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INSERTION OF A $\gamma-$CONTINUOUS FUNCTION

M. MIRMIRAN

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ABSTRACT. A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a $\gamma-$continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour, et. al. [12], while the concept of a locally dense set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [?]. A subset A of a topological space (X, τ) is called *semi-open* [11] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open, or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X, τ) is called $\gamma-$*open*, if $A \cap S$ is preopen, whenever S is preopen [1]. A set A is called $\gamma-$closed if its complement is $\gamma-$open, or equivalently if $A \cup S$ is preclosed, whenever S is preclosed.

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We have that if a set is $\gamma-$open, then it is semi-open and preopen.

Recall that a real-valued function f, defined on a topological space X, is called $A-$continuous [14] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subset of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts, the reader may refer to [5, 6].

Hence, a real-valued function f defined on a topological space X is called precontinuous (respectively semi-continuous or $\gamma-$continuous) if the preimage of every open subset of $\mathbb{R}$ is preopen (respectively semi-open or $\gamma-$open) subset of X.

Precontinuity was called by Ptak nearly continuity [15]. Nearly continuity or precontinuity is known also as almost continuity by Husain [7]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [2].

Results of Katětov [8, 9] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a sufficient condition for the insertion of a $\gamma-$continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, then we write $g \leq f$ (respectively $g < f$) in case $g(x) \leq f(x)$ (respectively $g(x) < f(x)$), for all x in X.

The following definitions are modifications of conditions considered in [10].

A property P defined relative to a real-valued function on a topological space is a $\gamma-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any $\gamma-$continuous function also has property P. If P_1 and P_2 are $\gamma-$property, then the following terminology is used: (i) A space X has the *weak* $\gamma-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a $\gamma-$continuous function h such that $g \leq h \leq f$, (ii) A space X has the $\gamma-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a $\gamma-$continuous function h such that $g < h < f$, and (iii) A space X has the *weakly* $\gamma-$*insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$

has property P_1, f has property P_2 and $f-g$ has property P_2, then there exists a $\gamma-$continuous function h such that $g<h<f$.

Here, we give a sufficient condition for the weak $\gamma-$insertion property. Also, for a space with the weak $\gamma-$insertion property, we give a necessary and sufficient condition for the space to have the $\gamma-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. **The Main Result**

Before giving a sufficient condition for insertability of a $\gamma-$continuous function, the necessary definitions and terminology are stated.

Let (X,τ) be a topological space. The family of all $\gamma-$open, $\gamma-$closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma O(X,\tau)$, $\gamma C(X,\tau)$, $sO(X,\tau)$, $sC(X,\tau)$, $pO(X,\tau)$ and $pC(X,\tau)$, respectively.

Definition 2.1. *Let A be a subset of a topological space (X,τ). Respectively, we define the $\gamma-$closure, $\gamma-$interior, s-closure, s-interior, p-closure and p-interior of a set A, denoted by $\gamma Cl(A), \gamma Int(A), sCl(A), sInt(A), pCl(A)$ and $pInt(A)$ as follows:*
$\gamma Cl(A)=\cap\{F : F\supseteq A, F\in \gamma C(X,\tau)\}$,
$\gamma Int(A)=\cup\{O : O\subseteq A, O\in \gamma O(X,\tau)\}$,
$sCl(A)=\cap\{F : F\supseteq A, F\in sC(X,\tau)\}$,
$sInt(A)=\cup\{O : O\subseteq A, O\in sO(X,\tau)\}$,
$pCl(A)=\cap\{F : F\supseteq A, F\in pC(X,\tau)\}$ *and*
$pInt(A)=\cup\{O : O\subseteq A, O\in pO(X,\tau)\}$.

Respectively, we have $\gamma Cl(A), sCl(A), pCl(A)$ are $\gamma-$closed, semi-closed, preclosed and $\gamma Int(A), sInt(A), pInt(A)$ are $\gamma-$open, semi-open, preopen.

The following first two definitions are modifications of conditions considered in [9, 10].

