$\mathrm{s}_R^2(\Sigma^{2n+1} R) \cong \Sigma^{4n+2} R/(2) \not\cong \Sigma^{4n+2} R \cong \Sigma^{4n+2} \mathrm{s}_R^2(R).$$

Contrast this with the behavior of $\mathrm{s}_R^2(\Sigma^{2n} X)$ and $\mathrm{S}_R^2(\Sigma^{2n} X)$ documented in (3.5.2).

The following properties are straightforward to verify and will be used frequently in the sequel.

Properties 3.5. Let X be an R-complex.

3.5.1. If 2 is a unit in R, then the natural morphism $\omega^X\colon \mathrm{s}^2_R(X) \to \mathrm{S}^2_R(X)$ is an isomorphism, and the morphism $\frac{1}{2}\alpha^X$ is idempotent.

3.5.2. For each integer n, there is a commutative diagram

$$\begin{array}{ccc} (\Sigma^{2n}X)\otimes_R(\Sigma^{2n}X) & \xrightarrow{\alpha^{\Sigma^{2n}X}} & (\Sigma^{2n}X)\otimes_R(\Sigma^{2n}X) \\ \cong\downarrow\beta & & \cong\downarrow\beta \\ \Sigma^{4n}(X\otimes_R X) & \xrightarrow{\Sigma^{4n}\alpha^X} & \Sigma^{4n}(X\otimes_R X) \end{array}$$

with $\beta(x\otimes y) = x\otimes y$. The resulting isomorphism of cokernels yields an isomorphism

$$\overline{\beta}\colon \mathrm{s}^2_R(\Sigma^{2n}X) \xrightarrow{\cong} \Sigma^{4n}\mathrm{s}^2_R(X)$$

given by $\overline{\beta}\,(\overline{x\otimes y}) = \overline{x\otimes y}$. In particular, the equality $\overline{\beta}\,(\overline{x\otimes x}) = \overline{x\otimes x}$ implies that $\overline{\beta}$ induces an isomorphism

$$\mathrm{S}^2_R(\Sigma^{2n}X) \cong \Sigma^{4n}\mathrm{S}^2_R(X).$$

3.5.3. There is an exact sequence

$$0 \to \mathrm{Ker}(\alpha^X) \xrightarrow{j^X} X\otimes_R X \xrightarrow{\alpha^X} X\otimes_R X \xrightarrow{p^X} \mathrm{s}^2_R(X) \to 0,$$

where, j^X and p^X are the natural injection and surjection, respectively.

3.5.4. A morphism of complexes $f\colon X\to Y$ yields a commutative diagram of morphisms

$$\begin{array}{ccc} X\otimes_R X & \xrightarrow{\alpha^X} & X\otimes_R X \\ f\otimes_R f\downarrow & & \downarrow f\otimes_R f \\ Y\otimes_R Y & \xrightarrow{\alpha^Y} & Y\otimes_R Y. \end{array}$$

Hence, the morphism $f\otimes_R f$ induces a well-defined morphism on cokernels $\mathrm{s}^2_R(f)\colon \mathrm{s}^2_R(X)\to \mathrm{s}^2_R(Y)$, given by $\mathrm{s}^2_R(f)\,(\overline{x\otimes y}) = \overline{f(x)\otimes f(y)}$. The equality $\mathrm{s}^2_R(f)\,(\overline{x\otimes x}) = \overline{f(x)\otimes f(x)}$ shows that $\mathrm{s}^2_R(f)$ induces a well-defined morphism $\mathrm{S}^2_R(f)\colon \mathrm{S}^2_R(X)\to \mathrm{S}^2_R(Y)$, given by $\mathrm{S}^2_R(f)\,(\overline{x\otimes y}) = \overline{f(x)\otimes f(y)}$. From the definition, one sees that the operators $\mathrm{s}^2_R(-)$ and

$\mathrm{S}^2_R(-)$ are functorial, but Example 5.7 shows that they are not additive, as one might expect.

The next two results show that the functors $\mathrm{s}^2_R(-)$ and $\mathrm{S}^2_R(-)$ interact well with basic constructions.

Proposition 3.6. *Let X be an R-complex.*

(a) *Given a ring homomorphism $\varphi\colon R \to S$, there are isomorphisms of S-complexes $\mathrm{s}^2_S(S \otimes_R X) \cong S \otimes_R \mathrm{s}^2_R(X)$ and $\mathrm{S}^2_S(S \otimes_R X) \cong S \otimes_R \mathrm{S}^2_R(X)$.*

(b) *If $\mathfrak{p} \subset R$ is a prime ideal, then there are isomorphisms of $R_{\mathfrak{p}}$-complexes $\mathrm{s}^2_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \cong \mathrm{s}^2_R(X)_{\mathfrak{p}}$ and $\mathrm{S}^2_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \cong \mathrm{S}^2_R(X)_{\mathfrak{p}}$.*

Proof. (a) The vertical isomorphisms in the following commutative diagram are given by $\beta((s \otimes x) \otimes (t \otimes y)) = (st) \otimes (x \otimes y)$:

$$\begin{array}{ccc}
(S \otimes_R X) \otimes_S (S \otimes_R X) & \xrightarrow{\alpha^{S\otimes_R X}} & (S \otimes_R X) \otimes_S (S \otimes_R X) \\
\cong \Big\downarrow \beta & & \cong \Big\downarrow \beta \\
S \otimes_R (X \otimes_R X) & \xrightarrow{S\otimes_R \alpha^X} & S \otimes_R (X \otimes_R X).
\end{array}$$

This diagram yields the first isomorphism in the next sequence. The second isomorphism is due to the right-exactness of $S \otimes_R -$, and the equalities are by definition,

$$\begin{aligned}
\mathrm{s}^2_S(S \otimes_R X) &= \mathrm{Coker}(\alpha^{S\otimes_R X}) \cong \mathrm{Coker}(S \otimes_R \alpha^X) \\
&\cong S \otimes_R \mathrm{Coker}(\alpha^X) = S \otimes_R \mathrm{s}^2_R(X).
\end{aligned}$$

By definition, the induced isomorphism $\overline{\beta}\colon \mathrm{s}^2_S(S \otimes_R X) \xrightarrow{\cong} S \otimes_R \mathrm{s}^2_R(X)$ is given by $\overline{\beta}\left(\overline{(s \otimes x) \otimes (t \otimes y)}\right) = \overline{(st) \otimes (x \otimes y)}$.

Let $Y \subseteq \mathrm{s}^2_S(S \otimes_R X)$ be the S-submodule generated by elements of the form $\overline{u \otimes u}$ such that $u \in S \otimes_R X$ has an odd degree. That is, $Y = \mathrm{Ker}(\omega^{S\otimes X})$ where $\omega^{S\otimes X}\colon \mathrm{s}^2_S(S \otimes_R X) \to \mathrm{S}^2_S(S \otimes_R X)$ is the natural surjection. It is straightforward to show that Y is generated over S by all elements of the form $\overline{(1 \otimes x) \otimes (1 \otimes x)}$.

Let $Z \subset \mathrm{s}^2_R(X)$ be the R-submodule generated by elements of the form $\overline{x \otimes x}$ with $x \in X$ of odd degree. That is, we have an exact sequence of

R-morphisms

$$0 \to Z \to \mathrm{s}^2_R(X) \xrightarrow{\omega^X} \mathrm{S}^2_R(X) \to 0.$$

Tensoring with S yields the next exact sequence of S-morphisms:

$$S \otimes_R Z \to S \otimes_R \mathrm{s}^2_R(X) \xrightarrow{S\otimes_R\omega^X} S \otimes_R \mathrm{S}^2_R(X) \to 0$$

and it follows that $\mathrm{Ker}(S \otimes_R \omega^X)$ is generated over S by all elements of the form $\overline{1 \otimes (x \otimes x)}$ with $x \in X$ of odd degree. Thus, the equality $\overline{\beta}\left(\overline{(1 \otimes x) \otimes (1 \otimes x)}\right) = \overline{1 \otimes (x \otimes x)}$ shows that $\overline{\beta}$ induces an S-isomorphism $\mathrm{S}^2_S(S \otimes_R X) \cong S \otimes_R \mathrm{S}^2_R(X)$.

(b) This follows from part (a) using the natural map $R \to R_{\mathfrak{p}}$. □

Proposition 3.7. *If X and Y are R-complexes, then there are isomorphisms*

$$\mathrm{s}^2_R(X \oplus Y) \cong \mathrm{s}^2_R(X) \oplus (X \otimes_R Y) \oplus \mathrm{s}^2_R(Y) \tag{3.7.1}$$

$$\mathrm{S}^2_R(X \oplus Y) \cong \mathrm{S}^2_R(X) \oplus (X \otimes_R Y) \oplus \mathrm{S}^2_R(Y). \tag{3.7.2}$$

Proof. (3.7.1) Tensor-distribution yields the horizontal isomorphisms in the following commutative diagram:

$$\begin{array}{ccc}
(X \oplus Y) \otimes (X \oplus Y) & \xrightarrow{\cong} & (X \otimes X) \oplus (X \otimes Y) \oplus (Y \otimes X) \oplus (Y \otimes Y) \\
\Big\downarrow{\scriptstyle \alpha^{X\oplus Y}} & & \Big\downarrow{\scriptstyle \begin{pmatrix} \alpha^X & 0 & 0 & 0 \\ 0 & \mathrm{id}_{X\otimes Y} & -\theta_{YX} & 0 \\ 0 & -\theta_{XY} & \mathrm{id}_{Y\otimes X} & 0 \\ 0 & 0 & 0 & \alpha^Y \end{pmatrix}} \\
(X \oplus Y) \otimes (X \oplus Y) & \xrightarrow{\cong} & (X \otimes X) \oplus (X \otimes Y) \oplus (Y \otimes X) \oplus (Y \otimes Y),
\end{array}$$

where, $\theta_{UV} \colon U \otimes_R V \to V \otimes_R U$ is the tensor-commutativity isomorphism given by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. This diagram yields the first isomorphism in the following sequence, while the first equality is by definition,

$$\begin{aligned}
\mathrm{s}^2_R(X \oplus Y) &= \mathrm{Coker}(\alpha^{X\oplus Y}) \\
&\cong \mathrm{Coker}\begin{pmatrix} \alpha^X & 0 & 0 & 0 \\ 0 & \mathrm{id}_{X\otimes_R Y} & -\theta_{YX} & 0 \\ 0 & -\theta_{XY} & \mathrm{id}_{Y\otimes_R X} & 0 \\ 0 & 0 & 0 & \alpha^Y \end{pmatrix} \\
&\cong \mathrm{Coker}(\alpha^X) \oplus \mathrm{Coker}\begin{pmatrix} \mathrm{id}_{X\otimes_R Y} & -\theta_{YX} \\ -\theta_{XY} & \mathrm{id}_{Y\otimes_R X} \end{pmatrix} \oplus \mathrm{Coker}(\alpha^Y) \\
&\cong \mathrm{s}^2_R(X) \oplus (X \otimes_R Y) \oplus \mathrm{s}^2_R(Y).
\end{aligned}$$

The second isomorphism is by elementary linear algebra. For the third isomorphism, using the definition of $\mathrm{s}^2_R(-)$, we need to prove $\mathrm{Coker}(\beta) \cong X \otimes_R Y$, where,

$$\beta = \begin{pmatrix} \mathrm{id}_{X\otimes_R Y} & -\theta_{YX} \\ -\theta_{XY} & \mathrm{id}_{Y\otimes_R X} \end{pmatrix} : (X \otimes_R Y) \oplus (Y \otimes_R X) \to (X \otimes_R Y) \oplus (Y \otimes_R X).$$

We set

$$\gamma = (\mathrm{id}_{X\otimes_R Y} \quad \theta_{YX},) \colon (X \otimes_R Y) \oplus (Y \otimes_R X) \to X \otimes_R Y,$$

which is a surjective morphism such that $\mathrm{Im}(\beta) \subseteq \mathrm{Ker}(\gamma)$. Thus, there is a well-defined surjective morphism $\overline{\gamma} \colon \mathrm{Coker}(\beta) \to X \otimes_R Y$, given by

$$\overline{\begin{pmatrix} x\otimes y \\ y'\otimes x' \end{pmatrix}} \mapsto x \otimes y + (-1)^{|x'||y'|} x' \otimes y'.$$

It remains to show that $\overline{\gamma}$ is injective. To this end, define $\delta \colon X \otimes_R Y \to \mathrm{Coker}(\beta)$ by the formula $x \otimes y \mapsto \overline{\begin{pmatrix} x\otimes y \\ 0 \end{pmatrix}}$. It is straightforward to show that δ is a well-defined morphism and that $\delta\overline{\gamma} = \mathrm{id}_{\mathrm{Coker}(\beta)}$. It follows that $\overline{\gamma}$ is injective, hence an isomorphism, as desired.

