

VARIOUS TOPOLOGICAL FORMS OF VON NEUMANN REGULARITY IN BANACH ALGEBRAS

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ABSTRACT. In this article we study topological von Neumann regularity and principal von Neumann regularity of Banach algebras. Our main objective is comparing these two types of Banach algebras with some other known Banach algebras and with each other. In particular we show that the class of topologically von Neumann regular Banach algebras contains all C^* -algebras, group algebras of compact abelian groups and certain weakly amenable Banach algebras while it excludes measure algebras of certain locally compact Abelian groups. Moreover we show that in a unital amenable Banach algebra principal regularity implies topological regularity. Finally we use topological regularity to obtain some information about hereditary C^* -subalgebras of a given C^* -algebra.

1. Introduction

Finding conditions which force a Banach algebra to be finite-dimensional has a fairly long history. See [3, 5, 8, 14, 21, 24–27] for some results in this direction. Kaplansky [14] showed that every von Neumann regular Banach algebra is finite dimensional. To the best of our knowledge, so

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far the only reference on the topological analogs of von Neumann regularity in the literature is the unpublished work of the first author [7]. The main motivation for introducing topological regularity was identifying conditions which force a Banach algebra to be finite dimensional. However existence of infinite dimensional topologically von Neumann regular Banach algebras motivated us to do a comparative study of the notions of topological von Neumann regularity and principal regularity. For instance it is well known that von Neumann regularity implies semisimplicity; While we show that topological von Neumann regularity implies semiprimeness. Roughly speaking topological von Neumann regularity is consistent with operator norm (Theorem 2.5 below) and inconsistent with the total variation norm in certain measure algebras (Corollary 2.4). Also between the two concepts, we can say that at least under certain conditions principal von Neumann regularity is stronger than topological von Neumann regularity as the main theorem of Section 3 shows.

This paper is organized as follows. Section 2 is devoted to the study of topological von Neumann regularity. In particular some basic facts and hereditary properties of topologically von Neumann regular algebras are discussed in this section. Then we find a criterion for topological regularity and the relationship between topological regularity and weak amenability. Moreover we prove that all C^* -algebras are topologically regular, while measure algebras of certain locally compact groups are not. In Section 3 we study another topological form of von Neumann regularity which we call principal regularity. In this section we compare the two concepts of topological von Neumann regularity and principal regularity. In the last section we apply some results of previous sections to identify hereditary C^* -subalgebras of an arbitrary C^* -algebra.

Before proceeding further, let us describe some notations which we rely on throughout this article. For the terms which are not introduced here the reader may refer to one of [2, 4, 13, 20].

Throughout \mathcal{A} and \mathcal{B} are Banach algebras, \mathcal{A} -module means Banach \mathcal{A} -bimodule and the term semisimple means Jacobson semisimple. If for every \mathcal{A} -bimodule \mathcal{X} every bounded derivation from \mathcal{A} into \mathcal{X}^* is inner, then \mathcal{A} is called *amenable*. If every bounded derivation from \mathcal{A} into \mathcal{A}^* is inner, then \mathcal{A} is called *weakly amenable*. We say that a short exact sequence

$$0 \longrightarrow \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \longrightarrow 0.$$

of \mathcal{A} -modules is *admissible* [resp. *splits*] if g has a bounded linear right inverse [resp. a bounded linear right inverse which is also an \mathcal{A} -module homomorphism]. This is equivalent to say that $f(\mathcal{X})$ has a Banach space [resp. \mathcal{A} -module] complement in \mathcal{Y} . We say that a left \mathcal{A} -module \mathcal{M} is *flat* if for every short exact sequence

$$0 \longrightarrow \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \longrightarrow 0.$$

of \mathcal{A} -modules, the following sequence remains exact, where $\widehat{\otimes}_{\mathcal{A}}$ denotes \mathcal{A} -module projective tensor product.

$$0 \longrightarrow \mathcal{X} \widehat{\otimes}_{\mathcal{A}} \mathcal{M} \xrightarrow{f \otimes 1} \mathcal{Y} \widehat{\otimes}_{\mathcal{A}} \mathcal{M} \xrightarrow{g \otimes 1} \mathcal{Z} \widehat{\otimes}_{\mathcal{A}} \mathcal{M} \longrightarrow 0.$$

The left annihilator [resp. right annihilator/ annihilator] of a subset E of \mathcal{A} which is denoted by $\ellan(E)$ [resp. $\text{ran}(E)/\text{ann}(E)$] is the set of all $x \in \mathcal{A}$ where $xE = 0$ [resp. $Ex = 0/ \text{ } Ex = xE = 0$]. When $E = \{a\}$, we denote $\ellan(E)$ [resp. $\text{ran}(E)/\text{ann}(E)$] simply by $\ellan(a)$ [resp. $\text{ran}(a)/\text{ann}(a)$].

