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CONVERGENCE OF PRODUCT INTEGRATION METHOD APPLIED FOR NUMERICAL SOLUTION OF LINEAR WEAKLY SINGULAR VOLTERRA SYSTEMS

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Abstract. We develop and apply the product integration method to a large class of linear weakly singular Volterra systems. We show that under certain sufficient conditions this method converges. Numerical implementation of the method is illustrated by a benchmark problem originated from heat conduction.

1. **Introduction**

Transformation of a differential system (such as heat conduction problem) to an equivalent system of integral equations is a powerful technique for deducing the existence and uniqueness of the solution [2]. We begin our study and the development of the product integration method from an equivalent system of Volterra integral equations. To start, suppose that a system of Volterra integral equations is given by

$$U(t) = F(t) + \int_0^t K(t, \tau, U(\tau))d\tau, \quad t \in [0, b], \tag{1.1}$$

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where, $b > 0$ and

$$U(t) = (u_1(t), ..., u_d(t))^T, \ F(t) = (f_1(t), ..., f_d(t))^T, \tag{1.2}$$

$$K(t, \tau, U(\tau)) := (k_1(t, \tau, U(\tau)), ..., k_d(t, \tau, U(\tau)))^T, \tag{1.3}$$

(1.4)
$$\int_0^t K(t, \tau, U(\tau))d\tau := \left(\int_0^t k_1(t, \tau, U(\tau))d\tau, ..., \int_0^t k_d(t, \tau, U(\tau))d\tau\right)^T,$$

$u_i(t), f_i(t), k_i(t, \tau, U(\tau))$, for $i = 1, 2, ..., d$, are real valued functions, and each k_i can be written as:

$$k_i(t, \tau, U(\tau)) = p_i(t, \tau)\widetilde{k}_i(t, \tau, U(\tau)), \tag{1.5}$$

where, every $\widetilde{k}_i$ is continuous and every p_i has one of the following form:

$$p(t, \tau) = |t - \tau|^{-\alpha}, \quad 0 < \alpha < 1, \tag{1.6}$$

or

$$p(t, \tau) = \log |t - \tau|. \tag{1.7}$$

A special case is $p \equiv 1$, whence there is no singularity.

2. **Product Integration Technique**

For simplicity suppose that all the p_i, for $i = 1, ..., d$, are the same, and consider the following form:

$$U(t) = F(t) + \int_0^t p(t, \tau)K(t, \tau, U(\tau))d\tau, \quad t \in [0, b]. \tag{2.1}$$

Further, suppose that all components of K are continuous. We introduce $N + 1$ grid points $0 \leq t_0 < t_1 < ... < t_N \leq b$ in t. Our goal is to compute $U(t)$ at the grid points and the numerical approximation to $U(t_n)$ is written as U_N^n. The basic point in the product integration technique are:
(i) Sample the system of Volterra integral equations at points t_n in the grid:

$$U(t_n) = F(t_n) + \int_0^{t_n} p(t_n, \tau)K(t_n, \tau, U(\tau))d\tau. \tag{2.2}$$

(ii) Use the Lagrange interpolation polynomial

$$L_N(K, t_n; \tau) = \sum_{j=0}^{N} l_{N,j}(\tau) K(t_n, t_j, U(t_j)), \tag{2.3}$$

to approximate $K(t_n, \tau, U(\tau))$ and obtain the following algorithm:

$$U_N^{(n)} = F(t_n) + \sum_{j=0}^{N} \omega_j(t_n) K(t_n, t_j, U_N^{(j)}), \tag{2.4}$$

where,

$$\omega_j(t) = \int_0^t p(t, \tau) l_{N,j}(\tau) d\tau. \tag{2.5}$$

Solving the system (2.4), we obtain $U_N(t)$ as a Nystrom approximation for $U(t)$:

$$U_N(t) = F(t) + \sum_{j=0}^{N} \omega_j(t) K(t, t_j, U_N^{(j)}). \tag{2.6}$$

Note that $U_N^{(n)} = U_N(t_n)$, for $n = 1, 2, ..., d$.