(3.7.2) The isomorphism $\beta \colon \mathrm{s}^2_R(X \oplus Y) \xrightarrow{\cong} \mathrm{s}^2_R(X) \oplus (X \otimes_R Y) \oplus \mathrm{s}^2_R(Y)$ from part (3.7.1) is given by the formula

$$\beta\left(\overline{(x, y) \otimes (x', y')}\right) = \left(\overline{x \otimes x'}, x \otimes y' + (-1)^{|x'||y|} x' \otimes y, \overline{y \otimes y'}\right).$$

Thus, for an element $(x, y) \in X \oplus Y$ of odd order $|x| = |(x, y)| = |y|$, we have

$$\begin{aligned}
\beta\left(\overline{(x, y) \otimes (x, y)}\right) &= \left(\overline{x \otimes x}, x \otimes y + (-1)^{|x||y|} x \otimes y, \overline{y \otimes y}\right) \\
&= (\overline{x \otimes x}, x \otimes y - x \otimes y, \overline{y \otimes y}) \\
&= (\overline{x \otimes x}, 0, \overline{y \otimes y}).
\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{S}^2_R(X\oplus Y) &\cong \frac{\mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus \mathrm{s}^2_R(Y)}{\langle(\overline{x\otimes x},0,\overline{y\otimes y})\mid x\in X \text{ and } y\in Y \text{ have odd degree}\rangle}\\ &\cong \frac{\mathrm{s}^2_R(X)}{\langle\overline{x\otimes x}\mid x\in X \text{ odd degree}\rangle}\oplus\frac{(X\otimes_R Y)}{0}\\ &\qquad\qquad\oplus\frac{\mathrm{s}^2_R(Y)}{\langle\overline{y\otimes y}\mid y\in Y \text{ odd degree}\rangle}\\ &\cong \mathrm{S}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{S}^2_R(Y),\end{aligned}$$

as desired. □

Example 3.4 shows why we need to assume that 2 is a unit in R in the next result.

Proposition 3.8. *Assume that 2 is a unit in R, and let X be an R-complex.*

(a) *The following exact sequences are split exact:*

$$0\to \mathrm{Ker}(\alpha^X)\xrightarrow{j^X} X\otimes_R X\xrightarrow{q^X}\mathrm{Im}(\alpha^X)\to 0$$

$$0\to \mathrm{Im}(\alpha^X)\xrightarrow{i^X} X\otimes_R X\xrightarrow{p^X}\mathrm{S}^2_R(X)\to 0,$$

where, i^X and j^X are the natural inclusions, p^X is the natural surjection, and q^X is induced by α^X. The splitting on the right of the first sequence is given by $\frac{1}{2}i^X$, and the splitting on the left of the second sequence is given by $\frac{1}{2}q^X$. In particular, there are isomorphisms

$$\mathrm{Im}(\alpha^X)\oplus\mathrm{Ker}(\alpha^X)\cong X\otimes_R X\cong \mathrm{Im}(\alpha^X)\oplus\mathrm{S}^2_R(X).$$

(b) *If X is a bounded-below complex of projective R-modules, then so are the complexes $\mathrm{Im}(\alpha^X)$, $\mathrm{Ker}(\alpha^X)$ and $\mathrm{S}^2_R(X)$.*

Proof. (a) The given exact sequences come from Properties (3.5.1) and (3.5.3). The fact that $\frac{1}{2}\alpha^X$ is idempotent tells us that i^X is a split injection with splitting given by $\frac{1}{2}q^X$ and q^X is a split surjection with splitting given by $\frac{1}{2}i^X$. The desired isomorphisms follow immediately from the splitting of the sequences.

(b) With the isomorphisms from part (a), the fact that $X\otimes_R X$ is a bounded-below complex of projective R-modules implies that $\mathrm{Im}(\alpha^X)$,

$\mathrm{Ker}(\alpha^X)$ and $\mathrm{S}^2_R(X)$ are also bounded-below complexes of projective R-modules. $\square$

The next two results explicitly describe the modules in $\mathrm{s}^2_R(X)$ and $\mathrm{S}^2_R(X)$. Note that the difference between parts (a)–(b) and part (c) shows that the behavior documented in Example 3.4 is, in a sense, the norm, not the exception.

Theorem 3.9. *Let X be a complex of R-modules. Fix an integer n and set $h = n/2$ and $V = \bigoplus_{m<h}(X_m \otimes X_{n-m})$.*

(a) *If n is odd, then $\mathrm{s}^2_R(X)_n \cong V$.*
(b) *If $n \equiv 0 \pmod 4$, then $\mathrm{s}^2_R(X)_n \cong V \bigoplus \mathrm{S}^2_R(X_h)$.*
(c) *Assume that $n \equiv 2 \pmod 4$.*
  (c1) *There is an isomorphism*
$$\mathrm{s}^2_R(X)_n \cong V \bigoplus \frac{X_h \otimes_R X_h}{\langle x \otimes x' + x' \otimes x \mid x, x' \in X_h\rangle}$$
  and there is a surjection $\tau\colon \mathrm{s}^2_R(X)_n \to V \oplus \wedge^2(X_h)$ with $\mathrm{Ker}(\tau)$ generated by $\{\overline{x \otimes x} \in \mathrm{s}^2_R(X)_n \mid x \in X_h\}$.
  (c2) *If X_h is projective, then $\mathrm{s}^2_R(X)_n \cong V \bigoplus \wedge^2(X_h) \bigoplus K$, for some R-module K that is a homomorphic image of $X_h/2X_h$.*
  (c3) *If X_h is projective and 2 is a unit in R, then $\mathrm{s}^2_R(X)_n \cong V \bigoplus \wedge^2(X_h)$.*

Proof. (a) Assume that n is odd. Let $\gamma\colon (X \otimes X)_n \to V \oplus V$ be given on generators by the formula
$$\gamma(x \otimes x') = \begin{cases} (x \otimes x', 0), & \text{if } |x| < h \\ (0, x' \otimes x), & \text{if } |x| > h. \end{cases}$$
Since n is odd, this is a well-defined isomorphism. Let $g\colon V \oplus V \to V \oplus V$ be given by $g(v, v') = (v - v', v' - v)$. This yields a commutative diagram

$$\text{(3.9.1)} \qquad \begin{array}{ccc} (X \otimes_R X)_n & \xrightarrow{\alpha^X_n} & (X \otimes_R X)_n \\ \cong\downarrow\gamma & & \cong\downarrow\gamma \\ V \oplus V & \xrightarrow{g} & V \oplus V. \end{array}$$

Note that the commutativity depends on the fact that n is odd, because it implies that $|x||x'|$ is even for each $x \otimes x' \in (X \otimes_R X)_n$.

The map $f\colon V \oplus V \to V$ given by $f(v, v') = v + v'$ is a surjective homomorphism with $\mathrm{Ker}(f) = \langle (v, 0) - (0, v) \mid v \in V\rangle = \mathrm{Im}(g)$. This explains the last isomorphism in the next sequence:

$$\mathrm{s}^2_R(X)_n = \mathrm{Coker}(\alpha^X_n) \cong \mathrm{Coker}(g) \cong V.$$

The other isomorphism follows from diagram (3.9.1).

(b)–(c) When n is even, we have a similar commutative diagram:

$$\begin{array}{ccc} (X \otimes_R X)_n & \xrightarrow{\alpha^X_n} & (X \otimes_R X)_n \\ \cong \downarrow \gamma' & & \cong \downarrow \gamma' \\ V \oplus V \oplus (X_h \otimes X_h) & \xrightarrow{g'} & V \oplus V \oplus (X_h \otimes X_h), \end{array} \tag{3.9.2}$$

where, γ' and g' are given by

$$\gamma'(x \otimes x') = \begin{cases} (x \otimes x', 0, 0) & \text{if } |x| < h \\ (0, x' \otimes x, 0) & \text{if } |x| > h \\ (0, 0, x \otimes x') & \text{if } |x| = h. \end{cases}$$

$$\begin{aligned} g'(v, v', x \otimes x') &= (v - v', v' - v, x \otimes x' - (-1)^{h^2} x' \otimes x) \\ &= (v - v', v' - v, x \otimes x' - (-1)^{h} x' \otimes x). \end{aligned}$$

In other words, we have $g' = g \oplus \widetilde{\alpha}$, where, $\widetilde{\alpha}\colon X_h \otimes_R X_h \to X_h \otimes_R X_h$ is given by

$$\widetilde{\alpha}(x \otimes x') := x \otimes x' - (-1)^h x' \otimes x.$$

The following sequence of isomorphisms follows directly:

$$\begin{aligned} \mathrm{s}^2_R(X)_n = \mathrm{Coker}(\alpha^X_n) &\cong \mathrm{Coker}(g') \\ &\cong \mathrm{Coker}(g) \oplus \mathrm{Coker}(\widetilde{\alpha}) \cong V \oplus \mathrm{Coker}(\widetilde{\alpha}). \end{aligned}$$

If $n \equiv 0 \pmod 4$, then h is even, and so we have

$$\mathrm{Coker}(\widetilde{\alpha}) \cong \frac{X_h \otimes_R X_h}{\langle x \otimes x' - x' \otimes x \mid x, x' \in X_h\rangle} \cong \mathrm{S}^2_R(X_h).$$

For the remainder of the proof, we assume that $n \equiv 2 \pmod 4$, that is, that h is odd. In this case, we have

$$\mathrm{Coker}(\widetilde{\alpha}) \cong \frac{X_h \otimes_R X_h}{\langle x \otimes x' + x' \otimes x \mid x, x' \in X_h\rangle}. \tag{3.9.3}$$

It is straightforward to show that

$$\langle x \otimes x' + x' \otimes x \mid x, x' \in X_h\rangle \subseteq \langle x \otimes x \mid x \in X_h\rangle.$$

Hence, there is an epimorphism

$$\tau_1\colon \operatorname{Coker}(\widetilde{\alpha}) \to \frac{X_h \otimes_R X_h}{\langle x \otimes x \mid x \in X_h \rangle} \cong \wedge^2(X_h)$$

such that

$$(3.9.4) \qquad \begin{aligned} \operatorname{Ker}(\tau_1) &= \langle \overline{x \otimes x} \in \operatorname{Coker}(\widetilde{\alpha}) \mid x \in X_h \rangle \\ &\cong \langle \overline{x \otimes x} \in \mathrm{s}^2_R(X)_n \mid x \in X_h \rangle. \end{aligned}$$

The conclusions of part (c1) follow from setting $\tau = \mathrm{id}_V \oplus \tau_1$.