We say that \mathcal{A} is *reduced* if for every $a \in \mathcal{A}$ the identity $a^2 = 0$ implies $a = 0$. We say \mathcal{A} is *semiprime* if $\{0\}$ is the only ideal \mathcal{I} of \mathcal{A} with $\mathcal{I}^2 = \{0\}$; However an ideal \mathcal{I} of \mathcal{A} is semiprime if \mathcal{A}/\mathcal{I} is semiprime. Note that being a semiprime ideal for an ideal \mathcal{I} of \mathcal{A} is different from being a semiprime algebra in its own right.

Let X be a compact Hausdorff space, $C(X)$ be the C^* -algebra of continuous functions on X and $f \in C(X)$. We denote the zero set of f by $Z(f)$. Let G be a locally compact group. The algebras $L^1(G)$, $M(G)$, $L^\infty(G)$, $UC(G)$, $LUC(G)$, $RUC(G)$ and $C_b(G)$ have their usual meanings.

2. Topological Regularity

In this section we study a natural topological analog of the concept of von Neumann regularity. Recall that an element $a \in \mathcal{A}$ is called *von Neumann regular* if there is an $x \in \mathcal{A}$ such that $a = axa$. A non-zero element $a \in \mathcal{A}$ is called *resp. weakly von Neumann regular* if there is a non-zero $x \in \mathcal{A}$ such that $x = xax$. If every non-zero $a \in \mathcal{A}$ is von Neumann regular [resp. weakly von Neumann regular], then \mathcal{A} is von Neumann regular [resp. weakly von Neumann regular]. It is well known that in a von Neumann regular Banach algebra every principal one sided ideal is generated by an idempotent. Also weak regularity is equivalent

to the following condition: Every non-zero one sided ideal contains a non-zero idempotent [14].

Definition. An element $a \in \mathcal{A}$ is called *topologically von Neumann regular* if $a \in \overline{a\mathcal{A}a}^{\|\cdot\|}$. If every element of \mathcal{A} is topologically von Neumann regular, then \mathcal{A} is called topologically von Neumann regular.

Convention. From now on by the suffix “regular” we mean “von Neumann regular”.

Examples 2.1. (i) Let \mathcal{H} be a Hilbert space. Every partial isometry and every Fredholm operator on \mathcal{H} is a regular element of $\mathcal{B}(\mathcal{H})$.

(ii) Every C^* -algebra \mathcal{A} is topologically regular, as we will see in the Theorem 2.6.

(iii) Assume L_a and R_a are left and right multiplication by an element a of \mathcal{A} , respectively. If a is topologically regular then L_a and R_a are topologically regular elements of $\mathcal{B}(\mathcal{A})$.

(iv) In every $*$ -Banach algebra \mathcal{A} with continuous involution, $a \in \mathcal{A}$ is topologically regular if and only if a^* is topologically regular.

In the following proposition whose proof is not difficult and is left to the reader, we have collected some basic facts about topological regularity.

Proposition 2.2. Suppose $\mathcal{B}_\lambda, \lambda \in \Lambda$ are Banach algebras and \mathcal{A} is topologically regular.

(i) If $e \neq 0$ is an idempotent in \mathcal{A} , then $e\mathcal{A}e$ is topologically regular.

(ii) If $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous epimorphism, then \mathcal{B} is topologically regular.

(iii) If the ℓ^∞ -direct sum $\bigoplus_\infty \mathcal{B}_\lambda$ is topologically regular, then every \mathcal{B}_λ is topologically regular.

(iv) If the ℓ^1 -direct sum $\bigoplus_1 \mathcal{B}_\lambda$ is topologically regular, then every \mathcal{B}_λ is topologically regular. If Λ is countable, the converse is also true.

(v) The unitization $\mathcal{B}^\#$ of \mathcal{B} is topologically regular if and only if \mathcal{B} is.