3. Convergence of Product Integration Technique

For convergence analysis, we examine the following linear test problem:

$$U(t) = F(t) + \int_0^t p(t, \tau) K(t, \tau) U(\tau) d\tau, \tag{3.1}$$

where, K is a $d \times d$ matrix with continuous components $k_{i,j}, i, j \in \{1, ..., d\}$, and $U(t)$ is the unknown vector, all components of vector $F(t)$ are continuous and $p(t, \tau)$ is defined as in (1.6).
For an arbitrary $t \in [0, b]$, we can write

$$\begin{aligned} U(t) - U_N(t) &= \int_0^t p(t, \tau) \left[K(t, \tau) U(\tau) - \sum_{j=0}^{N} K(t, t_j) l_{N,j}(\tau) U_N^{(j)} \right] d\tau \\ &= \int_0^t p(t, \tau) \left[\sum_{j=0}^{N} \left(K(t, \tau) U(\tau) - K(t, t_j) U(t_j) \right) l_{N,j}(\tau) \right] d\tau \end{aligned}$$

$$+\int_0^t p(t,\tau)\left[\sum_{j=0}^{N} K(t,t_j) l_{N,j}(\tau)\left(U(t_j)-U_N^{(j)}\right)\right]d\tau$$

$$= t_N(K,U,t) + A_N(U-U_N)(t) \tag{3.2}$$

Such that

$$t_N(K,U,t) := \int_0^t p(t,\tau)\left[\sum_{j=0}^{N} \left(K(t,\tau)U(\tau)-K(t,t_j)U(t_j)\right) l_{N,j}(\tau)\right]d\tau, \tag{3.3}$$

$$A_N(U-U_N)(t) := \int_0^t p(t,\tau)\left[\sum_{j=0}^{N} K(t,t_j) l_{N,j}(\tau)\left(U(t_j)-U_N^{(j)}\right)\right]d\tau. \tag{3.4}$$

We shall show $(I-A_N)^{-1}$ exists and so (3.2) is equivalent to:

$$(U-U_N)(t) = (I-A_N)^{-1} t_N(K,U,t). \tag{3.5}$$

Hence, if we show

$$\exists c>0 \quad \forall N\in\mathbb{N} \quad \|(I-A_N)^{-1}\|_\infty \le c, \tag{3.6}$$

$$\|t_N\|_\infty \to 0 \quad as \quad N\to\infty, \tag{3.7}$$

then the following uniform convergence holds:

$$\|U-U_N\|_\infty \le \|(I-A_N)^{-1}\|_\infty \|t_N\|_\infty \le c\|t_N\|_\infty \to 0, \quad as \quad N\to\infty. \tag{3.8}$$

Lemma 3.1. *Let $\{p_i\}_{i=1}^N$ be a sequence of orthogonal polynomials on* [-1,1] *with weight the function $\omega(x)$. Then, there is a sequence $\{q_i\}_{i=1}^N$ of orthogonal polynomials on* [a,b] *with weight function $\widetilde{\omega}(t)$, where,*

$$q_i(t) = p_i\left(\frac{2}{b-a}[t-\frac{b+a}{2}]\right), \qquad t\in[a,b], \tag{3.9}$$

$$\widetilde{\omega}(t) = \omega\left(\frac{2}{b-a}[t-\frac{b+a}{2}]\right), \qquad t\in[a,b]. \tag{3.10}$$

Proof. Put $x=\frac{2}{b-a}[t-\frac{b+a}{2}]$. Then, for $i,j\in\{1,...,N\}$ and $i\neq j$,

$$\int_a^b q_i(t)q_j(t)\widetilde{\omega}(t)dt = \frac{b-a}{2}\int_{-1}^1 p_i(x)p_j(x)\omega(x)dx = 0.$$

