

INSERTION OF A γ -CONTINUOUS FUNCTION

M. MIRMIRAN

Communicated by Fariborz Azarpanah

ABSTRACT. A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a γ -continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq \text{Int}(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(\text{Int}(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour, et. al. [12], while the concept of a locally dense set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [?]. A subset A of a topological space (X, τ) is called *semi-open* [11] if $A \subseteq Cl(\text{Int}(A))$. A set A is called *semi-closed* if its complement is semi-open, or equivalently if $\text{Int}(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X, τ) is called γ -open, if $A \cap S$ is preopen, whenever S is preopen [1]. A set A is called γ -closed if its complement is γ -open, or equivalently if $A \cup S$ is preclosed, whenever S is preclosed.

MSC(2000): Primary: 54C08; Secondary: 54C10, 54C50, 26A15.

Keywords: Insertion, strong binary relation, preopen set, semi-open set, γ -open set, lower cut set.

Received: 29 November 2009, Accepted: 16 February 2010.

© 2011 Iranian Mathematical Society.

We have that if a set is γ -open, then it is semi-open and preopen.

Recall that a real-valued function f , defined on a topological space X , is called A -continuous [14] if the preimage of every open subset of $\mathbb{R}$ belongs to A , where A is a collection of subset of X . Most of the definitions of function used throughout this paper are consequences of the definition of A -continuity. However, for unknown concepts, the reader may refer to [5, 6].

Hence, a real-valued function f defined on a topological space X is called precontinuous (respectively semi-continuous or γ -continuous) if the preimage of every open subset of $\mathbb{R}$ is preopen (respectively semi-open or γ -open) subset of X .

Precontinuity was called by Ptak nearly continuity [15]. Nearly continuity or precontinuity is known also as almost continuity by Husain [7]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [2].

Results of Katětov [8, 9] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a sufficient condition for the insertion of a γ -continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X , then we write $g \leq f$ (respectively $g < f$) in case $g(x) \leq f(x)$ (respectively $g(x) < f(x)$), for all x in X .

The following definitions are modifications of conditions considered in [10].

A property P defined relative to a real-valued function on a topological space is a γ -property provided that any constant function has property P and provided that the sum of a function with property P and any γ -continuous function also has property P . If P_1 and P_2 are γ -property, then the following terminology is used: (i) A space X has the *weak γ -insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a γ -continuous function h such that $g \leq h \leq f$, (ii) A space X has the *γ -insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f$, g has property P_1 and f has property P_2 , then there exists a γ -continuous function h such that $g < h < f$, and (iii) A space X has the *weakly γ -insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f$, g

has property P_1 , f has property P_2 and $f - g$ has property P_2 , then there exists a γ -continuous function h such that $g < h < f$.

Here, we give a sufficient condition for the weak γ -insertion property. Also, for a space with the weak γ -insertion property, we give a necessary and sufficient condition for the space to have the γ -insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a γ -continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space. The family of all γ -open, γ -closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma O(X, \tau)$, $\gamma C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ) . Respectively, we define the γ -closure, γ -interior, s -closure, s -interior, p -closure and p -interior of a set A , denoted by $\gamma Cl(A)$, $\gamma Int(A)$, $sCl(A)$, $sInt(A)$, $pCl(A)$ and $pInt(A)$ as follows:

$$\begin{aligned}\gamma Cl(A) &= \cap \{F : F \supseteq A, F \in \gamma C(X, \tau)\}, \\ \gamma Int(A) &= \cup \{O : O \subseteq A, O \in \gamma O(X, \tau)\}, \\ sCl(A) &= \cap \{F : F \supseteq A, F \in sC(X, \tau)\}, \\ sInt(A) &= \cup \{O : O \subseteq A, O \in sO(X, \tau)\}, \\ pCl(A) &= \cap \{F : F \supseteq A, F \in pC(X, \tau)\} \text{ and} \\ pInt(A) &= \cup \{O : O \subseteq A, O \in pO(X, \tau)\}.\end{aligned}$$

Respectively, we have $\gamma Cl(A)$, $sCl(A)$, $pCl(A)$ are γ -closed, semi-closed, preclosed and $\gamma Int(A)$, $sInt(A)$, $pInt(A)$ are γ -open, semi-open, preopen.

The following first two definitions are modifications of conditions considered in [9, 10].

Definition 2.2. If ρ is a binary relation in a set S , then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$, for any u and v in S .

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a strong binary relation in $P(X)$ in case ρ satisfies each of the following conditions:

- 1) If $A_i \rho B_j$, for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$, for any $i \in \{1, \dots, m\}$ and any $j \in \{1, \dots, n\}$.
- 2) If $A \subseteq B$, then $A \bar{\rho} B$.
- 3) If $A \rho B$, then $\gamma Cl(A) \subseteq B$ and $A \subseteq \gamma Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows.