In the following theorem we provide a criterion for topological regularity. But first we need to recall the following concept from ring theory. A ring R is called *semiprime* if for every ideal \mathcal{I} of R the identity $\mathcal{I}^2 = 0$ implies $\mathcal{I} = 0$. However an ideal \mathcal{I} of R is called semiprime if R/\mathcal{I} is

a semiprime ring. It is well known that regularity implies semisimplicity [11, page 443]. A topological analog of this fact is part (ii) of the following theorem.

Theorem 2.3. *Suppose \mathcal{A} is topologically regular. Then the following statements hold.*

(i) *For every right ideal I and every left ideal J in \mathcal{A} we have $\overline{IJ} = \overline{I} \cap \overline{J}$. If \mathcal{A} has an approximate identity, then the converse is also true.*

(ii) *\mathcal{A} is semiprime.*

(iii) *If \mathcal{I} is a closed ideal of \mathcal{A} , then \mathcal{I} and $\frac{\mathcal{A}}{\mathcal{I}}$ are topologically regular.*

Proof. (i) Suppose \mathcal{A} is topologically regular and I and J are right and left ideals in \mathcal{A} respectively. It suffices to show that $\overline{I} \cap \overline{J} \subseteq \overline{IJ}$, since the inclusion $\overline{IJ} \subseteq \overline{I} \cap \overline{J}$ holds trivially. Let $a \in \overline{I} \cap \overline{J}$ and $\{x_n\}$ be a sequence in \mathcal{A} such that $ax_n a \rightarrow a$. Then $a \in \overline{I \overline{J}} \subseteq \overline{IJ}$.

Conversely, assume that \mathcal{A} has an approximate identity and $\overline{IJ} = \overline{I} \cap \overline{J}$ for any right ideal I and any left ideal J . Then for any $a \in \mathcal{A}$, we have

$$a \in \overline{(a\mathcal{A})} \cap \overline{(\mathcal{A}a)} = \overline{(a\mathcal{A})(\mathcal{A}a)} = \overline{a\mathcal{A}a}.$$

Therefore a is topologically regular.

(ii) Suppose I is an ideal of \mathcal{A} such that $I^2 = 0$. By the first part, $\overline{I} = \overline{I^2} = 0$ and hence $I = 0$.

(iii) The statement regarding $\frac{\mathcal{A}}{\mathcal{I}}$ follows from Proposition 2.2(ii). Suppose \mathcal{A} is topologically regular and $a \in I$. Then there is a sequence $\{x_n\}$ in \mathcal{A} such that $a = \lim_n ax_n a$. By part (i) for every n we have

$$ax_n a \in aI \cap Ia \subseteq \overline{aI} \cap \overline{Ia} = \overline{aIIa} \subseteq \overline{aIa}.$$

Therefore $a \in \overline{aIa}$. □

Now let us consider concrete Banach algebras, in particular C^* -algebras and the algebras associated with a locally compact group G . Topological regularity of group algebras of compact abelian groups, C^* -algebras, the algebras $RUC(G)$, $LUC(G)$, $UC(G)$, $L^\infty(G)$ and $C_b(G)$ follows from Theorems 2.5 and 2.6 below. However for $M(G)$ the situation is totally different, even under restrictive assumptions, as we see in the next corollary.

Corollary 2.4. *The measure algebra $M(G)$ of every non-discrete metrizable locally compact group G is not topologically regular.*

Proof. Suppose in contrary that $M(G)$ is topologically regular. By Theorem 2.3 for every ideal I in $M(G)$ we have $\overline{I^2} = \overline{I}$. But by [4, Theorem 3.3.39] this is not the case for the ideal $M_C(G)$ of continuous measures on G . Therefore $M(G)$ is not topologically regular. \square

Theorem 2.5. *The group algebra $L^1(G)$ of every compact abelian group G is topologically regular.*

Proof. Let $a \in L^1(G)$. Then $I = \overline{aL^1(G)}$ is a closed ideal of $L^1(G)$ and hence has an approximate identity by [28, Proposition 1]. So $\overline{I^2} = I$ and hence $a \in I = \overline{I^2} \subseteq \overline{aL^1(G)a}$. \square