For the rest of the proof, we assume that X_h is projective. It follows that $\wedge^2(X_h)$ is also projective, and hence the surjection τ_1 splits. Setting $K = \operatorname{Ker}(\tau_1)$, we have $\mathrm{s}^2_R(X)_n \cong V \bigoplus \wedge^2(X_h) \oplus K$. Using (3.9.3) and (3), we see that the map $\pi\colon X_h \to \operatorname{Ker}(\tau_1)$, given by $x \mapsto \overline{x \otimes x}$, is surjective with $2X_h \subseteq \operatorname{Ker}(\pi)$. It follows that K is a homomorphic image of $X_h/2X_h$, which establishes part (c2). Finally, part (c3) follows directly from (c2): if 2 is a unit in R, then $X_h/2X_h = 0$. $\square$

Theorem 3.10. *Let X be a complex of R-modules. Fix an integer n and set $h = n/2$ and $V = \bigoplus_{m<h}(X_m \otimes X_{n-m})$.*

(a) *If n is odd, then $\mathrm{S}^2_R(X)_n \cong V$.*
(b) *If $n \equiv 0 \pmod 4$, then $\mathrm{S}^2_R(X)_n \cong V \bigoplus \mathrm{S}^2_R(X_h)$.*
(c) *If $n \equiv 2 \pmod 4$, then $\mathrm{S}^2_R(X)_n \cong V \bigoplus \wedge^2(X_h)$.*

Proof. Set $Y = \left\langle \overline{x \otimes x} \in \mathrm{s}^2_R(X) \,\middle|\, x \in X \text{ odd degree} \right\rangle \subseteq \mathrm{s}^2_R(X)$.

(a)–(b) If n is odd or $n \equiv 0 \pmod 4$, then $Y_n = 0$; hence, $\mathrm{S}^2_R(X)_n \cong \mathrm{s}^2_R(X)_n$, and the desired conclusions follow from Theorem 3.9(a)–(b).

(c) Assume that $n \equiv 2 \pmod 4$. The epimorphism $\tau\colon \mathrm{s}^2_R(X)_n \to V \oplus \wedge^2(X_h)$ from Theorem 3.9(c1) has $\operatorname{Ker}(\tau) = \langle \overline{x \otimes x} \in \mathrm{s}^2_R(X)_n \mid x \in X_h \rangle$; that is, $\operatorname{Ker}(\tau) = Y_n$, and so we have

$$V \bigoplus \wedge^2(X_h) \cong \mathrm{s}^2_R(X)_n / Y_n \cong \mathrm{S}^2_R(X)_n,$$

as desired. $\square$

We state the next result for $\mathrm{S}^2_R(X)$ only, because Theorem 3.9 shows that it is only reasonable to consider such formulas for $\mathrm{s}^2_R(X)$, when 2 is a unit; in this case, the formulas are the same because of the isomorphism $\mathrm{s}^2_R(X) \cong \mathrm{S}^2_R(X)$.

Corollary 3.11. *Let X be a bounded-below complex of finite rank free R-modules. For each integer l, set $r_l = \operatorname{rank}_R(X_l)$. Then, each R-module $\mathrm{S}^2_R(X)_n$ is free and*

$$\operatorname{rank}_R((\mathrm{S}^2_R(X)_n) = \begin{cases} \sum_{m<h} r_m r_{n-m}, & \text{if } n \text{ is odd} \\ \binom{r_h+1}{2} + \sum_{m<h} r_m r_{n-m}, & \text{if } n \equiv 0 \pmod 4 \\ \binom{r_h}{2} + \sum_{m<h} r_m r_{n-m}, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. Using the notation of Theorem 3.10, we have

$$V = \bigoplus_{m<h} (X_m \otimes X_{n-m}) \cong \bigoplus_{m<h} (R^{r_m} \otimes R^{r_{n-m}}) \cong \bigoplus_{m<h} R^{r_m r_{n-m}}$$

and, when n is even,

$$\mathrm{S}^2_R(X_h) \cong \mathrm{S}^2_R(R^{r_h}) \cong R^{\binom{r_h+1}{2}} \qquad \wedge^2(X_h) \cong \wedge^2(R^{r_h}) \cong R^{\binom{r_h}{2}}.$$

The desired formula now follows from Theorem 3.10. □

Remark 3.12. There are several ways to present the formula in Corollary 3.11. One other way to write it is the following:

$$\operatorname{rank}_R((\mathrm{S}^2_R(X)_n) = \begin{cases} \frac{1}{2}\operatorname{rank}_R((X \otimes_R X)_n), & \text{if } n \text{ is odd} \\ \frac{1}{2}\operatorname{rank}_R((X \otimes_R X)_n) + \frac{1}{2} r_h, & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{2}\operatorname{rank}_R((X \otimes_R X)_n) - \frac{1}{2} r_h, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Another way is in terms of generating functions: for a complex Y of free R-modules, set $P^R_Y(t) = \sum_n \operatorname{rank}_R(Y_n) t^n$. Note that this is not usually the same as the Poincaré series of Y. It is the same if and only if R is local and Y is minimal. Using the previous display, we can then write:

$$(3.12.1) \quad P^R_{\mathrm{S}^2_R(X)}(t) = \tfrac{1}{2}\left[P^R_{X\otimes_R X}(t) + P^R_X(-t^2)\right] = \tfrac{1}{2}\left[P^R_X(t)^2 + P^R_X(-t^2)\right].$$

We make use of this expression several times in what follows.

4. Homological Properties of $\mathrm{S}^2_R(X)$

This section documents the homological and homotopical aspects of the functor $\mathrm{S}^2_R(-)$. It also contains our proof of Theorem A from the introduction. We assume throughout this section that 2 is a unit in R, and it follows that $\mathrm{S}^2_R(X) \cong \mathrm{s}^2_R(X)$ via the natural map for all X.

We begin with the following result showing that $\mathrm{S}^2_R(X)$ exhibits properties similar to those for $X \otimes_R X$, noted as in Fact 2.11. Example 3.4 shows what goes wrong in part (b) when $\inf(X)$ is odd: we have $\mathrm{S}^2_R(\Sigma R) = 0$, so $\inf(\mathrm{S}^2_R(\Sigma R)) = \infty > 2 = 2\inf(\Sigma R)$. Note that we do not need R to be local in either part of this result.

Proposition 4.1. *Assume that 2 is a unit in R and let X be a bounded-below complex of projective R-modules with $i = \inf(X)$.*

(a) *There is an inequality $\inf(\mathrm{S}^2_R(X)) \geqslant 2i$, and there is an isomorphism*

$$\mathrm{H}_{2i}(\mathrm{S}^2_R(X)) \cong \begin{cases} \mathrm{S}^2_R(\mathrm{H}_i(X)), & \text{if } i \text{ is even,} \\ \dfrac{\mathrm{H}_i(X) \otimes \mathrm{H}_i(X)}{\langle x \otimes y + y \otimes x \mid x, y \in \mathrm{H}_i(X)\rangle}, & \text{if } i \text{ is odd.} \end{cases}$$

(b) *Assume that R is noetherian and that $\mathrm{H}_i(X)$ is finitely generated. If i is even, then $\inf(\mathrm{S}^2_R(X)) = 2i$.*

Proof. (a) Proposition 3.8(b) yields an isomorphism:

$$\mathrm{Im}(\alpha^X) \oplus \mathrm{S}^2_R(X) \cong X \otimes_R X.$$

This isomorphism yields the first inequality in the next sequence

$$\inf(\mathrm{S}^2_R(X)) \geqslant \inf(X \otimes_R X) \geqslant 2i,$$

while the second inequality is from Fact 2.11.

The split exact sequences from Proposition 3.8(a) fit together in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\alpha^X) & \xrightarrow{j^X} & X \otimes_R X & \xrightarrow{q^X} & \mathrm{Im}(\alpha^X) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle q^X} & \searrow^{\alpha^X} & \downarrow{\scriptstyle i^X} & & \\ 0 & \longrightarrow & \mathrm{Im}(\alpha^X) & \xrightarrow{i^X} & X \otimes_R X & \xrightarrow{p^X} & \mathrm{S}^2_R(X) & \longrightarrow & 0. \end{array} \tag{4.1.1}$$

Define $\widetilde{\alpha}\colon \mathrm{H}_i(X) \otimes_R \mathrm{H}_i(X) \to \mathrm{H}_i(X) \otimes_R \mathrm{H}_i(X)$ by the formula

$$\overline{x} \otimes \overline{x'} \mapsto \overline{x} \otimes \overline{x'} - (-1)^{i^2}\overline{x'} \otimes \overline{x} = \overline{x} \otimes \overline{x'} - (-1)^{i}\overline{x'} \otimes \overline{x}.$$

It is straightforward to show that the following diagram commutes:

$$(4.1.2)\qquad \begin{array}{ccc} \mathrm{H}_{2i}(X \otimes_R X) & \xrightarrow{\mathrm{H}_{2i}(\alpha^X)} & \mathrm{H}_{2i}(X \otimes_R X) \\ \cong \downarrow \gamma & & \gamma \downarrow \cong \\ \mathrm{H}_i(X) \otimes_R \mathrm{H}_i(X) & \xrightarrow{\widetilde{\alpha}} & \mathrm{H}_i(X) \otimes_R \mathrm{H}_i(X), \end{array}$$

where, the isomorphism γ is from Fact 2.11. Together, diagrams (**??**) and (4.1.2) yield the next commutative diagram:

$$\begin{array}{ccccccc} \mathrm{H}_i(X) \otimes \mathrm{H}_i(X) & \xrightarrow{\mathrm{H}_{2i}(q^X)\gamma^{-1}} & \mathrm{H}_{2i}(\mathrm{Im}(\alpha^X)) & \longrightarrow & 0 & & \\ \downarrow \mathrm{H}_{2i}(q^X)\gamma^{-1} & \searrow^{\widetilde{\alpha}} & \downarrow \gamma\,\mathrm{H}_{2i}(i^X) & & & & \\ \mathrm{H}_{2i}(\mathrm{Im}(\alpha^X)) & \xrightarrow{\gamma\,\mathrm{H}_{2i}(i^X)} & \mathrm{H}_i(X) \otimes \mathrm{H}_i(X) & \xrightarrow{\mathrm{H}_{2i}(p^X)\gamma^{-1}} & \mathrm{H}_{2i}(\mathrm{S}^2_R(X)) & \longrightarrow & 0, \end{array}$$

whose rows are exact because the rows of diagram (**??**) are split exact. A straightforward diagram-chase yields the equality $\mathrm{Ker}(\mathrm{H}_{2i}(p^X)\gamma^{-1}) = \mathrm{Im}(\widetilde{\alpha})$, and so

$$\begin{aligned} \mathrm{H}_{2i}(\mathrm{S}^2_R(X)) &\cong \frac{\mathrm{H}_i(X) \otimes_R \mathrm{H}_i(X)}{\mathrm{Im}(\widetilde{\alpha})} \\ &\cong \begin{cases} \mathrm{S}^2_R(\mathrm{H}_i(X)), & \text{if } i \text{ is even} \\ \dfrac{H_i(X) \otimes H_i(X)}{\langle x \otimes y + y \otimes x \mid x, y \in \mathrm{H}_i(X)\rangle}, & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

(b) Using part (a), it suffices to to show that $\mathrm{S}^2_R(\mathrm{H}_i(X)) \neq 0$, where $i = \inf(X)$. Fix a maximal ideal $\mathfrak{m} \in \mathrm{Supp}_R(\mathrm{H}_i(X))$, and set $k = R/\mathfrak{m}$. Using the isomorphisms

$$k \otimes_R \mathrm{H}_i(X) \cong (k \otimes_{R_\mathfrak{m}} R_\mathfrak{m}) \otimes_R \mathrm{H}_i(X) \cong k \otimes_{R_\mathfrak{m}} \mathrm{H}_i(X)_\mathfrak{m} \cong k \otimes_{R_\mathfrak{m}} \mathrm{H}_i(X_\mathfrak{m}),$$

Nakayama's Lemma implies that $k \otimes_R \mathrm{H}_i(X)$ is a nonzero k-vector space of finite rank, say $k \otimes_R \mathrm{H}_i(X) \cong k^r$. In the following sequence, the first and third isomorphisms are well-known; see, e.g., [7, (A2.2.b) and (A2.3.c)]:

$$k \otimes_R \mathrm{S}^2_R(\mathrm{H}_i(X)) \cong \mathrm{S}^2_k(k \otimes_R \mathrm{H}_i(X)) \cong \mathrm{S}^2_k(k^r) \cong k^{\binom{r+1}{r-1}} \neq 0.$$

It follows that $\mathrm{S}^2_R(\mathrm{H}_i(X)) \neq 0$, as desired. □

The next result establishes the homotopy-theoretic properties of the functor $\mathrm{S}^2_R(-)$. Example 5.6 shows that conclusion fails when 2 is not a unit in R. Note that we cannot reduce part (a) to the case $g = 0$ by replacing f by $f - g$, as Example 5.7 shows that $\mathrm{S}^2_R(f - g)$ might not equal $\mathrm{S}^2_R(f) - \mathrm{S}^2_R(g)$.