□

Theorem 3.2. *Let $\{t_j\}_{j=0}^N$ be the zeros of the (N+1)st degree member of a set of polynomials that are orthogonal on $[0,b]$ with the weight function $\omega(t)$,*

$$\omega(t) = u(\frac{2t}{b}-1)(2-\frac{2t}{b})^{\alpha}(\frac{2t}{b})^{\beta}, \quad -1 < \alpha \le \frac{3}{2}, \beta > -\frac{1}{2}, \tag{3.11}$$

where, $u(t)$ is positive and continuous in $[0,b]$ and the modulus of continuity φ of u satisfies $\int_0^1 \varphi(u,\delta)\frac{d\delta}{\delta} < \infty$.
Let $L_N(f;\tau)$ denote the interpolating polynomial of degree $\le N$ that coincides with the function f at the nodes $\{t_j\}_{j=0}^N$. Then, for every function f containing only endpoint singularity of the type $\tau^{\sigma}, \sigma > -1$ (not an integer), and in particular for every function $f \in C[0,b]$, the following holds:

$$\lim_{N\to\infty} \left\| \int_0^t p(t,\tau)[f(\tau) - L_N(f;\tau)]d\tau \right\|_{\infty} = 0. \tag{3.12}$$

In particular, we have the bound

$$\left\| t_N(|t-\tau|^{-\overline{\alpha}}, f, t) \right\|_{\infty} = O\{(N+1)^{-2-2\sigma+2\overline{\alpha}} \log(N+1)\}, 0 < \overline{\alpha} < 1. \tag{3.13}$$

Proof. See Theorem 1 of [6], Theorem 5 of [3] and Lemma 3.1. □

Indeed, from (3.12) we show that the maximum norm of every component of t_N tends to zero, and hence (3.7) is valid. Using the following theorem, we prove (3.6) is valid.

Theorem 3.3. *Let $A : Y \to Y$ be a compact linear operator over a Banach space Y such that $I-A$ is injective, and assume that the sequence $A_N : Y \to Y$ of linear operators is collectively compact and pointwise convergent, i.e., $A_N U \to AU$, as $N \to \infty$, for all $U \in Y$. Then, for sufficiently large N, the inverse operator $(I-A_N)^{-1} : Y \to Y$ exists and is uniformly bounded. This means that (3.6) is satisfied.*

Proof. See theorem 12.10 in [5].

By definition, $A : X^d \to X^d, X := C[0,b]$,and

$$AU(t) = \int_0^t p(t,\tau)K(t,\tau)U(\tau)d\tau. \tag{3.14}$$

We establish that A is a compact linear operator over a Banach space X^d such that $I-A$ is injective, and the sequence A_N, by definition (3.4), is collectively compact and pointwise convergent. □

3.1. **The operator A is compact.** Consider the sequence of operators

$$\begin{cases} A_{ij}: X \to X \quad X := C[0,b], \quad i,j \in \{1,2,...,d\} \\ A_{ij}u(t) = \int_0^t p(t,\tau)k_{ij}(t,\tau)u(\tau)d\tau, \quad u \in X. \end{cases} \tag{3.15}$$

where, every A_{ij} is compact (see [1], page 75). We show that A is a compact operator on a Banach space $Y := X^d$, with the following norm:

$$\|U\|_\infty = \max\{\|u_1\|_\infty, ..., \|u_d\|_\infty\}, \quad U = (u_1, ..., u_d)^T. \tag{3.16}$$

Suppose $\{U^{(n)}\}_{n=1}^\infty$ is a bounded sequence in Y. It is sufficient to show that $\{AU^{(n)}\}_{n=1}^\infty$ has a subsequence converging to a point of Y. Boundedness of $\{U^{(n)}\}_{n=1}^\infty$ implies that

$$\exists C > 0 \quad \sup_{n\in\mathbb{N}} \|U^{(n)}\|_\infty \leq C. \tag{3.17}$$

Then, with $U^{(n)} = (u_1^{(n)}, ..., u_d^{(n)})^T$ we have

$$\sup_{n\in\mathbb{N}} \|u_i^{(n)}\|_\infty \leq C \quad i = 1, ...d. \tag{3.18}$$