Definition 2.2. *If ρ is a binary relation in a set S, then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$, for any u and v in S.*

Definition 2.3. *A binary relation ρ in the power set $P(X)$ of a topological space X is called a strong binary relation in $P(X)$ in case ρ satisfies each of the following conditions:*

1) If $A_i \;\rho\; B_j$, *for any* $i \in \{1, \dots, m\}$ *and for any* $j \in \{1, \dots, n\}$, *then there exists a set* C *in* $P(X)$ *such that* $A_i \;\rho\; C$ *and* $C \;\rho\; B_j$, *for any* $i \in \{1, \dots, m\}$ *and any* $j \in \{1, \dots, n\}$.

2) If $A \subseteq B$, *then* $A \;\bar{\rho}\; B$.

3) If $A \;\rho\; B$, *then* $\gamma Cl(A) \subseteq B$ *and* $A \subseteq \gamma Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows.

Definition 2.4. *If* f *is a real-valued function defined on a space* X *and if* $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$, *for a real number* ℓ, *then* $A(f, \ell)$ *is called a lower indefinite cut set in the domain of* f *at the level* ℓ.

We now give the following main result.

Theorem 2.5. *Let* g *and* f *be real-valued functions on a topological space* X *with* $g \leq f$. *If there exists a strong binary relation* ρ *on the power set of* X *and if there exist lower indefinite cut sets* $A(f, t)$ *and* $A(g, t)$ *in the domain of* f *and* g *at the level* t *for each rational number* t *such that if* $t_1 < t_2$, *then* $A(f, t_1) \;\rho\; A(g, t_2)$, *then there exists a* $\gamma-$*continuous function* h *defined on* X *such that* $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis, there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1) \;\rho\; A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \;\bar{\rho}\; F(t_2), G(t_1) \;\bar{\rho}\; G(t_2)$, and $F(t_1) \;\rho\; G(t_2)$. By lemmas 1 and 2 of [9], it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \;\rho\; H(t_2), H(t_1) \;\rho\; H(t_2)$ and $H(t_1) \;\rho\; G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: if x is in $H(t)$, then x is in $G(t')$, for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence, $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$, for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence, $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \gamma Int(H(t_2)) \setminus \gamma Cl(H(t_1))$. Hence, $h^{-1}(t_1, t_2)$ is a $\gamma-$open subset of X, i.e., h is a $\gamma-$continuous function on X. □

The above proof used the technique of the proof of Theorem 1 in [8].

Theorem 2.6. *Let P_1 and P_2 be $\gamma-$property and X be a space that satisfies the weak $\gamma-$insertion property for (P_1, P_2). Also, assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the $\gamma-$insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by $\gamma-$continuous functions.*

Proof. See Theorem 2.1 in [13]. □

3. Applications

The abbreviations *pc* and *sc* are used for precontinuous and semicontinuous, respectively.

Corollary 3.1. *If for each pair of disjoint preclosed (respectively semi-closed) sets F_1, F_2 of X, there exist $\gamma-$open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then X has the weak $\gamma-$insertion property for (pc, pc) (resoectively (sc, sc)).*

proof. Let g and f be real-valued functions defined on X such that f and g are *pc* (respectively *sc*), and $g \leq f$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preclosed (respectively semi-closed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (respectively semi-open) set, it follows that $pCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (respectively $sCl(A(f, t_1)) \subseteq sInt(A(g, t_2)))$. Hence, $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.5. □

Corollary 3.2. *If for each pair of disjoint preclosed (respectively semi-closed) sets F_1, F_2, there exist $\gamma-$open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then every precontinuous (respectively semi-continuous) function is $\gamma-$continuous.*

Proof. Let f be a real-valued precontinuous (respectively semi-continuous) function defined on X. Set $g = f$. Then, by Corollary 3.1, there exists a $\gamma-$continuous function h such that $g = h = f$. □

Corollary 3.3. *If for each pair of disjoint preclosed (respectively semi-closed) sets F_1, F_2 of X, there exist $\gamma-$open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then X has the $\gamma-$insertion property for (pc, pc) (respectively (sc, sc)).*

Proof. Let g and f be real-valued functions defined on X such that f and g are pc (respectively sc), and $g < f$. Set $h = (f + g)/2$. Thus, $g < h < f$, and by Corollary 3.2, since g and f are $\gamma-$continuous functions, hence h is a $\gamma-$continuous function. □

Corollary 3.4. *If for each pair of disjoint subsets F_1 and F_2 of X such that F_1 is preclosed and F_2 is semi-closed, there exist $\gamma-$open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then X has the weak $\gamma-$insertion property for (pc, sc) and (sc, pc).*