Theorem 4.2. *Assume that 2 is a unit in R, and let X and Y be R-complexes. Fix morphisms $f, g\colon X \to Y$ and $h\colon Y \to X$.*

(a) *If f and g are homotopic, then $\mathrm{S}^2_R(f)$ and $\mathrm{S}^2_R(g)$ are homotopic.*
(b) *If f is a homotopy equivalence with homotopy inverse h, then $\mathrm{S}^2_R(f)$ is a homotopy equivalence with homotopy inverse $\mathrm{S}^2_R(h)$.*

Proof. (a) Fix a homotopy s from f to g as in Definition 2.3. Define

$$f \otimes_R s + s \otimes_R g = \{(f \otimes_R s + s \otimes_R g)_n\colon (X \otimes_R X)_n \to (Y \otimes_R Y)_{n+1}\}$$
$$g \otimes_R s + s \otimes_R f = \{(g \otimes_R s + s \otimes_R f)_n\colon (X \otimes_R X)_n \to (Y \otimes_R Y)_{n+1}\}$$

on each generator $x \otimes x' \in (X \otimes_R X)_n$ by the formulas

$$(f \otimes_R s + s \otimes_R g)_n(x \otimes x') := (-1)^p f_p(x) \otimes s_q(x') + s_p(x) \otimes g_q(x')$$
$$(g \otimes_R s + s \otimes_R f)_n(x \otimes x') := (-1)^p g_p(x) \otimes s_q(x') + s_p(x) \otimes f_q(x'),$$

where, $p = |x|$ and $q = |x'|$. One checks readily that the sequences $f \otimes_R s + s \otimes_R g$ and $g \otimes_R s + s \otimes_R f$ are homotopies from $f \otimes_R f$ to $g \otimes_R g$. As 2 is a unit in R, it follows that the sequence

$$\sigma = \tfrac{1}{2}(f \otimes_R s + s \otimes_R g + g \otimes_R s + s \otimes_R f)$$

is also a homotopy from $f \otimes_R f$ to $g \otimes_R g$. It is straightforward to show that $\sigma_n \alpha^X_n = \alpha^Y_{n+1}\sigma_n$, for all n. Using the fact that σ is a homotopy from $f \otimes_R f$ to $g \otimes_R g$, it is thus straightforward to show that σ induces a homotopy $\overline{\sigma}$ from $\mathrm{S}^2_R(f)$ to $\mathrm{S}^2_R(g)$ by the formula $\overline{\sigma}_n\left(\overline{x \otimes x'}\right) = \overline{\sigma_n(x \otimes x')}$.

(b) By hypothesis, the composition hf is homotopic to id_X. Part (a) implies that $\mathrm{S}^2_R(hf) = \mathrm{S}^2_R(h)\mathrm{S}^2_R(f)$ is homotopic to $\mathrm{S}^2_R(\mathrm{id}_X) = \mathrm{id}_{\mathrm{S}^2_R(X)}$. The same logic implies that $\mathrm{S}^2_R(f)\mathrm{S}^2_R(h)$ is homotopic to $\mathrm{id}_{\mathrm{S}^2_R(Y)}$, and hence the desired conclusions are reached. □

For the next results, Examples 5.5 and 5.6 show why we need to assume that X and Y are bounded-below complexes of projective R-modules and 2 is a unit in R.

Corollary 4.3. *Assume that 2 is a unit in R, and let X and Y be bounded-below complexes of projective R-modules.*

(a) *If* $f\colon X \to Y$ *is a quasiisomorphism, then so is* $\mathrm{S}^2_R(f)\colon \mathrm{S}^2_R(X) \to \mathrm{S}^2_R(Y)$.

(b) *If* $X \simeq Y$, *then* $\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(Y)$.

Proof. (a) Our assumptions imply that f is a homotopy equivalence by Fact 2.5, so the desired conclusion follows from Theorem 4.2(b).

(b) Assume $X \simeq Y$. Because X and Y are bounded-below complexes of projective R-modules, there is a quasiisomorphism $f\colon X \xrightarrow{\simeq} Y$. Now, apply part (a). □

Corollary 4.4. *If 2 is a unit in* R *and* X *is a bounded-below complex of projective* R*-modules, then there is a containment* $\mathrm{Supp}_R(\mathrm{S}^2_R(X)) \subseteq \mathrm{Supp}_R(X)$.

Proof. Fix a prime ideal $\mathfrak{p} \notin \mathrm{Supp}_R(X)$. It suffices to show that $\mathfrak{p} \notin \mathrm{Supp}_R(\mathrm{S}^2_R(X))$. The first isomorphism in the following sequence is from Proposition 3.6(b):

$$\mathrm{S}^2_R(X)_{\mathfrak{p}} \cong \mathrm{S}^2_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) \simeq \mathrm{S}^2_{R_{\mathfrak{p}}}(0) = 0.$$

The quasiisomorphism is from Corollary 4.3(b), because $X_{\mathfrak{p}} \simeq 0$. □

The following result is a key for our proof of Theorem A.

Theorem 4.5. *Assume that* R *is noetherian and local and that 2 is a unit in* R. *Let* X *be a bounded-below complex of finite-rank free* R*-modules. The following conditions are equivalent:*

(i) *the surjection* $p^X\colon X \otimes_R X \to \mathrm{S}^2_R(X)$ *is a quasiisomorphism;*

(ii) $\mathrm{Im}(\alpha^X) \simeq 0$;

(iii) *the injection* $j^X\colon \mathrm{Ker}(\alpha^X) \to X \otimes_R X$ *is a quasiisomorphism;*

(iv) *either* $X \simeq 0$ *or* $X \simeq \Sigma^{2n} R$, *for some integer* n.

Proof. (i) The biimplications (i) $\iff$ (ii) $\iff$ (iii) follow easily from the long exact sequences associated with the exact sequences in Proposition 3.8(a).

(iv) $\implies$ (i). If $X \simeq 0$, then $X \otimes_R X \simeq 0 \simeq \mathrm{S}^2_R(X)$ and so p^X is trivially a quasiisomorphism; see Fact 2.11 and Example 3.3.

Assuming that $X \simeq \Sigma^{2n}R$, there is a quasiisomorphism $\gamma\colon R \xrightarrow{\simeq} \Sigma^{-2n}X$. The commutative diagrams from (3.5.2) and (3.5.4) can be combined and augmented to form the following commutative diagram:

$$\begin{array}{ccccccc}
R\otimes R & \xrightarrow{\alpha^R} & R\otimes R & \xrightarrow[\cong]{p^R} & \mathrm{S}^2_R(R) & \longrightarrow & 0 \\
\simeq\big\downarrow{\scriptstyle\gamma\otimes\gamma} & & \simeq\big\downarrow{\scriptstyle\gamma\otimes\gamma} & & \simeq\big\downarrow{\scriptstyle\mathrm{S}^2(\gamma)} & & \\
(\Sigma^{-2n}X)\otimes(\Sigma^{-2n}X) & \xrightarrow{\alpha^{\Sigma^{-2n}X}} & (\Sigma^{-2n}X)\otimes(\Sigma^{-2n}X) & \xrightarrow{p^{\Sigma^{-2n}X}} & \mathrm{S}^2_R(\Sigma^{-2n}X) & \longrightarrow & 0 \\
\cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow & & \\
\Sigma^{-4n}(X\otimes X) & \xrightarrow{\Sigma^{-4n}\alpha^X} & \Sigma^{-4n}(X\otimes X) & \xrightarrow{\Sigma^{-4n}p^X} & \Sigma^{-4n}\mathrm{S}^2_R(X) & \longrightarrow & 0.
\end{array}$$

The morphism $\gamma\otimes\gamma$ is a quasiisomorphism by Fact 2.11, and $\mathrm{S}^2(\gamma)$ is a quasiisomorphism by Corollary 4.3(a). One checks readily that $\alpha^R = 0$, and so p^R is an isomorphism. The diagram shows that $p^{\Sigma^{-2n}X}$ is a quasiisomorphism, and hence so is $\Sigma^{-4n}p^X$. It follows that p^X is a quasiisomorphism, as desired.

(i) $\Longrightarrow$ (iv). Assume that the surjection $p^X\colon X\otimes_R X \to \mathrm{S}^2_R(X)$ is a quasiisomorphism and $X \not\simeq 0$.

Case 1: X is minimal. This implies that $X\otimes_R X$ is minimal. Also, since $\mathrm{S}^2_R(X)$ is a direct summand of $X\otimes_R X$, it follows that $\mathrm{S}^2_R(X)$ is also minimal. The fact that p^X is a quasiisomorphism implies that it is an isomorphism; see Fact 2.9. This explains the second equality in the next sequence:

$$P^R_X(t)^2 = P^R_{X\otimes_R X}(t) = P^R_{\mathrm{S}^2_R(X)}(t) = \tfrac{1}{2}\left[P^R_X(t)^2 + P^R_X(-t^2)\right].$$

The third equality is from equation (3.12). It follows that

$$P^R_X(t)^2 = P^R_X(-t^2). \tag{4.5.1}$$

Let $i = \inf(X)$ and note that $r_i \geqslant 1$. Set $r_n = \operatorname{rank}_R(X_{n-i})$, for each n, and $Q(t) = \sum_{n=0}^{\infty} r_{n+i}t^n$, so that we have $P^R_X(t) = t^iQ(t)$. Equation (4) then reads as $t^{2i}Q(t)^2 = (-1)^i t^{2i}Q(-t^2)$, that is, we have

$$Q(t)^2 - (-1)^iQ(-t^2) = 0. \tag{4.5.2}$$

If i were odd, then this would say $Q(t)^2 + Q(-t^2) = 0$, contradicting Lemma 2.12(a). It follows that $i = 2n$, for some n. Equation (4) then says $Q(t)^2 - Q(-t^2) = 0$, and so Lemma 2.12(b) implies that $Q(t) = 1$. This says that $P^R_X(t) = t^i = t^{2n}$, and so $X \cong \Sigma^{2n}R$, as desired.