This means that $\{u_i^{(n)}\}_{n=1}^\infty$, $i = 1, ..., d$, is a bounded sequence in $X = C[0,b]$. The ith component of $AU^{(n)}(t)$ is:

$$\begin{aligned} \left[AU^{(n)}(t)\right]_i &= \int_0^t p(t,\tau)\sum_{j=1}^d k_{ij}(t,\tau)u_j^{(n)}(\tau)d\tau \\ &= \sum_{j=1}^d A_{ij}u_j^{(n)}(t), \end{aligned} \tag{3.19}$$

where, $A_{ij}u_j^{(n)}$ has a convergence subsequence (A_{ij} is compact). Without loss of generality, again we denote this subsequence by $A_{ij}u_j^{(n)}$. Hence, there is a function $u_j \in X = C[0,b]$ such that

$$\lim_{n\to\infty} A_{ij}u_j^{(n)} = A_{ij}u_j. \tag{3.20}$$

And from (3.14) and (3.15), we have

$$\lim_{n\to\infty} AU^{(n)} = (\sum_{j=1}^{d} A_{ij}u_j, \sum_{j=1}^{d} A_{ij}u_j, ..., \sum_{j=1}^{d} A_{ij}u_j)^T$$

$$= AU, \quad U := (u_1, ..., u_d)^T. \tag{3.21}$$

This means that A is compact, as required.

3.2. **The operator $I - A$ is one-to-one.** The operator A is compact. Then, from the Fredholm alternative theorem, it is sufficient to show that $(I - A)U = 0$ has only a trivial solution. It is equivalent to show that $AU = U$ has only a trivial solution $U = 0$. But, $U = 0$ is a fixed point of A. To show that A has one and only one fixed point, it is sufficient to show

$$\exists m \in \mathbb{N} \quad \|A^m(U) - A^m(W)\|_\infty \le \theta \|U - W\|_\infty, \quad U, W \in X^d. \tag{3.22}$$

where, $0 \le \theta < 1$ is called contraction factor and $X = C[0, b]$. Now, we define some notation. For $a = (a_1, ..., a_d)^T \in \mathbb{R}^d$, define

$$|a| := (|a_1|, ..., |a_d|)^T, \tag{3.23}$$

$$|a|_i := \textit{the} \quad \textit{ith} \quad \textit{component} \quad \textit{of} \quad |a| = |a_i|, \quad i = 1, 2, ..., d, \tag{3.24}$$

and write $(a_1, ..., a_d)^T \le (b_1, ..., b_d)$, if and only if $a_1 \le b_1, ..., a_d \le b_d$. Every $k_{ij}(t, \tau)$, $t \in [0, b], 0 \le \tau \le t$, is continuous, and hence

$$M := \max\{|k_{ij}(t, \tau)| : t \in [0, b], 0 \le \tau \le t, i, j \in \{1, 2, ..., d\}\} < \infty. \tag{3.25}$$

For arbitrary $i \in \{1, ..., d\}, U, W \in X^d$ we, have

$$|AU(t)|_i : = \left| \sum_{j=1}^{d} \int_0^t p(t, \tau) k_{ij}(t, \tau) u_j(\tau) d\tau \right|$$

$$\le Md \int_0^t p(t, \tau) \max_{1\le j\le d} |u_j(\tau)| d\tau.$$

Hence, we can write

$$\max_{1\le i\le d} |AU(t)|_i \le Md \int_0^t p(t, \tau) \max_{1\le j\le d} |u_j(\tau)| d\tau. \tag{3.26}$$

In our analysis, we let $p(t,\tau) = \frac{1}{|t-\tau|^\alpha}$. Then, (3.20) implies:

$$\begin{aligned} \max_{1\le i\le d} |(AU-AW)(t)|_i &= \max_{1\le i\le d} |A(U-W)(t)|_i (A \quad is \quad linear) \\ &\le Md \int_0^t \frac{1}{|t-\tau|^\alpha} \max_{1\le i\le d} |A(U-W)(\tau)|_i d\tau \\ &= \frac{Md}{1-\alpha} t^{1-\alpha} \|U-W\|_\infty. \end{aligned} \tag{3.27}$$