Proof. Let g and f be real-valued functions defined on X such that g is pc (respectively sc) and f is sc (respectively pc), with $g \leq f$. If a binary relation ρ is defined by $A \; \rho \; B$ in case $sCl(A) \subseteq pInt(B)$ (respectively $pCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a semi-closed (respectively preclosed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (respectively semi-open) set, it follows that $sCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (respectively $pCl(A(f, t_1)) \subseteq sInt(A(g, t_2)))$. Hence, $t_1 < t_2$ implies that $A(f, t_1) \; \rho \; A(g, t_2)$. The proof follows from Theorem 2.5. □

Before stating the consequences of Theorem 2.6, we state and prove some necessary lemmas.

Lemma 3.5. *The following conditions on the space X are equivalent.*

(i) For each pair of disjoint subsets F_1 and F_2 of X such that F_1 is preclosed and F_2 is semi-closed, there exist $\gamma-$open subsets G_1, G_2 of X such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$.

(ii) If F is a semi-closed (respectively preclosed) subset of X which is contained in a preopen (respectively semi-open) subset G of X, then there exists a $\gamma-$open subset H of X such that $F \subseteq H \subseteq \gamma Cl(H) \subseteq G$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where F and G are semi-closed (respectively preclosed) and preopen (respectively semi-open) subsets of X, respectively. Hence, G^c is a preclosed (respectively semi-closed) and $F \cap G^c = \emptyset$.

By (i), there exist two disjoint $\gamma-$open subsets G_1 and G_2 of X such that, $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c.$$

Hence,

$$F \subseteq G_1 \subseteq G_2^c \subseteq G,$$

and since G_2^c is a $\gamma-$closed set containing G_1, we conclude that $\gamma Cl(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq \gamma Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that F_1 and F_2 are two disjoint subsets of X such that F_1 is preclosed and F_2 is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and F_1^c is a preopen subset of X. Hence, by (ii) there exists a $\gamma-$open set H such that, $F_2 \subseteq H \subseteq \gamma Cl(H) \subseteq F_1^c$. But

$$H \subseteq \gamma Cl(H) \Rightarrow H \cap (\gamma Cl(H))^c = \emptyset$$

and

$$\gamma Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\gamma Cl(H))^c.$$

Furthermore, $(\gamma Cl(H))^c$ is a $\gamma-$open set of X. Hence, $F_2 \subseteq H, F_1 \subseteq (\gamma Cl(H))^c$ and $H \cap (\gamma Cl(H))^c = \emptyset$. This means that condition (i) holds. □

Lemma 3.6. *Suppose that X is a topological space. If each pair of disjoint subsets F_1 and F_2 of X, where F_1 is preclosed and F_2 is semi-closed, can separate by $\gamma-$open subsets of X, then there exists a $\gamma-$continuous function $h : X \to [0, 1]$ such that, $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint subsets of X, where F_1 is preclosed and F_2 is semi-closed. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. In particular, since F_1^c is a preopen subset of X containing semi-closed subset F_2 of X, by Lemma 3.5, there exists a $\gamma-$open subset $H_{1/2}$ of X such that,

$$F_2 \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also a preopen subset of X and contains F_2, and F_1^c is a preopen subset of X and contains a semi-closed subset $\gamma Cl(H_{1/2})$ of X. Hence, by Lemma 3.5, there exist $\gamma-$open subsets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_2 \subseteq H_{1/4} \subseteq \gamma Cl(H_{1/4}) \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq H_{3/4} \subseteq \gamma Cl(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers with their denominators being powers of 2, we obtain $\gamma-$open subsets H_t of X with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$, for $x \notin F_1$ and $h(x) = 1$, for $x \in F_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, F_2 \subseteq H_t$; hence, $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that h is a $\gamma-$continuous function on X. For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$, then $\{x \in X : h(x) < \beta\} = \emptyset$ and if $0 < \beta$, then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are $\gamma-$open subsets of X. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$, then $\{x \in X : h(x) > \beta\} = \cup\{(\gamma Cl(H_t))^c : t > \beta\}$; hence, every one of them is a $\gamma-$open subset of X. Consequently, h is a $\gamma-$continuous function. □