Case 2: the general case. Let $\delta\colon P \xrightarrow{\simeq} X$ be a minimal free resolution. We again augment the commutative diagram from (3.5.4):

$$\begin{array}{ccccccc}
P \otimes_R P & \xrightarrow{\alpha^P} & P \otimes_R P & \xrightarrow{p^P} & \mathrm{S}^2_R(P) & \longrightarrow & 0 \\
\simeq \downarrow \delta\otimes\delta & & \simeq \downarrow \delta\otimes\delta & & \simeq \downarrow \mathrm{S}^2(\delta) & & \\
X \otimes_R X & \xrightarrow{\alpha^X} & X \otimes_R X & \xrightarrow[\simeq]{p^X} & \mathrm{S}^2_R(X) & \longrightarrow & 0.
\end{array}$$

This implies that p^P is a quasiisomorphism. Since P is minimal, Case 1 implies that either $P \simeq 0$ or $P \simeq \Sigma^{2n} R$, for some integer n. Since we have $X \simeq P$, the desired conclusion follows. □

Remark 4.6. One can remove the local assumption and change the word "free" to "projective" in Theorem 4.5, if one replaces condition (iv) with the following condition: (iv') for every maximal ideal $\mathfrak{m} \subset R$, one has either $X_\mathfrak{m} \simeq 0$ or $X_\mathfrak{m} \simeq \Sigma^{2n} R_\mathfrak{m}$, for some integer n. (Here, the integer n depends on the choice of $\mathfrak{m}$.) While this gives the illusion of greater generality, this version is equivalent to Theorem 4.5 because each of the conditions (i)–(iii) and (iv') is local. Hence, we state only the local versions of our results, with the knowledge that nonlocal versions are direct consequences. On the other hand, Example 5.8 shows that one needs to take care when removing the local hypotheses from our results.

We next show how Theorem A is a consequence of Theorem 4.5.

4.7. *Proof of Theorem A.* The assumption $X_\mathfrak{p} \simeq S_\mathfrak{p} \neq 0$, for each $\mathfrak{p} \in \mathrm{Ass}(R)$, implies $X \not\simeq 0$ and $\inf(X) \leqslant \inf(X_\mathfrak{p}) = 0$. On the other hand, since $X_n = 0$, for all $n < 0$, we know $\inf(X) \geqslant 0$, and so $\inf(X) = 0$.

Consider the split exact sequence from Proposition 3.8(a):

$$0 \to \mathrm{Im}(\alpha^X) \xrightarrow{i^X} X \otimes_S X \xrightarrow{p^X} \mathrm{S}^2_S(X) \to 0. \tag{4.7.1}$$

This sequence splits, and so $\mathrm{H}_n(\mathrm{Im}(\alpha^X)) \hookrightarrow \mathrm{H}_n(X \otimes_S X)$, for each n; hence,

$$\mathrm{Ass}_R(\mathrm{H}_n(\mathrm{Im}(\alpha^X))) \subseteq \mathrm{Ass}_R(\mathrm{H}_n(X \otimes_S X)) \subseteq \mathrm{Ass}(R). \tag{4.7.2}$$

For each $\mathfrak{p} \in \operatorname{Ass}(R)$, localization of (4.7) yields the exactness of the rows of the following commutative diagram; see also Proposition 3.6(b):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}(\alpha^X)_\mathfrak{p} & \xrightarrow{(i^X)_\mathfrak{p}} & (X \otimes_S X)_\mathfrak{p} & \xrightarrow{(p^X)_\mathfrak{p}} & \mathrm{S}^2_S(X)_\mathfrak{p} & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \operatorname{Im}(\alpha^{X_\mathfrak{p}}) & \xrightarrow{i^{X_\mathfrak{p}}} & X_\mathfrak{p} \otimes_{S_\mathfrak{p}} X_\mathfrak{p} & \xrightarrow{p^{X_\mathfrak{p}}} & \mathrm{S}^2_{S_\mathfrak{p}}(X_\mathfrak{p}) & \longrightarrow & 0 \end{array}$$

The quasiisomorphism $X_\mathfrak{p} \simeq S_\mathfrak{p}$ implies that $p^{X_\mathfrak{p}}$ is also a quasiisomorphism by Theorem 4.5, and so the previous sequence implies $\operatorname{Im}(\alpha^X)_\mathfrak{p} \cong \operatorname{Im}(\alpha^{X_\mathfrak{p}}) \simeq 0$, for each $\mathfrak{p} \in \operatorname{Ass}(R)$. For each n and $\mathfrak{p}$, this implies $\mathrm{H}_n(\operatorname{Im}(\alpha^X))_\mathfrak{p} \cong \mathrm{H}_n(\operatorname{Im}(\alpha^X)_\mathfrak{p}) = 0$; the containment in (4.7) implies $\mathrm{H}_n(\operatorname{Im}(\alpha^X)) = 0$, for each n, that is, $\operatorname{Im}(\alpha^X) \simeq 0$. Hence, Theorem 4.5 implies $X \simeq S$. □

The next result is a companion to Theorem 4.5.

Theorem 4.8. *Assume that R is noetherian and local, and that 2 is a unit in R. Let X be a bounded-below complex of finite rank free R-modules. The following conditions are equivalent:*

(i) *the morphism $\alpha^X \colon X \otimes_R X \to X \otimes_R X$ is a quasiisomorphism;*
(ii) *the surjection $q^X \colon X \otimes_R X \to \operatorname{Im}(\alpha^X)$ is a quasiisomorphism;*
(iii) *the injection $i^X \colon \operatorname{Im}(\alpha^X) \to X \otimes_R X$ is a quasiisomorphism;*
(iv) $\mathrm{S}^2_R(X) \simeq 0$*;*
(v) $\operatorname{Ker}(\alpha^X) \simeq 0$*;*
(vi) *$X \simeq 0$ or $X \simeq \Sigma^{2n+1} R$, for some integer n.*

Proof. The biimplications (ii) $\iff$ (v) and (iii) $\iff$ (iv) follow easily from the long exact sequences associated with the exact sequences in Proposition 3.8(a).

For the remainder of the proof, we use the easily verified fact that the exact sequences from Proposition 3.8(a) fit together in the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \operatorname{Ker}(\alpha^X) & \xrightarrow{j^X} & X \otimes_R X & \xrightarrow{q^X} & \operatorname{Im}(\alpha^X) & \longrightarrow & 0 & & \\ & & & & {\scriptstyle q^X}\downarrow & \searrow^{\alpha^X} & \downarrow{\scriptstyle i^X} & & & & \\ & & 0 & \longrightarrow & \operatorname{Im}(\alpha^X) & \xrightarrow{i^X} & X \otimes_R X & \xrightarrow{p^X} & \mathrm{S}^2_R(X) & \longrightarrow & 0 \end{array} \tag{4.8.1}$$

and we recall that these exact sequences split.

(i) $\implies$ (iv). Assume that α^X is a quasiisomorphism.

Case 1: X is minimal. In this case, the complex $X \otimes_R X$ is minimal, and so the fact that α^X is a quasiisomorphism implies that α^X is an isomorphism; see Fact 2.9. Hence, we have $\mathrm{S}^2_R(X) = \mathrm{Coker}(\alpha^X) = 0$.

Case 2: the general case. Let $f\colon P \xrightarrow{\simeq} X$ be a minimal free resolution. The commutative diagram from (3.5.4),

$$\begin{array}{ccc} P \otimes_R P & \xrightarrow{\alpha^P} & P \otimes_R P \\ {\scriptstyle f\otimes_R f}\downarrow{\scriptstyle \simeq} & & {\scriptstyle f\otimes_R f}\downarrow{\scriptstyle \simeq} \\ X \otimes_R X & \xrightarrow[\simeq]{\alpha^X} & X \otimes_R X \end{array}$$

shows that α^P is a quasiisomorphism; see Fact 2.11. Using Corollary 4.3(a), Case 1 implies that $\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(P) = 0$.

(iv) $\implies$ (v) and (iv) $\implies$ (i) and (iv) $\implies$ (vi). Assume $\mathrm{S}^2_R(X) \simeq 0$.

Case 1: X is minimal. In this case, $X \otimes_R X$ is also minimal. The bottom row of (4) is split exact, and so this implies that $\mathrm{S}^2_R(X)$ is also minimal. Hence, the condition $\mathrm{S}^2_R(X) \simeq 0$ implies that $\mathrm{S}^2_R(X) = 0$. Hence, the following sequence is split exact:

$$0 \to \mathrm{Ker}(\alpha^X) \xrightarrow{j^X} X \otimes_R X \xrightarrow{\alpha^X} X \otimes_R X \to 0.$$

Since each R-module $\mathrm{Ker}(\alpha^X)_n$ is free of finite rank, the additivity of rank implies that $\mathrm{Ker}(\alpha^X)_n = 0$, for all n, that is, $\mathrm{Ker}(\alpha^X) = 0$. The displayed sequence then shows that α^X is an isomorphism.

Assume for the rest of this case that $X \not\simeq 0$ and set $i = \inf(X)$. If i is even, then Proposition 4.1 implies that $\infty = \inf(\mathrm{S}^2_R(X)) = 2i < \infty$, a contradiction. Thus, i is odd. As before, there is a formal power series $Q(t) = \sum_{i=0}^{\infty} r_i t^i$ with nonnegative integer coefficients such that $r_0 \neq 0$ and $P^R_X(t) = t^i Q(t)$. Since $\mathrm{S}^2_R(X) = 0$, the following formal equalities are from (3.12):

$$0 = P^R_{\mathrm{S}^2_R(X)}(t) = \tfrac{1}{2}\left[P^R_X(t)^2 + P^R_X(-t^2)\right] = \tfrac{1}{2}\left[t^{2i}Q(t)^2 - t^{2i}Q(-t^2)\right].$$

It follows that $Q(t)^2 - Q(-t^2) = 0$, and so Lemma 2.12(b) implies that $Q(t) = 1$. This implies that $P^R_X(t) = t^i$, and so $X \cong \Sigma^i R$.

Case 2: the general case. Let $f\colon P \to X$ be a minimal free resolution. Corollary 4.3 implies that $\mathrm{S}^2_R(P) \simeq \mathrm{S}^2_R(X) \simeq 0$, and so Case 1 also implies that either $X \simeq P \simeq 0$ or $X \simeq P \simeq \Sigma^{2n+1}R$, for some integer n.

Case 1 also implies that $\operatorname{Ker}(\alpha^P) = 0$ and α^P is an isomorphism. The commutative diagram from (3.5.4),

$$\begin{array}{ccc} P \otimes_R P & \xrightarrow[\cong]{\alpha^P} & P \otimes_R P \\ {\scriptstyle f\otimes_R f}\downarrow{\scriptstyle \simeq} & & {\scriptstyle f\otimes_R f}\downarrow{\scriptstyle \simeq} \\ X \otimes_R X & \xrightarrow{\alpha^X} & X \otimes_R X \end{array}$$

shows that α^X is a quasiisomorphism; see Fact 2.11. Since $\mathrm{S}^2_R(X) \simeq 0$, the bottom row of (4) shows that i^X is a quasiisomorphism. Since α^X is also a quasiisomorphism, the commutativity of (4) shows that q^X is a quasiisomorphism as well. Hence, the top row of (4) implies that $\operatorname{Ker}(\alpha^X) \simeq 0$.

(v) $\Longrightarrow$ (iv). Argue as in the proof of the implication (iv) $\Longrightarrow$ (v).