Application of (3.26) and (3.27) yield:

$$\begin{aligned} \max_{1\le i\le d} |(A^2U-A^2W)(t)|_i &= \max_{1\le i\le d} |A(A(U-W))(t)|_i (A^2 \quad is \quad linear) \\ &\le Md \int_0^t \frac{1}{|t-\tau|^\alpha} \max_{1\le i\le d} |A(U-W)(\tau)|_i d\tau \\ &\le \frac{(Md)^2}{1-\alpha}.\frac{\Gamma(1-\alpha)\Gamma(2-\alpha)}{\Gamma(3-2\alpha)} t^{2-2\alpha} \|U-W\|_\infty. \end{aligned} \tag{3.28}$$

It is easy to show by induction that

$$\begin{aligned} \max_{1\le i\le d} |(A^nU-A^nW)(t)|_i &= \max_{1\le i\le d} |A(A^{n-1}(U-W))(t)|_i (A^n \ is \ linear) \\ &\le \frac{\Gamma(2-\alpha)}{(1-\alpha)\Gamma(1-\alpha)}.\frac{(Mdb^{1-\alpha}\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha)+1)} \\ &\quad \|U-W\|_\infty. \end{aligned} \tag{3.29}$$

We know that

$$\lim_{n\to\infty} \frac{\Gamma(2-\alpha)}{(1-\alpha)\Gamma(1-\alpha)}.\frac{(Mdb^{1-\alpha}\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha)+1)} = 0. \tag{3.30}$$

Then, for sufficiently large $m \in \mathbb{N}$, there exists $\theta \in [0,1)$ such that $\|A^mU - A^mW\|_\infty \le \theta \|U-W\|_\infty$. This means that A has one and only one fixed point(see, [1] exercise 4.1.2)

3.3. **The sequence $A_n : Y \to Y$ defined by (3.4) is collectively compact and pointwise convergent.** This is a special case of the Arzela-Ascoli theorem [4].

Theorem 3.4. *Suppose $S \subseteq X^d, X = C[a,b]$. Then, S is relatively sequentially compact if and only if it is bounded and equicontinuous; i.e., if there exists a constant C such that*

$$\max_{i=1,\ldots d} |u_i(x)| \le C, \tag{3.31}$$

for all $x \in C[a,b]$ *and all* $U = (u_1, ..., u_d) \in S$*, and for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\max_{i=1,...d} |u_i(x) - u_i(y)| < \epsilon, \tag{3.32}$$

for all $x, y \in C[a,b]$ *with* $|x - y| < \delta$ *and all* $U = (u_1, ..., u_d) \in S$*.*

Proof. See [4].

For collectively compactness of A_n, it is sufficient to show that, for some $N \in \mathbb{N}$,

$$S := \{A_n U : U \in X^d, \|U\|_\infty \leq 1, n \geq N\} \tag{3.33}$$

satisfies the hypothesis of Theorem 3.4. Let $U = (u_1, ..., u_d)^T \in X^d, \|U\|_\infty \leq 1, t \in [0,b], i \in \{1, ..., d\}$. In (3.12), for $f(\tau) \equiv i$th component of $K(t,\tau)U(\tau)$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\left| \int_0^t p(t,\tau)(L_n(K(t,.)U(.);\tau) - K(t,\tau)U(\tau))d\tau \right|_i < 1. \tag{3.34}$$

Hence,

$$\begin{aligned} |A_n U(t)|_i &= \left| \int_0^t p(t,\tau) L_n(K(t,.)U(.);\tau) d\tau \right|_i \\ &\leq \left| \int_0^t p(t,\tau) K(t,\tau) U(\tau) d\tau \right|_i \\ &\quad + \left| \int_0^t p(t,\tau)(L_n(K(t,.)U(.);\tau) - K(t,\tau)U(\tau))d\tau \right|_i \\ &\leq Md \int_0^t p(t,\tau) d\tau + 1, \end{aligned} \tag{3.35}$$

where, M is defined in (3.25). Thus,

$$\|A_n\|_\infty \leq Md \int_0^b p(b,\tau) d\tau + 1 < \infty. \tag{3.36}$$

This proves the first assertion of Theorem 3.4. For the second condition, let $\epsilon > 0$. Similar to (3.34), there exist $N \in \mathbb{N}$ such that, for all $n \geq N$, and $\widehat{t} \in [0,b]$,

$$\left| \int_0^{\widehat{t}} p(\widehat{t},\tau)(L_n(K(\widehat{t},.)U(.);\tau) - K(\widehat{t},\tau)U(\tau))d\tau \right|_i < \frac{\epsilon}{4}. \tag{3.37}$$