Theorem 2.6. *Every C^* -algebra is topologically regular.*

Proof. Let \mathcal{A} be an arbitrary C^* -algebra and $a \in \mathcal{A}$. Since \mathcal{A} has a bounded approximate identity, then $a \in \overline{a\mathcal{A}} \cap \overline{\mathcal{A}a}$. By [13, Proposition 4.2.9] there is a $c \in \overline{\mathcal{A}a} \cap \mathcal{A}^+$ and a $b \in \mathcal{A}$ such that $a = bc$. Theorem 3.1.2 of [19] implies that $\overline{a\mathcal{A}}$ has a bounded left approximate identity $\{u_\lambda\}$. So

$$a = \lim_{\lambda} u_\lambda a = \lim_{\lambda} (u_\lambda b)c \in \overline{a\mathcal{A}} \overline{\mathcal{A}a} = \overline{a\mathcal{A}a} = \overline{a\mathcal{A}a}.$$

Therefore a is topologically regular for all $a \in \mathcal{A}$. \square

Theorem 2.7. *Suppose \mathcal{A} is commutative and has a bounded approximate identity. Then \mathcal{A} is topologically regular if and only if every closed ideal of \mathcal{A} is semiprime.*

Proof. Suppose every closed ideal of \mathcal{A} is semiprime and $a \in \mathcal{A}$. By assumption $\mathcal{I} = \overline{a^2\mathcal{A}}$ is a semiprime ideal of \mathcal{A} . So $\mathcal{B} = \frac{\mathcal{A}}{\mathcal{I}}$ is a semiprime commutative Banach algebra. Let \mathcal{J} be the principal ideal $(a + \mathcal{I})\mathcal{B}$. Since \mathcal{A} has a bounded approximate identity, then $a^2 \in \overline{a^2\mathcal{A}} = \mathcal{I}$ and hence $\mathcal{J}^2 = 0$. Since \mathcal{B} is semiprime, then $\mathcal{J} = 0$ which implies that $a \in \overline{a\mathcal{A}a}$. Therefore \mathcal{A} is topologically regular. The converse statement follows from Theorem 2.3. \square

Theorem 2.8. *Let \mathcal{A} be a commutative weakly amenable Banach algebra with a bounded approximate identity. Then \mathcal{A} is topologically regular if and only if every closed ideal of \mathcal{A} is weakly amenable.*

Proof. Suppose \mathcal{A} is topologically regular. By Theorem 2.3 (i) for every closed ideal I in \mathcal{A} we have $\overline{I^2} = I$. So by [9, Proposition 2.2] I is weakly amenable.

Conversely, suppose every closed ideal in \mathcal{A} is weakly amenable. Then $\overline{a\mathcal{A}}$ is weakly amenable and hence $(a\mathcal{A})^2 = \overline{a\mathcal{A}}$ by [9, Proposition 2.2]. Since \mathcal{A} has a bounded approximate identity, then $a \in \overline{a\mathcal{A}}$. Thus

$$a \in \overline{a\mathcal{A}} = \overline{(a\mathcal{A})^2} = \overline{(a\mathcal{A})(a\mathcal{A})} = \overline{(a\mathcal{A})(\mathcal{A}a)} = \overline{a\mathcal{A}^2a} = \overline{a\mathcal{A}a}.$$

Therefore a is topologically regular. □

3. Principal Regularity

If $a \in \mathcal{A}$ is regular, then the principal left [resp. right] ideal generated by a has an idempotent generator [11, page 443] and hence is complemented. So another natural generalization of the notion of regularity could be existence of a Banach space complement for the closure of one sided principal ideals which we call principal regularity. In this section we study this notion, specially in the category of unital amenable Banach algebras. We begin with a precise definition of this concept.

Definition. An element $a \in \mathcal{A}$ is called *principally left* [resp. *right*] *von Neumann regular* if $\overline{a\mathcal{A}}^{\|\cdot\|}$ [resp. $\overline{\mathcal{A}a}^{\|\cdot\|}$] is complemented. If every element of \mathcal{A} is both principally left and right von Neumann regular, then \mathcal{A} is called *principally von Neumann regular*.

Notation. As in the previous section we use the term “...regular” instead of “...von Neumann regular”.