(vi) $\Longrightarrow$ (iv). If $X \simeq 0$, then $\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(0) = 0$ by Example 3.3 and Corollary 4.3(b). If $X \simeq \Sigma^{2n+1}R$, for some integer n, then Corollary 4.3(b) explains the first quasiisomorphism in the next sequence:

$$\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(\Sigma^{2n+1}R) \simeq \mathrm{S}^2_R(\Sigma^{2n}(\Sigma R)) \simeq \Sigma^{4n}\mathrm{S}^2_R(\Sigma R) \simeq 0.$$

The second quasiisomorphism is because of the isomorphism $\Sigma^{2n+1}R \cong \Sigma^{2n}(\Sigma R)$; the third quasiisomorphism is from (3.5.2); and the last quasiisomorphism follows from Example 3.4. $\square$

Corollary 4.9. *Assume that R is noetherian and local, and that 2 is a unit in R. Let X be a bounded-below complex of finite rank free R-modules. The complex $\mathrm{S}^2_R(X)$ has finite projective dimension if and only if X has finite projective dimension.*

Proof. Assume first that $\operatorname{pd}_R(X)$ is finite, and let $P \xrightarrow{\simeq} X$ be a bounded free resolution. It follows that $P \otimes_R P$ is a bounded complex of free R-modules. Hence, the isomorphism $P \otimes_R P \cong \mathrm{S}^2_R(P) \oplus \operatorname{Im}(\alpha^P)$ from Proposition 3.8(b) implies that $\mathrm{S}^2_R(P)$ is a bounded complex of free R-modules. The quasiisomorphism $\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(P)$ from Corollary 4.3(b) implies that $\mathrm{S}^2_R(X)$ has finite projective dimension.

For the converse, assume that X has infinite projective dimension. Let $P \xrightarrow{\simeq} X$ be a minimal free resolution, which is necessarily unbounded. As noted previously, the fact that P is minimal implies that $\mathrm{S}^2_R(P) \xrightarrow{\simeq} \mathrm{S}^2_R(X)$ is a minimal free resolution, and so it suffices to show that $\mathrm{S}^2_R(P)$ is unbounded; see Fact 2.9.

Set $r_n = \operatorname{rank}_R(P_n)$, for each integer n. Since P is unbounded, we know that, for each integer n, there exist integers p and q such that $q > p > n$ and such that the free R-modules P_p and P_q are nonzero, that is, such that $r_p r_q \neq 0$. The inequality $q > p$ implies $p < (p+q)/2$. For each $n \geqslant 0$, we then have $p + q > 2n$ and

$$\operatorname{rank}_R(\mathrm{S}^2_R(P)_{p+q}) \geqslant \sum_{m<(p+q)/2} r_m r_{p+q-m} \geqslant r_p r_q > 0.$$

The first inequality is from Corollary 3.11; the second inequality follows from the inequality $p < (p+q)/2$; and the third inequality follows from the assumption $r_p r_q \neq 0$. This shows that for each $n \geqslant 0$, there is an integer $m = p + q > n$ such that $\mathrm{S}^2_R(P)_m \neq 0$. This means that $\mathrm{S}^2_R(P)$ is unbounded, as desired. □

The final result of this section is a refinement of the previous result. It characterizes the complexes X such that $\mathrm{S}^2_R(X) \simeq, \Sigma^j R$, for some integer j.

Corollary 4.10. *Assume that R is noetherian and local, and that 2 is a unit in R. Let X be a bounded-below complex of finite rank free R-modules. The folowing conditions are equivalent:*

(i) *$X \simeq \Sigma^{2n} R$, for some n, or $X \simeq (\Sigma^{2n+1} R) \oplus (\Sigma^{2m+1} R)$, for some n and m;*
(ii) *$\mathrm{S}^2_R(X) \simeq \Sigma^j R$, for some even integer j;*
(iii) *$\mathrm{S}^2_R(X) \simeq \Sigma^j R$, for some integer j.*

Proof. (i) $\Longrightarrow$ (ii). If $X \simeq \Sigma^{2n} R$, then we have

$$\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(\Sigma^{2n} R) \cong \Sigma^{4n} \mathrm{S}^2_R(R) \cong \Sigma^{4n} R$$

by (3.5.2), Example 3.3 and Corollary 4.3(b). In the case when $X \simeq (\Sigma^{2n+1} R) \oplus (\Sigma^{2m+1} R)$, Proposition 3.7 implies:

$$\mathrm{S}^2_R(X) \simeq \mathrm{S}^2_R(\Sigma^{2n+1} R) \oplus \left[(\Sigma^{2n+1} R) \otimes_R (\Sigma^{2m+1} R)\right] \oplus \mathrm{S}^2_R(\Sigma^{2m+1} R).$$

Example 3.4 implies that the first and last summands on the right side are 0, and so

$$\mathrm{S}^2_R(X) \cong \Sigma^{2n+1} R \otimes_R \Sigma^{2m+1} R \cong \Sigma^{2n+2m+2} R.$$

(ii) $\Longrightarrow$ (iii). This is trivial.

(iii) $\implies$ (i). Assume that $S^2_R(X) \simeq \Sigma^j R$, which implies that $j = \inf(S^2_R(X))$. Use Corollary 4.3(b) to replace X with a minimal free resolution in order to assume that X is minimal. As noted before, this implies that $S^2_R(X)$ is minimal, and so the quasiisomorphism $S^2_R(X) \simeq \Sigma^j R$ implies $S^2_R(X) \cong \Sigma^j R$; see Fact 2.9.

For each integer n, set $r_n = \operatorname{rank}_R(X_n)$. Also, set $i = \inf(X)$, and note that Proposition 4.1 implies that $j \geqslant 2i$. Write $Q(t) = \sum_{n=0}^{\infty} r_{n-i} t^n$; this is a formal power series with nonnegative integer coefficients and constant term $r_i \geqslant 1$ such that $P^R_X(t) = t^i Q(t)$. Since $S^2_R(X) \cong \Sigma^j R$, equation (3.12) can be written as:

(4.10.1)

$$t^j = \tfrac{1}{2}\left[(t^i Q(t))^2 + (-t^2)^i Q(-t^2)\right] = \tfrac{1}{2} t^{2i}\left[Q(t)^2 + (-1)^i Q(-t^2)\right].$$

Case 1: $j = 2i$. In this case, equation (4.10.1) reads as:

$$t^{2i} = \tfrac{1}{2} t^{2i}\left[Q(t)^2 + (-1)^i Q(-t^2)\right],$$

and so $2 = Q(t)^2 + (-1)^i Q(-t^2)$. Lemma 2.12 implies that

$$Q(t) = \begin{cases} 1, & \text{if } i \text{ is even} \\ 2, & \text{if } i \text{ is odd.} \end{cases}$$

When i is even, this translates to $P^R_X(t) = t^i$, and so $X \cong \Sigma^i R = \Sigma^{2n} R$, where, $n = i/2$. When i is odd, we have $P^R_X(t) = 2$, and so $X \cong \Sigma^i R^2 \cong \Sigma^{2n+1} R \oplus \Sigma^{2n+1} R$, where, $n = (i-1)/2$.

Case 2: $j > 2i$. In this case, Proposition 4.1 implies that i is odd, and equation (4.10.1) translates to:

$$\text{(4.10.2)} \quad 2t^{j-2i} = Q(t)^2 - Q(-t^2)$$

$$2t^{j-2i} = (r_i^2 - r_i) + 2r_{i+1} r_i t + (2r_{i+2} r_i + r_{i+1}^2 + r_{i+1}) t^2 + \cdots.$$

Since $j > 2i$, we equate coefficients in degree 0 to find $0 = r_i^2 - r_i$, and so $r_i = 1$. Thus, equation (4.2.10) reads as:

$$\text{(4.10.3)} \quad 2t^{j-2i} = 2r_{i+1} t + (2r_{i+2} + r_{i+1}^2 + r_{i+1}) t^2 + \cdots.$$

We claim that $j > 2i + 1$. Indeed, supposing that $j \leqslant 2i + 1$, our assumption $j > 2i$ implies $j = 2i + 1$. Equating degree 1 coefficients in equation (4) yields $r_{i+1} = 1$. The coefficients in degree 2 show that

$$0 = 2r_{i+2} r_i + r_{i+1}^2 + r_{i+1} = 2r_{i+2} + 2.$$

Hence, $r_{i+2} = -1$, which is a contradiction.

Since we have $j > 2i+1$, the degree 1 coefficients in equation (4) imply $r_{i+1} = 0$. It follows that

$$X \cong \Sigma^i R \oplus Y, \tag{4.10.4}$$

where, Y is a bounded-below minimal complex of finitely generated free R-modules such that $Y_n = 0$, for all $n < i+2$. With the isomorphism in (4), Proposition 3.7 gives the second isomorphism in the next sequence:

$$\Sigma^j R \cong \mathrm{S}^2_R(X) \cong \mathrm{S}^2_R(\Sigma^i R) \oplus \left[(\Sigma^i R) \otimes_R Y\right] \oplus \mathrm{S}^2_R(Y) \cong \Sigma^i Y \oplus \mathrm{S}^2_R(Y).$$

The final isomorphism comes from Example 3.4, since i is odd. In particular, it follows that $Y \not\simeq 0$. The complex $\Sigma^j R$ is indecomposable, because R is local, and so the displayed sequence implies that $\mathrm{S}^2_R(Y) = 0$ and $\Sigma^i Y \simeq \Sigma^j R$. Because of the conditions $\mathrm{S}^2_R(Y) = 0$ and $Y \not\simeq 0$, Theorem 4.8 implies that $Y \simeq \Sigma^{2m+1} R$, for some m. Hence, the isomorphism in (4) reads as $X \cong \Sigma^{2n+1} R \oplus \Sigma^{2m+1} R$, where, $n = (i-1)/2$, as desired. □

5. Examples

We begin this section with three explicit computations of the complexes $\mathrm{S}^2_R(X)$ and $\mathrm{s}^2_R(X)$ and their homologies. As a consequence, we show that Buchbaum and Eisenbud's construction differs from those in [6, 11]. We also provide examples showing the need for certain hypotheses in the results of the previous sections.

Example 5.1. *Fix an element $x \in R$ and let K denote the Koszul complex $K^R(x)$ which has the following form, where the basis is listed in each degree:*

$$K = \qquad 0 \to \underbrace{R}_{e_1} \xrightarrow{(x)} \underbrace{R}_{e_0} \to 0.$$

The tensor product $K \otimes_R K$ has the form:

$$K \otimes_R K = \qquad 0 \to \underbrace{R}_{e_1 \otimes e_1} \xrightarrow{\left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} \underbrace{R^2}_{\substack{e_0 \otimes e_1 \\ e_1 \otimes e_0}} \xrightarrow{(x\ x)} \underbrace{R}_{e_0 \otimes e_0} \to 0. \tag{5.1.1}$$

Using this representation, the exact sequence in (3.5.3) *has the form:*

$$0 \longrightarrow \operatorname{Ker}(\alpha^X) \longrightarrow K \otimes_R K \xrightarrow{\alpha^K} K \otimes_R K \longrightarrow \mathrm{s}^2_R(K) \longrightarrow 0$$

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ann}_R(2) & \longrightarrow & R & \xrightarrow{(2)} & R & \longrightarrow & R/(2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (x)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} x \\ -x \end{smallmatrix}\right)} & & \downarrow{\scriptstyle (0)} & & \\
0 & \longrightarrow & R & \xrightarrow{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} & R^2 & \xrightarrow{\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)} & R^2 & \xrightarrow{(1\ 1)} & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (2x)} & & \downarrow{\scriptstyle (x\ x)} & & \downarrow{\scriptstyle (x\ x)} & & \downarrow{\scriptstyle (x)} & & \\
0 & \longrightarrow & R & \xrightarrow{(1)} & R & \xrightarrow{(0)} & R & \xrightarrow{(1)} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0. & &
\end{array}$$

From the rightmost column of this diagram, we have

$$\mathrm{H}_2(\mathrm{s}^2_R(K)) \cong R/(2) \quad \mathrm{H}_1(\mathrm{s}^2_R(K)) \cong \operatorname{Ann}_R(x) \quad \mathrm{H}_0(\mathrm{s}^2_R(K)) \cong R/(x),$$

and $\mathrm{H}_i(\mathrm{s}^2_R(K)) = 0$, *when* $i \notin \{0, 1, 2\}$.