Note that $k_{ij}(t,\tau), i,j \in \{1,...,d\}$, are uniformly continuous on $t \in [0,b], 0 \leq \tau \leq t$. Then, there exists $\delta_1 > 0$ such that, for every $t, \widetilde{t} \in [0,b], 0 \leq \tau \leq \widetilde{t} \leq t, |t-\widetilde{t}| < \delta_1$, we have

$$|k_{ij}(t,\tau) - k_{ij}(\widetilde{t},\tau)| < \frac{\epsilon}{8dP(b)}, \tag{3.38}$$

where, $P(t) := \int_0^t p(t,\tau)d\tau$. For all $t \in [0,b]$, $\lim_{\widetilde{t}\to t} \int_0^{\widetilde{t}} |p(t,\tau) - p(\widetilde{t},\tau)|d\tau = 0$. Then, there exists $\delta_2 > 0$ such that, for every $t, \widetilde{t} \in [0,b], |t-\widetilde{t}| < \delta_2$, we have

$$\int_0^{\widetilde{t}} |p(t,\tau) - p(\widetilde{t},\tau)|d\tau < \frac{\epsilon}{8dM}. \tag{3.39}$$

For all $t \in [0,b]$, $\lim_{\widetilde{t}\to t} \int_{\widetilde{t}}^t p(t,\tau)d\tau = 0$. then, there exists $\delta_3 > 0$ such that for every $t, \widetilde{t} \in [0,b], |t-\widetilde{t}| < \delta_3$, we have

$$\int_{\widetilde{t}}^t p(t,\tau)d\tau < \frac{\epsilon}{4dM}. \tag{3.40}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, for every $t, \widetilde{t} \in [0,b], \widetilde{t} < t < \widetilde{t} + \delta$,

$$
\begin{aligned}
(3.41)\qquad & |A_nU(t) - A_nU(\widetilde{t})|_i = \Big| \int_0^t p(t,\tau)L_n(K(t,.)U(.);\tau)d\tau \\
& - \int_0^{\widetilde{t}} p(\widetilde{t},\tau)L_n(K(\widetilde{t},.)U(.);\tau)d\tau\Big|_i \\
\leq\ & \left|\int_0^t p(t,\tau)[L_n(K(t,.)U(.);\tau) - K(t,\tau)U(\tau)]d\tau\right|_i \\
+\ & \left|\int_0^{\widetilde{t}} p(\widetilde{t},\tau)[L_n(K(\widetilde{t},.)U(.);\tau) - K(\widetilde{t},\tau)U(\tau)]d\tau\right|_i \\
+\ & \left|\int_0^t p(t,\tau)K(t,\tau)U(\tau)d\tau - \int_0^{\widetilde{t}} p(\widetilde{t},\tau)K(\widetilde{t},\tau)U(\tau)d\tau\right|_i \\
(3.42)\qquad <\ & \frac{\epsilon}{4} + \frac{\epsilon}{4} + \left|\int_0^{\widetilde{t}} [p(t,\tau)K(t,\tau) - p(\widetilde{t},\tau)K(\widetilde{t},\tau)]U(\tau)d\tau\right|_i \\
& + \left|\int_{\widetilde{t}}^t p(t,\tau)K(t,\tau)U(\tau)d\tau\right|_i \\
(3.43)\qquad \leq\ & \frac{\epsilon}{2} + Md\int_0^{\widetilde{t}} |p(t,\tau) - p(\widetilde{t},\tau)|d\tau \\
& + \left|\int_0^{\widetilde{t}} p(\widetilde{t},\tau)[K(t,\tau) - K(\widetilde{t},\tau)]U(\tau)d\tau\right|_i + dM\int_{\widetilde{t}}^t p(t,\tau)d\tau \\
<\ & \frac{\epsilon}{2} + Md\frac{\epsilon}{8Md} + \int_0^{\widetilde{t}} p(\widetilde{t},\tau)\sum_{j=1}^d \frac{\epsilon}{8dP(b)}d\tau + Md\frac{\epsilon}{4dM} \leq \epsilon. \\
(3.44)\qquad &
\end{aligned}
$$

This means that S is collectively compact. Similar arguments gives that $A_nU \to AU$; i.e., A_nU is pointwise convergent. □

4. **Numerical Results**

Consider the following theorem.