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$, for a real number ℓ , then $A(f, \ell)$ is called a lower indefinite cut set in the domain of f at the level ℓ .

We now give the following main result.

Theorem 2.5. Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1) \rho A(g, t_2)$, then there exists a γ -continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis, there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1) \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2)$, $G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By lemmas 1 and 2 of [9], it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any x in X , let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: if x is in $H(t)$, then x is in $G(t')$, for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence, $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$, for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence, $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \gamma \text{Int}(H(t_2)) \setminus \gamma \text{Cl}(H(t_1))$. Hence, $h^{-1}(t_1, t_2)$ is a γ -open subset of X , i.e., h is a γ -continuous function on X . $\square$

The above proof used the technique of the proof of Theorem 1 in [8].

Theorem 2.6. *Let P_1 and P_2 be γ -property and X be a space that satisfies the weak γ -insertion property for (P_1, P_2) . Also, assume that g and f are functions on X such that $g < f$, g has property P_1 and f has property P_2 . The space X has the γ -insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each n , $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by γ -continuous functions.*

Proof. See Theorem 2.1 in [13]. $\square$

3. Applications

The abbreviations pc and sc are used for precontinuous and semicontinuous, respectively.

Corollary 3.1. *If for each pair of disjoint preclosed (respectively semi-closed) sets F_1, F_2 of X , there exist γ -open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then X has the weak γ -insertion property for (pc, pc) (respectively (sc, sc)).*

proof. Let g and f be real-valued functions defined on X such that f and g are pc (respectively sc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $pCl(A) \subseteq pInt(B)$ (resp. $sCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preclosed (respectively semi-closed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (respectively semi-open) set, it follows that $pCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (respectively $sCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence, $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.5. $\square$

Corollary 3.2. *If for each pair of disjoint preclosed (respectively semi-closed) sets F_1, F_2 , there exist γ -open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then every precontinuous (respectively semi-continuous) function is γ -continuous.*

Proof. Let f be a real-valued precontinuous (respectively semi-continuous) function defined on X . Set $g = f$. Then, by Corollary 3.1, there exists a γ -continuous function h such that $g = h = f$. $\square$

Corollary 3.3. *If for each pair of disjoint preclosed (respectively semi-closed) sets F_1, F_2 of X , there exist γ -open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then X has the γ -insertion property for (pc, pc) (respectively (sc, sc)).*

Proof. Let g and f be real-valued functions defined on X such that f and g are pc (respectively sc), and $g < f$. Set $h = (f + g)/2$. Thus, $g < h < f$, and by Corollary 3.2, since g and f are γ -continuous functions, hence h is a γ -continuous function. $\square$

Corollary 3.4. *If for each pair of disjoint subsets F_1 and F_2 of X such that F_1 is preclosed and F_2 is semi-closed, there exist γ -open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, then X has the weak γ -insertion property for (pc, sc) and (sc, pc) .*

Proof. Let g and f be real-valued functions defined on X such that g is pc (respectively sc) and f is sc (respectively pc), with $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $sCl(A) \subseteq pInt(B)$ (respectively $pCl(A) \subseteq sInt(B)$), then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a semi-closed (respectively preclosed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (respectively semi-open) set, it follows that $sCl(A(f, t_1)) \subseteq pInt(A(g, t_2))$ (respectively $pCl(A(f, t_1)) \subseteq sInt(A(g, t_2))$). Hence, $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.5. $\square$

Before stating the consequences of Theorem 2.6, we state and prove some necessary lemmas.

Lemma 3.5. *The following conditions on the space X are equivalent.*

(i) *For each pair of disjoint subsets F_1 and F_2 of X such that F_1 is preclosed and F_2 is semi-closed, there exist γ -open subsets G_1, G_2 of X such that $F_1 \subseteq G_1, F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$.*

(ii) *If F is a semi-closed (respectively preclosed) subset of X which is contained in a preopen (respectively semi-open) subset G of X , then there exists a γ -open subset H of X such that $F \subseteq H \subseteq \gamma Cl(H) \subseteq G$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where F and G are semi-closed (respectively preclosed) and preopen (respectively semi-open) subsets of X , respectively. Hence, G^c is a preclosed (respectively semi-closed) and $F \cap G^c = \emptyset$.