Lemma 3.1. *Let I be a closed left ideal in \mathcal{A} with a bounded left approximate identity $\{e_\alpha\}$. If \mathcal{A} is principally left regular, then so is I . The right analog of this result is also true.*

Proof. Let $a \in I$. By assumption there is a closed subspace \mathcal{M} of \mathcal{A} such that $\mathcal{A} = \overline{\mathcal{A}a} \oplus \mathcal{M}$. Let $x \in \mathcal{A}$. Then $xa = x(\lim_\alpha e_\alpha a) = \lim_\alpha (xe_\alpha)a \in \overline{\mathcal{A}a}$. So $\mathcal{A}a \subseteq \overline{\mathcal{A}a}$ and hence $\overline{\mathcal{A}a} = \overline{\mathcal{A}a}$. Now let $x \in I$ and x_1 and x_2 be

the unique elements of \overline{Ia} and \mathcal{M} respectively such that $x = x_1 + x_2$. Then $x_2 \in \mathcal{M} \cap I$ and hence $I = \overline{Ia} + (\mathcal{M} \cap I)$. Since $\overline{Ia} \cap (\mathcal{M} \cap I) = \{0\}$, then $I = \overline{Ia} \oplus (\mathcal{M} \cap I)$. Therefore a is principally regular as an element of I . \square

Now we prove the main result of this section in which we compare the two notions of topological regularity and principal regularity.

Theorem 3.2. Let \mathcal{A} be a unital amenable Banach algebra. If $a \in \mathcal{A}$ is principally right or left regular, then a is topologically regular.

Proof. We use an algebraic argument. Suppose a is principally right regular. The sequence

$$0 \longrightarrow \overline{a\mathcal{A}} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\overline{a\mathcal{A}} \longrightarrow 0.$$

is admissible as $\overline{a\mathcal{A}}$ is complemented. Moreover since \mathcal{A} is amenable, then every left or right \mathcal{A} module is flat [22, page 136]. So the following sequence is exact, since $\overline{\mathcal{A}a}$ is flat.

$$0 \longrightarrow \overline{a\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a} \xrightarrow{i \otimes 1} \mathcal{A} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a} \xrightarrow{\pi \otimes 1} \mathcal{A}/\overline{a\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a} \longrightarrow 0.$$

We show that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{a\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a} & \xrightarrow{i \otimes 1} & \mathcal{A} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a} & \xrightarrow{\pi \otimes 1} & \mathcal{A}/\overline{a\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a} \longrightarrow 0 \\ & & \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 \\ 0 & \longrightarrow & \overline{a\mathcal{A}} \cap \overline{\mathcal{A}a} & \xrightarrow{i} & \overline{\mathcal{A}a} & \xrightarrow{\gamma} & \mathcal{A}/\overline{a\mathcal{A}} \longrightarrow 0 \end{array}$$

where i is the inclusion map, π_j , $j = 1, 2, 3$ are canonical projections and γ is defined by $\gamma(x) = \overline{a\mathcal{A}} + x$. The first row is exact, as we showed. For the second row, we need only to show that $\ker(\gamma) = \text{image}(i) = \overline{a\mathcal{A}} \cap \overline{\mathcal{A}a}$. Let $x \in \overline{\mathcal{A}a}$. Then $\gamma(x) = 0$ if and only if $x \in \overline{a\mathcal{A}}$ if and only if $x \in \overline{a\mathcal{A}} \cap \overline{\mathcal{A}a}$. Therefore the second row is exact.

Now suppose $\sum_{i=1}^{\infty} \bar{x}_i \otimes y_i a \in \mathcal{A}/\overline{a\mathcal{A}} \widehat{\otimes}_{\mathcal{A}} \overline{\mathcal{A}a}$ is such that $\pi_3 \left(\sum_{i=1}^{\infty} \bar{x}_i \otimes y_i a \right) = 0$. Then we have

$$0 = \sum_{i=1}^{\infty} \overline{x_i y_i a} = \overline{\sum_{i=1}^{\infty} x_i y_i a} = \left(\overline{\sum_{i=1}^{\infty} x_i y_i} \right) a.$$

So

$$0 = \left(\overline{\sum_{i=1}^{\infty} x_i y_i} \right) a \otimes 1 = \left(\overline{\sum_{i=1}^{\infty} x_i y_i} \right) \otimes a = \sum_{i=1}^{\infty} (\overline{x_i y_i}) \otimes a = \sum_{i=1}^{\infty} \overline{x_i} \otimes y_i a.$$