Theorem 4.1. *For piecewise-continuous f, g and h, the solution u of*

$$u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t, \tag{4.1}$$

$$u(x,0) = f(x), \quad 0 < x < 1, \tag{4.2}$$

$$u(0,t) = g(t), \quad 0 < t, \tag{4.3}$$

$$u_x(1,t) = h(t), \quad 0 < t, \tag{4.4}$$

has the form

$$u(x,t) = v(x,t) - 2\int_0^t \frac{\partial G}{\partial x}(x, t-\tau)\phi_1(\tau)d\tau + 2\int_0^t G(x-1, t-\tau)\phi_2(\tau)d\tau, \tag{4.5}$$

where

$$v(x,t) = \int_{-\infty}^{\infty} G(x-\xi, t) f(\xi) d\xi, \tag{4.6}$$

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\{-\frac{x^2}{4t}\}, \tag{4.7}$$

G is called the fundamental solution of heat equation and f here is a smooth, bounded extension of the f above, if and only if ϕ_1 and ϕ_2 are piecewise continuous solutions of

$$g(t) = v(0,t) + \phi_1(t) + 2\int_0^t G(-1, t-\tau)\phi_2(\tau)d\tau, \tag{4.8}$$

$$h(t) = v_x(1,t) + \phi_2(t) - 2\int_0^t \frac{\partial^2 G}{\partial^2 x}(1, t-\tau)\phi_1(\tau)d\tau. \tag{4.9}$$

Proof. see [2]. □

For $f(x) = 1, g(t) = Erf(\frac{1}{2\sqrt{t}})$, and $h(t) = Exp(-\frac{1}{t})/\sqrt{\pi t}$, the exact solution of (4.1) − (4.4) is $u(x,t) = Erf(\frac{x+1}{2\sqrt{t}})$, and the exact solutions of (4.8) − (4.9) are:

$$\phi_1(t) = -Erfc(\frac{1}{2\sqrt{t}}), \quad \phi_2(t) = 0. \tag{4.10}$$

Table 1 shows relative errors of $\underset{\sim}{\phi_1}$ at $t = 0.01i, i = 1, ..., 10$ with $b = 0.1$, ϕ_1 is exact solution and $\widetilde{\phi_1}$ is evaluated by the product integration technique. Absolute error of ϕ_2 is negligible, and since $\phi_2 = 0$ then the relative error of ϕ_2 is not computable.

Table 1.

i	$\lvert \frac{\phi_1 - \widetilde{\phi_1}}{\phi_1} \rvert_{t=0.01i}$
1	0.0000149272
2	3.54138×10^{-11}
3	4.63845×10^{-13}
4	1.47064×10^{-13}
5	8.07575×10^{-14}
6	8.13343×10^{-15}
7	3.99896×10^{-14}
8	6.6627×10^{-14}
9	1.85675×10^{-12}
10	6.51736×10^{-12}

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References

[1] K. Atkinson and W. Han, *Theoritical Numerical Analysis: A Functional Analysis Framwork*, Springer, New York, 2001.

[2] J. R. Cannon, *The One-dimensional Heat Equation*, Addison-Wesley Publishing Company, 1984.

[3] G. Criscuolo, G. Mastroianni and G.Monegato, Convergence properties of a class of product formulas for weakly singular integral equations, *Math. Comp.* **55** (1990) 213-230.

[4] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Mathematics in Science and Engineering, 132. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1977.

[5] R. Kress, *Numerical Analysis*, Springer-Verlag, New York, 1998.

[6] A. P. Orsi, Product integration for Volterra integral equations of the second kind with weakly singular kernels, *Math. Comp.* **65**(215) (1996) 1201-1212.

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