By (i), there exist two disjoint γ -open subsets G_1 and G_2 of X such that, $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c.$$

Hence,

$$F \subseteq G_1 \subseteq G_2^c \subseteq G,$$

and since G_2^c is a γ -closed set containing G_1 , we conclude that $\gamma Cl(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq \gamma Cl(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that F_1 and F_2 are two disjoint subsets of X such that F_1 is preclosed and F_2 is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and F_1^c is a preopen subset of X . Hence, by (ii) there exists a γ -open set H such that, $F_2 \subseteq H \subseteq \gamma Cl(H) \subseteq F_1^c$. But

$$H \subseteq \gamma Cl(H) \Rightarrow H \cap (\gamma Cl(H))^c = \emptyset$$

and

$$\gamma Cl(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\gamma Cl(H))^c.$$

Furthermore, $(\gamma Cl(H))^c$ is a γ -open set of X . Hence, $F_2 \subseteq H, F_1 \subseteq (\gamma Cl(H))^c$ and $H \cap (\gamma Cl(H))^c = \emptyset$. This means that condition (i) holds. $\square$

Lemma 3.6. *Suppose that X is a topological space. If each pair of disjoint subsets F_1 and F_2 of X , where F_1 is preclosed and F_2 is semi-closed, can separate by γ -open subsets of X , then there exists a γ -continuous function $h : X \rightarrow [0, 1]$ such that, $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint subsets of X , where F_1 is preclosed and F_2 is semi-closed. Since $F_1 \cap F_2 = \emptyset$, hence $F_2 \subseteq F_1^c$. In particular, since F_1^c is a preopen subset of X containing semi-closed

subset F_2 of X , by Lemma 3.5, there exists a γ -open subset $H_{1/2}$ of X such that,

$$F_2 \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also a preopen subset of X and contains F_2 , and F_1^c is a preopen subset of X and contains a semi-closed subset $\gamma Cl(H_{1/2})$ of X . Hence, by Lemma 3.5, there exist γ -open subsets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_2 \subseteq H_{1/4} \subseteq \gamma Cl(H_{1/4}) \subseteq H_{1/2} \subseteq \gamma Cl(H_{1/2}) \subseteq H_{3/4} \subseteq \gamma Cl(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers with their denominators being powers of 2, we obtain γ -open subsets H_t of X with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$, for $x \notin F_1$ and $h(x) = 1$, for $x \in F_1$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., h maps X into $[0, 1]$. Also, we note that for any $t \in D$, $F_2 \subseteq H_t$; hence, $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that h is a γ -continuous function on X . For every $\beta \in \mathbb{R}$, we have if $\beta \leq 0$, then $\{x \in X : h(x) < \beta\} = \emptyset$ and if $0 < \beta$, then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are γ -open subsets of X . Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leq \beta$, then $\{x \in X : h(x) > \beta\} = \cup\{(\gamma Cl(H_t))^c : t > \beta\}$; hence, every one of them is a γ -open subset of X . Consequently, h is a γ -continuous function. $\square$

Lemma 3.7. *Suppose that X is a topological space such that every two disjoint semi-closed and preclosed subsets of X can be separated by γ -open subsets of X . The following conditions are equivalent.*

(i) *Every countable converging of semi-open (respectively preopen) subsets of X has a refinement consisting of preopen (respectively semi-open) subsets of X such that for every $x \in X$, there exists a γ -open subset of X containing x such that it intersects only finitely many members of the refinement.*

(ii) *Corresponding to every decreasing sequence $\{F_n\}$ of semi-closed (respectively preclosed) subsets of X with empty intersection there exists a decreasing sequence $\{G_n\}$ of preopen (respectively semi-open) subsets of X such that $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}$, $F_n \subseteq G_n$.*

proof. (i) $\Rightarrow$ (ii) Suppose that $\{F_n\}$ be a decreasing sequence of semi-closed (respectively preclosed) subsets of X with empty intersection.

Then, $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of semi-open (respectively preopen) subsets of X . By hypothesis (i) and Lemma 3.5, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that, every V_n is a γ -open subset of X and $\gamma Cl(V_n) \subseteq F_n^c$. By setting $G_n = (\gamma Cl(V_n))^c$, we obtain a decreasing sequence of γ -open subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now, if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of semi-open (respectively preopen) subsets of X , and we set for $n \in \mathbb{N}$, $F_n = (\bigcup_{i=1}^n H_i)^c$, then $\{F_n\}$ is a decreasing sequence of semi-closed (respectively preclosed) subsets of X with empty intersection. By (ii), there exists a decreasing sequence $\{G_n\}$ consisting of preopen (respectively semi-open) subsets of X such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}$, $F_n \subseteq G_n$. Now, we define the subsets W_n of X in the following manner.

W_1 is a γ -open subset of X such that, $G_1^c \subseteq W_1$ and $\gamma Cl(W_1) \cap F_1 = \emptyset$.

W_2 is a γ -open subset of X such that, $\gamma Cl(W_1) \cup G_2^c \subseteq W_2$ and $\gamma Cl(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 3.5, W_n exists.)