Therefore π_3 is one to one. Clearly π_2 is onto. Now by the five Lemma [11, page 180] since π_2 is onto and π_3 is one to one, then π_1 is onto. But $\pi_1(\overline{a\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}a}) = \overline{a\mathcal{A}a}$. So $\overline{a\mathcal{A}a} = \overline{a\mathcal{A}} \cap \overline{\mathcal{A}a}$ and hence $a \in \overline{a\mathcal{A}a}$. \square

The following result is a partial converse of the the preceding theorem. Recall that a linear map T on \mathcal{A} is called a *multiplier* if for every $a, b \in \mathcal{A}$ the identity $aT(b) = T(a)b$ holds.

Proposition 3.3. Suppose \mathcal{A} is commutative and topologically regular. If $a \in \mathcal{A}$ is such that $a\mathcal{A}$ is closed, then a is principally regular.

Proof. First we show that for a multiplier T on \mathcal{A} with closed range, we have $T(\mathcal{A}) = T^2(\mathcal{A})$ if and only if $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$.

To see this fact, suppose $T(\mathcal{A}) = T^2(\mathcal{A})$ and $y \in T(\mathcal{A}) \cap Ker(T)$. Let $x \in \mathcal{A}$ be such that $y = T(x)$. Then $y^2 = yT(x) = T(y)x = 0$ which together with Theorem 2.3(ii) implies that $y = 0$. Thus $T(\mathcal{A}) \cap Ker(T) = \{0\}$. On the other hand if $x \in \mathcal{A}$, then $T(x) = T^2(w)$ for some $w \in \mathcal{A}$ and hence $x - T(w) \in Ker(T)$. Thus $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$. Conversely, suppose $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$ and $y \in T(\mathcal{A})$. Let $x \in \mathcal{A}$, $x_1 \in Ker(T)$ and $x_2 \in T(\mathcal{A})$ be such that $y = T(x)$ and $x = x_1 + x_2$. Then $y = T(x_2)$ and hence $y \in T^2(\mathcal{A})$. Therefore $T(\mathcal{A}) = T^2(\mathcal{A})$.

Now let L_a be the left translation operator by a and $\{x_n\}$ be a sequence in \mathcal{A} such that $a = \lim_n ax_n a$. Then for every $x \in \mathcal{A}$ we have

$$ax = \lim_n ax_n ax = \lim_n a^2 x_n x \in \overline{a^2 \mathcal{A}} \subseteq \overline{a\mathcal{A}} = a\mathcal{A}.$$

Thus $\overline{a^2 \mathcal{A}} = a\mathcal{A}$ and hence $a^2 \mathcal{A} = \overline{a\mathcal{A}} = \overline{a^2 \mathcal{A}} = a\mathcal{A}$, that is, $L_a(\mathcal{A}) = L_a^2(\mathcal{A})$ and by the fact that we just proved $\mathcal{A} = L_a(\mathcal{A}) \oplus Ker(L_a)$. Therefore a is principally regular. \square

4. Applications to Hereditary C^* -Subalgebras

In this section we obtain some information about hereditary C^* -subalgebras of a C^* -algebra. Recall that a closed self-adjoint subalgebra \mathcal{B} of a C^* -algebra \mathcal{A} is called a *hereditary C^* -subalgebra* if for $a \in \mathcal{A}^+$

and $b \in \mathcal{B}^+$ the inequality $a \leq b$ implies that $a \in \mathcal{B}$. The following result was known only for positive elements of a C^* -algebra [19, page 85]. Here we extend it to self-adjoint elements, with a different argument, based on Theorem 2.6.

Theorem 4.1. *Suppose \mathcal{A} is an arbitrary C^* -algebra and $a \in \mathcal{A}$ is self-adjoint. Then $\overline{a\mathcal{A}a}$ is the hereditary C^* -subalgebra of \mathcal{A} generated by a .*