A similar computation shows that

$$\mathrm{S}^2_R(K) = \qquad 0 \to R \xrightarrow{(x)} R \to 0,$$

and thus

$$\mathrm{H}_1(\mathrm{S}^2_R(K)) \cong \operatorname{Ann}_R(x), \qquad\qquad \mathrm{H}_0(\mathrm{S}^2_R(K)) \cong R/(x)$$

and $\mathrm{H}_i(\mathrm{S}^2_R(K)) = 0$, *when* $i \notin \{0, 1\}$.

Example 5.2. *Fix elements* $x, y \in R$ *and let* K *denote the Koszul complex* $K^R(x, y)$ *which has the following form, where the ordered basis is listed in each degree:*

$$(5.2.1) \qquad K = \qquad 0 \to \underbrace{R}_{e_2} \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} \underbrace{R^2}_{\substack{e_{11} \\ e_{12}}} \xrightarrow{(x\ y)} \underbrace{R}_{e_0} \to 0.$$

Using the same format, the complex $K \otimes_R K$ has the form:

$$K \otimes_R K =$$

$$0 \to \underbrace{R}_{e_2\otimes e_2} \xrightarrow{\partial_4^{K\otimes_R K}} \underbrace{R^4}_{\substack{e_2\otimes e_{11}\\ e_2\otimes e_{12}\\ e_{11}\otimes e_2\\ e_{12}\otimes e_2}} \xrightarrow{\partial_3^{K\otimes_R K}} \underbrace{R^6}_{\substack{e_2\otimes e_0\\ e_{11}\otimes e_{11}\\ e_{11}\otimes e_{12}\\ e_{12}\otimes e_{11}\\ e_{12}\otimes e_{12}\\ e_0\otimes e_2}} \xrightarrow{\partial_2^{K\otimes_R K}} \underbrace{R^4}_{\substack{e_{11}\otimes e_0\\ e_{12}\otimes e_0\\ e_0\otimes e_{11}\\ e_0\otimes e_{12}}} \xrightarrow{\partial_1^{K\otimes_R K}} \underbrace{R}_{e_0\otimes e_0} \to 0$$

with differentials given by the following matrices:

$$\partial_4^{K\otimes_R K} = \begin{pmatrix} y \\ -x \\ y \\ -x \end{pmatrix} \qquad \partial_3^{K\otimes_R K} = \begin{pmatrix} x & y & 0 & 0 \\ y & 0 & -y & 0 \\ 0 & y & x & 0 \\ -x & 0 & 0 & -y \\ 0 & -x & 0 & x \\ 0 & 0 & x & y \end{pmatrix}$$

$$\partial_2^{K\otimes_R K} = \begin{pmatrix} y & -x & -y & 0 & 0 & 0 \\ -x & 0 & 0 & -x & -y & 0 \\ 0 & x & 0 & y & 0 & y \\ 0 & 0 & x & 0 & y & -x \end{pmatrix} \qquad \partial_1^{K\otimes_R K} = (x \;\; y \;\; x \;\; y).$$

Under the same bases, the morphism $\alpha^K\colon K\otimes_R K \to K\otimes_R K$ is described by the following matrices:

$$\alpha_3^K = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \qquad \alpha_2^K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_1^K = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \qquad \alpha_4^K = (0) = \alpha_0^K.$$

As in Example 5.1, it follows that $\mathrm{S}^2_R(K)$ has the form:

$$\mathrm{S}^2_R(K) =$$

$$0 \to \underbrace{R}_{f_4} \xrightarrow{\partial_4^{\mathrm{S}^2_R(K)}} \underbrace{R^2}_{\substack{f_{31}\\ f_{32}}} \xrightarrow{\partial_3^{\mathrm{S}^2_R(K)}} \underbrace{R^2}_{\substack{f_{21}\\ f_{22}}} \xrightarrow{\partial_2^{\mathrm{S}^2_R(K)}} \underbrace{R^2}_{\substack{f_{11}\\ f_{12}}} \xrightarrow{\partial_1^{\mathrm{S}^2_R(K)}} \underbrace{R}_{f_0} \to 0$$

where the basis vectors are described as:

$$f_4 = \overline{e_2 \otimes e_2} \qquad f_{31} = \overline{e_2 \otimes e_{11}} = \overline{e_{11} \otimes e_2}$$
$$f_{32} = \overline{e_2 \otimes e_{12}} = \overline{e_{12} \otimes e_2} \qquad f_{21} = \overline{e_2 \otimes e_0} = \overline{e_0 \otimes e_2}$$
$$f_{22} = \overline{e_{11} \otimes e_{12}} = -\overline{e_{12} \otimes e_{11}} \qquad f_{11} = \overline{e_{11} \otimes e_0} = \overline{e_0 \otimes e_{11}}$$
$$f_{12} = \overline{e_{12} \otimes e_0} = \overline{e_0 \otimes e_{12}} \qquad f_0 = \overline{e_0 \otimes e_0}.$$

(Note also that $\overline{e_{11}\otimes e_{11}} = 0 = \overline{e_{12}\otimes e_{12}}$*.) Under these bases, the differentials* $\partial_n^{\mathrm{S}^2_R(K)}$ *are described by the following matrices:*
(5.2.2)

$$\partial_4^{\mathrm{S}^2_R(K)} = \begin{pmatrix} 2y \\ -2x \end{pmatrix} \qquad \partial_3^{\mathrm{S}^2_R(K)} = \begin{pmatrix} x & y \\ x & y \end{pmatrix}$$
$$\partial_2^{\mathrm{S}^2_R(K)} = \begin{pmatrix} y & -y \\ -x & x \end{pmatrix} \qquad \partial_1^{\mathrm{S}^2_R(K)} = \begin{pmatrix} x & y \end{pmatrix}.$$

Similar computations show that $\mathrm{s}^2_R(K) \cong \mathrm{S}^2_R(K) \oplus \Sigma^2(R/(2))^2$.

Example 5.3. *Let* $x, y \in R$ *be an* R*-regular sequence and continue with the notation of Example 5.2. We verify the following isomorphisms:*

$$\mathrm{H}_0(S^0_R(K)) \cong \mathrm{H}_2(\mathrm{S}^2_R(K)) \cong R/(x,y) \qquad \mathrm{H}_1(\mathrm{S}^2_R(K)) = 0$$
$$\mathrm{H}_3(\mathrm{S}^2_R(K)) \cong R/(2) \qquad \mathrm{H}_4(\mathrm{S}^2_R(K)) \cong \mathrm{Ann}_R(2).$$

The computation of $\mathrm{H}_0(\mathrm{S}^2_R(K))$ *follows from the description of* $\partial_1^{\mathrm{S}^2_R(K)}$ *in* (5.2).

For $\mathrm{H}_1(\mathrm{S}^2_R(K))$*, the second equality in the following sequence comes from the exactness of* K *in degree 1:*

$$\begin{aligned} \mathrm{Ker}\left(\partial_1^{\mathrm{S}^2_R(K)}\right) &= \mathrm{Ker}\left(\partial_1^K\right) = \mathrm{Im}\left(\partial_2^K\right) \\ &= \mathrm{Span}_R\left\{\begin{pmatrix} y \\ -x \end{pmatrix}\right\} = \mathrm{Im}\left(\partial_2^{\mathrm{S}^2_R(K)}\right) \end{aligned}$$

and the others come from the descriptions of K *and* $\mathrm{S}^2_R(K)$ *in* (5.2) *and* (5.2).

For $\mathrm{H}_2(\mathrm{S}^2_R(K))$*, use the fact that* x *is* R*-regular to check the first equality in the next display; the others follow from* (5.2)*:*

$$\mathrm{Ker}\left(\partial_2^{\mathrm{S}^2_R(K)}\right) = \mathrm{Span}_R\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$
$$\mathrm{Im}\left(\partial_3^{\mathrm{S}^2_R(K)}\right) = \mathrm{Span}_R\left\{\begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix}\right\} = (x,y)\,\mathrm{Span}_R\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

The isomorphism $\mathrm{H}_2(\mathrm{S}^2_R(K)) \cong R/(x,y)$ *now follows.*

For $\mathrm{H}_3(\mathrm{S}^2_R(K))$*, the second equality in the following sequence comes from the exactness of* K *in degree 1:*

$$\mathrm{Ker}\left(\partial_3^{\mathrm{S}^2_R(K)}\right) = \mathrm{Ker}\left(\partial_1^K\right) = \mathrm{Im}\left(\partial_2^K\right) = \mathrm{Span}_R\left\{\begin{pmatrix} y \\ -x \end{pmatrix}\right\}$$

$$\mathrm{Im}\left(\partial_4^{\mathrm{S}^2_R(K)}\right) = (2)\,\mathrm{Span}_R\left\{\begin{pmatrix} y \\ -x \end{pmatrix}\right\}$$

and the others come from the descriptions of K *and* $\mathrm{S}^2_R(K)$ *in* (5.2) *and* (5.2)*. The isomorphism* $\mathrm{H}_3(\mathrm{S}^2_R(K)) \cong R/(2)$ *now follows.*

Similarly, for $\mathrm{H}_4(\mathrm{S}^2_R(K))$*, we have*

$$\mathrm{H}_4(\mathrm{S}^2_R(K)) = \mathrm{Ker}\left(\partial_4^{\mathrm{S}^2_R(K)}\right) = (\mathrm{Ker}\left(\partial_2^K\right) : 2) = (0 :_R 2) = \mathrm{Ann}_R(2).$$

This completes the example.

As a first consequence of the previous computations, we next observe that $\mathrm{S}^2_R(X)$ is generally not isomorphic to Dold and Puppe's [6] construction $\mathcal{D}_{\mathrm{S}^2}(X)$ and not isomorphic to Tchernev and Weyman's [11] construction $\mathcal{C}_{\mathrm{S}^2}(X)$.

Example 5.4. *Assume that 2 is a unit in* R*. Fix an element* $x \in R$ *and let* K *denote the Koszul complex* $K^R(x)$*. Example 5.1 yields the following computation of* $\mathrm{S}^2_R(K)$*:*

$$\mathrm{S}^2_R(K) = \qquad 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

$$\mathcal{D}_{\mathrm{S}^2}(K) \cong \mathcal{C}_{\mathrm{S}^2}(K) = \quad 0 \longrightarrow R \xrightarrow{\begin{pmatrix} 1 \\ -x \end{pmatrix}} R^2 \xrightarrow{(x^2 \;\; x)} R \longrightarrow 0.$$

The fact that $\mathcal{D}_{\mathrm{S}^2}(K)$ *and* $\mathcal{C}_{\mathrm{S}^2}(K)$ *have the displayed form can be deduced from* [11, (11.2) and (14.4)]*; the maps were computed for us by Tchernev. In particular, in this case we have* $\mathcal{D}_{\mathrm{S}^2}(K) \cong \mathcal{C}_{\mathrm{S}^2}(K) \not\cong \mathrm{S}^2_R(K)$*.*

More generally, if we have

$$X = \quad 0 \to R^m \to R^n \to 0,$$

then Corollary 3.11 and [11, (11.2) and (14.4)] *yield:*

$$\mathrm{S}^2_R(X) = \quad 0 \longrightarrow R^{\binom{m}{2}} \longrightarrow R^{mn} \longrightarrow R^{\binom{n+1}{2}} \longrightarrow 0$$

$$\mathcal{D}_{\mathrm{S}^2}(X) \cong \mathcal{C}_{\mathrm{S}^2}(X) = \quad 0 \longrightarrow R^{m^2} \longrightarrow R^{\binom{m+1}{2}+mn} \longrightarrow R^{\binom{n+1}{2}} \longrightarrow 0.$$

Hence, we have $\mathcal{C}_{\mathrm{S}^2}(X) \cong \mathrm{S}^2_R(X)$ if and only if $m = 0$, i.e., if and only if $X \cong R^n$.