Then, since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X , hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of γ -open subsets of X . Moreover, we have

(i) $\gamma Cl(W_n) \subseteq W_{n+1}$,

(ii) $G_n^c \subseteq W_n$,

(iii) $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now, suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus \gamma Cl(W_{n-1})$.

Then, since $\gamma Cl(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of γ -open subsets of X and covers X . Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$$\begin{array}{l} S_1 \cap H_1, \quad S_1 \cap H_2 \\ S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3 \\ S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4 \\ \vdots \\ S_i \cap H_1, \quad S_i \cap H_2, \quad S_i \cap H_3, \quad S_i \cap H_4, \quad \dots, \quad S_i \cap H_{i+1} \\ \vdots \end{array}$$

These sets are γ -open subsets of X , cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below the row.

Hence, if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a γ -open subset X containing x that intersects at most finitely many sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \dots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are γ -open subsets of X , and for every point in X , we can find a γ -open subset of X containing the point that intersects only finitely many elements of that refinement.

Corollary 3.8. *If every two disjoint semi-closed and preclosed subsets of X can be separated by γ -open subsets of X , and in addition, every countable covering of semi-open (respectively preopen) subsets of X has a refinement that consists of preopen (respectively semi-open) subsets of X such that for every point of X we can find a γ -open subset containing that point such that, it intersects only a finite number of refining members, then X has the weakly γ -insertion property for (pc, sc) (respectively (sc, pc)).*

proof Since every two disjoint sets semi-closed and preclosed can be separated by γ -open subsets of X , therefore by Corollary 3.4, X has the weak γ -insertion property for (pc, sc) and (sc, pc) . Now, suppose that f and g are real-valued functions on X with $g < f$, such that, g is pc (respectively sc), f is sc (respectively pc) and $f - g$ is sc (respectively pc). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is sc (respectively pc), hence $A(f - g, 3^{-n+1})$ is a semi-closed (respectively preclosed) subset of X . Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of semi-closed (respectively preclosed) subsets of X and furthermore, since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset$. Now, by Lemma 3.7, there exists a decreasing sequence $\{D_n\}$ of preopen (respectively semi-open) subsets of X s. t., $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But, by Lemma 3.6, $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of semi-closed (respectively preclosed) and pre-closed (respectively semi-closed) subsets of X can be completely separated by γ -continuous functions. Hence, by Theorem 2.6, there exists a γ -continuous function h defined on X such that, $g < h < f$, i.e., X has the weakly γ -insertion property for (pc, sc) (respectively (sc, pc)). $\square$

Acknowledgments

This research was partially supported by Centre of Excellence for Mathematics, University of Isfahan.

REFERENCES

- [1] D. Andrijevic, On topology generated by preopen sets, *Mat. Vesnik* **39** (1987) 367-376.
- [2] H. Blumberg, New properties of all real functions, *Trans. Amer. Math. Soc.* **24** (1922) 113-128.
- [3] F. Brooks, Indefinite cut sets for real functions, *Amer. Math. Monthly* **78** (1971) 1007-1010.
- [4] H. H. Corson and E. Michael, Metrizability of certain countable unions, *Illinois J. Math.* **8** (1964) 351-360.
- [5] J. Dontchev, Characterization of some peculiar topological space via α - and β -sets, *Acta Math. Hungar.* **69**(1-2) (1995) 67-71.
- [6] M. Ganster and I. Reilly, A decomposition of continuity, *Acta Math. Hungar.* **56**(3-4) (1990) 299-301.
- [7] T. Husain, Almost continuous mappings, *Prace Mat.* **10** (1966) 1-7.
- [8] M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* **38** (1951) 85-91.
- [9] M. Katětov, Correction to, "On real-valued functions in topological spaces", (Fund. Math. 38 (1951), pp. 85-91) *Fund. Math.* **40** (1953) 203-205.
- [10] E. Lane, Insertion of a continuous function, *Pacific J. Math.* **66** (1976) 181-190.
- [11] N. Levine, Semi-open sets and semi-continuity in topological space, *Amer. Math. Monthly* **70** (1963) 36-41.
- [12] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt* **53** (1982) 47-53.
- [13] M. Mirmiran, Insertion of a function belonging to a certain subclass of $\mathbb{R}^X$, *Bull. Iranian Math. Soc.* **28**(2) (2002) 19-27.
- [14] M. Przemski, A decomposition of continuity and α -continuity, *Acta Math. Hungar.* **61**(1-2) (1993) 93-98.
- [15] V. Ptak, On complete topological linear spaces, (*Russian*), *Čechoslovack. Mat. Ž.* **3**(78) (1953) 301-364.

M. Mirmiran

Department of Mathematics, University of Isfahan, P.O. Box 81746-73441, Isfahan, Iran

Email: mirmir@sci.ui.ac.ir