Proof. By Theorem 2.6 \mathcal{A} is topologically regular. So $a \in \overline{a\mathcal{A}a}$. Moreover if $\mathcal{I} = a\mathcal{A}$ and $\mathcal{L} = \overline{\mathcal{I}}$, then by Theorem 2.3(i) we have $\overline{a\mathcal{A}a} = \overline{\mathcal{I}\mathcal{I}^*} = \overline{\mathcal{I}} \cap \overline{\mathcal{I}^*} = \mathcal{L} \cap \mathcal{L}^*$ where the later is a hereditary C^* -subalgebra of \mathcal{A} by [19, Theorem 3.2.1]. Thus $\overline{a\mathcal{A}a}$ is a hereditary C^* -subalgebra of \mathcal{A} , containing a . To show that it is the smallest such algebra, suppose \mathcal{B} is a hereditary C^* -subalgebra of \mathcal{A} containing a . Then by [19, Theorem 3.2.2] for every $x \in \mathcal{A}$, $axa \in \mathcal{B}$ and hence $\overline{a\mathcal{A}a} \subseteq \mathcal{B}$. \square

Theorem 4.2. *Let \mathcal{A} be a principally left [resp. right] regular C^* -algebra. Then every hereditary C^* -subalgebra \mathcal{B} of \mathcal{A} is principally left [resp. right] regular.*

Proof. By [19, Theorem 3.2.1] there is a closed left ideal \mathcal{L} in \mathcal{A} such that $\mathcal{B} = \mathcal{L} \cap \mathcal{L}^*$. Let $b \in \mathcal{B}$ and $x \in \mathcal{A}$. By assumption there is a closed subspace \mathcal{M} of \mathcal{A} such that $\mathcal{A} = \overline{\mathcal{A}b} \oplus \mathcal{M}$. Then with the same argument as in the proof of Lemma 3.1, we get $\overline{\mathcal{A}b} = \overline{\mathcal{L}b}$ and $\mathcal{L} = \overline{\mathcal{L}b} \oplus (\mathcal{M} \cap \mathcal{L})$. Now $\mathcal{L} \cap \mathcal{L}^*$ is a closed right ideal in \mathcal{L} . Again using the argument of Lemma 3.1 we see that $(\overline{\mathcal{L} \cap \mathcal{L}^*})b = \overline{\mathcal{L}b}$ and $\mathcal{L} \cap \mathcal{L}^* = (\overline{\mathcal{L} \cap \mathcal{L}^*})b \oplus (\mathcal{M} \cap \mathcal{L} \cap \mathcal{L}^*)$. Therefore \mathcal{B} is principally left regular. The statement for principal right regularity can be proved first by constructing a right version of [19, Theorem 3.2.1] and then using the above argument. \square

We close our discussion with the following proposition which sharpens the conclusion of [23, Proposition 1.3.1], as $\overline{C(X)f}$ is not an AW^* -algebra in general. Moreover the approximation can be done by using a special class of projections.

Proposition 4.3. *Let X be a compact Stonean space and $f \in C(X)$. Then every element of $\overline{C(X)f}$ can be uniformly approximated by finite linear combinations of projections in $C(X)f$.*

Proof. Let $g \in \overline{C(X)f}$ and $\varepsilon > 0$ be given. There is a $h \in C(X)f$ such that $\|h - g\| < \frac{\varepsilon}{2}$. Let $E = Z(f)^C$ and $k \in \mathbb{N}$ be such that $\frac{\|h\|}{k} < \frac{\varepsilon}{2}$. If

$$E_j = \{x : |h(x)| > j \frac{\|h\|}{k}\}, \quad j = 0, \dots, k$$

then

$$\Phi = E_k \subseteq \overline{E_{k-1}} \subseteq \dots \subseteq \overline{E_1} \subseteq E_0 = Z(h)^C \subseteq E.$$

Let $e_j = \chi_{(\overline{E_j} \setminus \overline{E_{j+1}})}$, $j = 1, \dots, k-1$. Define the function u on X by $u = \sum_{j=1}^{k-1} j \frac{\|h\|}{k} e_j$. Since $\overline{E_j}$ s are clopen, then so are $\overline{E_j} \setminus \overline{E_{j+1}}$. Thus $e_j^2 = e_j \in C(X)$. Moreover if for $j = 1, \dots, k-1$ we define the function f_j to be $f_j(x) = \frac{1}{f(x)}$ on $\overline{E_j} \setminus \overline{E_{j+1}}$ and zero elsewhere, then $f_j \in C(X)$. So $e_j = f_j f_j \in C(X)f$ and hence u is a linear combination of projections in $C(X)f$. Finally observe that $\|h - u\| \leq \frac{\varepsilon}{2}$ and hence $\|g - u\| < \varepsilon$. \square

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