We next show why we need to assume that X and Y are bounded-below complexes of projective R-modules in Corollary 4.3. It also shows that $\mathrm{S}^2_R(X)$ can have nontrivial homology, even when X is a minimal free resolution of a module of finite projective dimension.

Example 5.5. *Let $x, y \in R$ be an R-regular sequence and continue with the notation of Example 5.2. The computations in Example 5.3 show that* $\mathrm{H}_2(\mathrm{S}^2_R(K)) \cong R/(x,y) \neq 0 = \mathrm{H}_2(\mathrm{S}^2_R(R/(x,y)))$, and so $\mathrm{S}^2_R(K) \not\simeq \mathrm{S}^2_R(R/(x,y))$ even though $K \simeq R/(x,y)$.

The next example shows why we need to assume that 2 is a unit in R for Theorem 4.2 and Corollaries 4.3 and 4.4.

Example 5.6. *Assume that 2 is not a unit in R and let K denote the Koszul complex $K^R(1,1)$. Then, K is split exact, and so the zero map $z \colon K \to K$ is a homotopy equivalence, it is homotopic to id_K, and it is a quasiisomorphism. Example 5.2 shows that $\mathrm{H}_3(\mathrm{S}^2_R(K)) = R/(2) \neq 0$. On the other hand, the morhpism $\mathrm{S}^2_R(z) \colon \mathrm{S}^2_R(K) \to \mathrm{S}^2_R(K)$ is the zero morphism, and so the nonvanishing of $\mathrm{H}_2(\mathrm{S}^2_R(K))$ implies that $\mathrm{S}^2_R(z)$ is not a quasiisomorphism. It follows that $\mathrm{S}^2_R(z)$ is neither a homotopy equivalence nor homotopic to $\mathrm{id}_{\mathrm{S}^2_R(K)}$. This shows why we need to assume that 2 is a unit in R for Theorem 4.2 and Corollary 4.3(a). For Corollary 4.3(b), simply note that $K \simeq 0$ and $\mathrm{S}^2_R(K) \not\simeq 0 \simeq \mathrm{S}^2_R(0)$. For Corollary 4.4, note that this shows that $\mathrm{Supp}_R(\mathrm{S}^2_R(K)) \neq \emptyset = \mathrm{Supp}_R(K)$.*

Our next example shows that the functors $\mathrm{s}^2_R(-)$ and $\mathrm{S}^2_R(-)$ are not additive, even when 2 is a unit in R and we restrict to bounded complexes of finite rank free R-modules.

Example 5.7. *Let X and Y be nonzero R-complexes. Consider the natural surjections and injections:*

$$X \oplus Y \xrightarrow{\tau_1} X \xrightarrow{\epsilon_1} X \oplus Y, \qquad X \oplus Y \xrightarrow{\tau_2} Y \xrightarrow{\epsilon_2} X \oplus Y,$$

and set $f_i = \epsilon_i \tau_i \colon X \oplus Y \to X \oplus Y$. The equality $f_1 + f_2 = \mathrm{id}_{X \oplus Y}$ is immediate.

We claim that $\mathrm{s}^2_R(f_1+f_2) \neq \mathrm{s}^2_R(f_1)+\mathrm{s}^2_R(f_2)$*. To see this, first note that the equalities* $\mathrm{s}^2_R(f_1+f_2) = \mathrm{s}^2_R(\mathrm{id}_{X\oplus Y}) = \mathrm{id}_{\mathrm{s}^2_R(X\oplus Y)}$ *show that it suffices to verify* $\mathrm{s}^2_R(f_1)+\mathrm{s}^2_R(f_2) \neq \mathrm{id}_{\mathrm{s}^2_R(X\oplus Y)}$*. One checks that there is a commutative diagram:*

$$\begin{array}{ccc} (X\oplus Y)\otimes(X\oplus Y) & \xrightarrow{\cong} & (X\otimes X)\oplus(X\otimes Y)\oplus(Y\otimes X)\oplus(Y\otimes Y) \\ \Big\downarrow{\scriptstyle f_1\otimes f_1} & & \Big\downarrow{\scriptstyle \begin{pmatrix} \mathrm{id}_{X\otimes X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \\ (X\oplus Y)\otimes(X\oplus Y) & \xrightarrow{\cong} & (X\otimes X)\oplus(X\otimes Y)\oplus(Y\otimes X)\oplus(Y\otimes Y), \end{array}$$

wherein the horizontal maps are the natural distributivity isomorphisms. The proof of Proposition 3.7 yields another commutative diagram:

$$\begin{array}{ccc} \mathrm{s}^2_R(X\oplus Y) & \xrightarrow{\cong} & \mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{s}^2_R(Y) \\ \Big\downarrow{\scriptstyle \mathrm{s}^2_R(f_1)} & & \Big\downarrow{\scriptstyle \begin{pmatrix} \mathrm{id}_{\mathrm{s}^2_R(X)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \\ \mathrm{s}^2_R(X\oplus Y) & \xrightarrow{\cong} & \mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{s}^2_R(Y). \end{array}$$

Similarly, there is another commutative diagram:

$$\begin{array}{ccc} \mathrm{s}^2_R(X\oplus Y) & \xrightarrow{\cong} & \mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{s}^2_R(Y) \\ \Big\downarrow{\scriptstyle \mathrm{s}^2_R(f_2)} & & \Big\downarrow{\scriptstyle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{\mathrm{s}^2_R(Y)} \end{pmatrix}} \\ \mathrm{s}^2_R(X\oplus Y) & \xrightarrow{\cong} & \mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{s}^2_R(Y). \end{array}$$

This implies that $\mathrm{s}^2_R(f_1)+\mathrm{s}^2_R(f_2)$ *is equivalent to the morphism*

$$\mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{s}^2_R(Y) \xrightarrow{\begin{pmatrix} \mathrm{id}_{\mathrm{s}^2_R(X)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{\mathrm{s}^2_R(Y)} \end{pmatrix}} \mathrm{s}^2_R(X)\oplus(X\otimes_R Y)\oplus\mathrm{s}^2_R(Y),$$

and so cannot equal $\mathrm{id}_{\mathrm{s}^2_R(X\oplus Y)}$*.*

Similarly, we have $\mathrm{S}^2_R(f_1+f_2) = \mathrm{S}^2_R(\mathrm{id}_{X\oplus Y}) = \mathrm{id}_{\mathrm{S}^2_R(X\oplus Y)} \neq \mathrm{S}^2_R(f_1)+\mathrm{S}^2_R(f_2)$*.*

Our final example shows that one needs to be careful about removing the local hypotheses from the results of Section 4. Specifically, it shows that, without the local hypotheses, the implication (i) $\implies$ (iv) fails in Theorem 4.5.

Example 5.8. *Let K and L be fields, and set $R = K \times L$. The prime ideals of R are all maximal, and they are precisely the ideals $\mathfrak{m} = K \times 0$ and $\mathfrak{n} = 0 \times L$. Furthermore, we have $R_\mathfrak{m} \cong L$ and $R_\mathfrak{n} \cong K$. Assume that $\operatorname{char}(K) \neq 2$ and $\operatorname{char}(L) \neq 2$, so that 2 is a unit in R.*

First, consider the complex $Y = (K \times 0) \oplus \Sigma^2(0 \times L)$. Then, Y is a bounded-below complex of finitely generated projective R-modules such that $Y_\mathfrak{m} \cong \Sigma^2 L \cong \Sigma^2 R_\mathfrak{m}$ and $Y_\mathfrak{n} \cong K \cong R_\mathfrak{n}$. Hence, Remark 4.6 implies that the surjection $p^Y : Y \otimes_R Y \to \mathrm{S}^2_R(Y)$ is a quasiisomorphism. However, the fact that Y has nonzero homology in degrees 2 and 0 implies that, $Y \not\simeq 0$ and $Y \not\simeq \Sigma^{2t} R$, for each integer t.

Next, we provide an example of a bounded-below complex X of finitely generated free *R-modules with the same behavior. The following complex describes a free resolution F of $K \times 0$:*

$$\cdots \xrightarrow{(e)} R \xrightarrow{(f)} R \xrightarrow{(e)} R \xrightarrow{(f)} \cdots \xrightarrow{(f)} R \to 0,$$

where, $e = (1,0) \in R$ and $f = (0,1) \in R$. An R-free resolution G for $0 \times L$ is constructed similarly. The complex $X = F \oplus \Sigma^2 G$ yields a degreewise-finite R-free resolution of $g \colon X \xrightarrow{\simeq} Y$. Corollary 4.3(a) implies that $\mathrm{S}^2_R(g)$ is a quasiisomorphism. Hence, the next commutative diagram shows that the surjection $p^X : X \otimes_R X \to \mathrm{S}^2_R(X)$ is also a quasiisomorphism:

$$\begin{array}{ccc} X \otimes_R X & \xrightarrow{p^X} & \mathrm{S}^2_R(X) \\ \simeq \downarrow g\otimes g & & \simeq \downarrow \mathrm{S}^2(g) \\ Y \otimes_R Y & \xrightarrow[\simeq]{p^Y} & \mathrm{S}^2_R(Y). \end{array}$$

However, we have $X \simeq Y$, and so $X \not\simeq 0$ and $X \not\simeq \Sigma^{2t} R$, for each integer t.

Acknowledgments

We are grateful to Frank Moore and Alex Tchernev for helpful conversations about the existing literature on this subject. We are grateful to the referee for thoughtful comments. Research of the second author was supported in part by a grant from the NSA.

References

1. D. Apassov, *Complexes and differential graded modules*, Ph.D. thesis, Lund University, 1999.
2. L. L. Avramov, R.-O. Buchweitz and L. M. Şega, Extensions of a dualizing complex by its ring: commutative versions of a conjecture of Tachikawa, *J. Pure Appl. Algebra* **201**(1-3) (2005) 218-239.
3. L. L. Avramov, H.-B. Foxby and S. Halperin, Differential graded homological algebra, in preparation.
4. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.
5. D. A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99**(3) (1977) 447-485.
6. A. Dold and D. Puppe, Homologie nicht-additiver Funktoren. Anwendungen, (German), *Ann. Inst. Fourier Grenoble* **11** (1961) 201-312.
7. D. Eisenbud, *Commutative Algebra*, With a view toward algebraic geometry, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995.
8. Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, 205, Springer-Verlag, New York, 2001.
9. H.-B. Foxby, *Hyperhomological Algebra & Commutative Rings*, In preparation.
10. A. J. Frankild, S. Sather-Wagstaff and A. Taylor, Relations between semidualizing complexes, *J. Commut. Algebra* **1**(3) (2009) 393-436.
11. A. Tchernev and J. Weyman, Free resolutions for polynomial functors, *J. Algebra* **271**(1) (2004) 22-64.

Sean Sather-Wagstaff
Department of Mathematics, NDSU Dept #2750, P.O. Box 6050, Fargo, ND 58108-6050, USA
Email: Sean.Sather-Wagstaff@ndsu.edu

Amelia Taylor
Colorado College, 14 E. Cache La Poudre St., Colorado Springs, CO 80903, USA
Email: amelia.taylor@coloradocollege.edu