Lemma 3.7. *Suppose that X is a topological space such that every two disjoint semi-closed and preclosed subsets of X can be separated by $\gamma-$open subsets of X. The following conditions are equivalent.*

(i) Every countable convering of semi-open (respectively preopen) subsets of X has a refinement consisting of preopen (respectively semi-open) subsets of X such that for every $x \in X$, there exists a $\gamma-$open subset of X containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{F_n\}$ of semi-closed (respectively preclosed) subsets of X with empty intersection there exists a decreasing sequence $\{G_n\}$ of preopen (respectively semi-open) subsets of X such that $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.

proof.(i) ⇒ (ii) Suppose that $\{F_n\}$ be a decreasing sequence of semi-closed (respectively preclosed) subsets of X with empty intersection.

Then, $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of semi-open (respectively preopen) subsets of X. By hypothesis (i) and Lemma 3.5, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that, every V_n is a $\gamma-$open subset of X and $\gamma Cl(V_n) \subseteq F_n^c$. By setting $G_n = (\gamma Cl(V_n))^c$, we obtain a decreasing sequence of $\gamma-$open subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now, if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of semi-open (respectively preopen) subsets of X, and we set for $n \in \mathbb{N}, F_n = (\bigcup_{i=1}^n H_i)^c$, then $\{F_n\}$ is a decreasing sequence of semi-closed (respectively preclosed) subsets of X with empty intersection. By (ii), there exists a decreasing sequence $\{G_n\}$ consisting of preopen (respectively semi-open) subsets of X such that, $\bigcap_{n=1}^\infty G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now, we define the subsets W_n of X in the following manner.

W_1 is a $\gamma-$open subset of X such that, $G_1^c \subseteq W_1$ and $\gamma Cl(W_1) \cap F_1 = \emptyset$.

W_2 is a $\gamma-$open subset of X such that, $\gamma Cl(W_1) \cup G_2^c \subseteq W_2$ and $\gamma Cl(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 3.5, W_n exists.)

Then, since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of $\gamma-$open subsets of X. Moreover, we have

(i) $\gamma Cl(W_n) \subseteq W_{n+1}$,

(ii) $G_n^c \subseteq W_n$,

(iii) $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now, suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus \gamma Cl(W_{n-1})$.

Then, since $\gamma Cl(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $\gamma-$open subsets of X and covers X. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$$
\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \cdots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}
$$

These sets are $\gamma-$open subsets of X, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below the row.

Hence, if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $\gamma-$open subset X containing x that intersects at most finitely many sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $\gamma-$open subsets of X, and for every point in X, we can find a $\gamma-$open subset of X containing the point that intersects only finitely many elements of that refinement.

Corollary 3.8. *If every two disjoint semi-closed and preclosed subsets of X can be separated by $\gamma-$open subsets of X, and in addition, every countable covering of semi-open* (*respectively preopen*) *subsets of X has a refinement that consists of preopen* (*respectively semi-open*) *subsets of X such that for every point of X we can find a $\gamma-$open subset containing that point such that, it intersects only a finite number of refining members, then X has the weakly $\gamma-$insertion property for* (pc, sc) (*respectively* (sc, pc)).

proof Since every two disjoint sets semi-closed and preclosed can be separated by $\gamma-$open subsets of X, therefore by Corollary 3.4, X has the weak $\gamma-$insertion property for (pc, sc) and (sc, pc). Now, suppose that f and g are real-valued functions on X with $g < f$, such that, g is pc (respectively sc), f is sc (respectively pc) and $f - g$ is sc (respectively pc). For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is sc (respectively pc), hence $A(f-g, 3^{-n+1})$ is a semi-closed (respectively preclosed) subset of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of semi-closed (respectively preclosed) subsets of X and furthermore, since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset$. Now, by Lemma 3.7, there exists a decreasing sequence $\{D_n\}$ of preopen (respectively semi-open) subsets of X s. t., $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But, by Lemma 3.6, $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of semi-closed (respectively preclosed) and preclosed (respectively semi-closed) subsets of X can be completely separated by $\gamma-$continuous functions. Hence, by Theorem 2.6, there exists a $\gamma-$continuous function h defined on X such that, $g < h < f$, i.e., X has the weakly $\gamma-$insertion property for (pc, sc) (respectively (sc, pc)). $\square$

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M. Mirmiran
Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran
Email: `mirmir@sci.ui.ac.ir`