MONOMIAL IRREDUCIBLE $\mathfrak{sl}_n(\mathbb{C})$-MODULES

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ABSTRACT. In this article, we introduce monomial irreducible representations of the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We will show that this kind of representations have bases for which the action of the Chevalley generators of the Lie algebra on the basis elements can be given by a simple formula.

1. **Introduction**

Let L be a finite dimensional complex simple Lie algebra with a Cartan subalgebra H and the Cartan decomposition

$$L = H \oplus \sum_{r \in \Phi}^{\cdot} L_r,$$

where Φ is the corresponding root system and $L_r = < x_r >$ is the one dimensional root space associated with r. We fix a basis

$$h_1, h_2, \ldots, h_l$$

for H. For any functional $\lambda \in H^*$, we define a left ideal K_λ in the universal enveloping algebra $\mathfrak{U}(L)$. In fact, K_λ is generated by all x_r, $r \in \Phi^+$ and elements $h_i - \lambda(h_i)$, for $1 \le i \le l$. Now, define the associated Verma module of λ to be the quotient $\mathfrak{U}(L)/K_\lambda$. We denote this Verma

module by $M(\lambda)$. Although $M(\lambda)$ is not irreducible in general, it has a unique maximal submodule $Z(\lambda)$. So, the quotient $L(\lambda) = M(\lambda)/Z(\lambda)$ is irreducible. It is well-known that $L(\lambda)$ is finite dimensional if and only if λ is an *integral dominant weight* of L, i.e., $\lambda(h_i)$ is a non-negative integer for all $1 \le i \le l$. Also, every finite dimensional irreducible L-module can be produced by this way. However, it is very hard to extract useful information concerning $L(\lambda)$ as it is a quotient of a quotient of the universal enveloping algebra of L.

One of the most important problems concerning representations of simple Lie algebras, is considered in this article: to find an ordered basis for $L(\lambda)$, such that one can obtain the matrix representations of elements of L with respect to this ordered basis. It is trivial that handling with matrix representations are more flexible than working with L-modules, especially in practise.

It is the aim of this article to introduce such a suitable basis for $L(\lambda)$. In the present work we do this for *monomial weights* of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. Note that every dominant integral weight λ is associated with a partition π. We say that λ is monomial, iff χ_π, the corresponding character of π, is monomial character. In this case, by a paper of the author and A. Madadi, (see [7]), there is a subgroup $G \le S_m$ and a linear character χ of G, such that

$$L(\lambda) \cong V_\chi(G),$$

where $V_\chi(G)$ is the *symmetry class of tensors* associated with G and χ over $V = \mathbb{C}^n$, (see [7]).

The symmetry class of tensors $V_\chi(G)$ has an orthonormal basis, consisting of decomposable symmetrized tensors. To describe this basis, we need to introduce some notations. Let Γ_n^m be the set of all m-tuples of integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $1 \le \alpha_i \le n$. The permutation group G acts on Γ_n^m and so we can perform a set of representatives of orbits of this action, say Δ. Let

$$\bar{\Delta} = \{\alpha \in \Delta : G_\alpha \subseteq \ker \chi\},$$

where G_α is the stablizer subgroup of α. Now, it is well-known that $V_\chi(G)$ has an orthonormal basis, say

$$E = \{|\alpha\rangle : \alpha \in \bar{\Delta}\}$$

such that $|\alpha^\sigma\rangle = \chi(\sigma^{-1})|\alpha\rangle$, for all $\sigma \in G$. This is just the basis we need, because for Chevalley generators of $\mathfrak{sl}_n(\mathbb{C})$, we will prove that

$$H_i.|\alpha\rangle = \mu_\alpha|\alpha\rangle,$$

$$X_i.|\alpha\rangle = \sum_{r=1}^{m} \delta_{i+1,\alpha_r} \chi(\sigma_r^{-1})|(\alpha - \epsilon_r)^{\sigma_r}\rangle.$$

For details and notations, see Section 3.

In this article, we first give a brief review of the theory of symmetry classes of tensors. Then, we will show any irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ can be constructed as a symmetry class of tensors over the standard $\mathfrak{sl}_n(\mathbb{C})$-module. Finally, we define monomial weights and we give a basis for the corresponding monomial modules, as well as a compact formula concerning the action of Chevalley generators on the basis elements. The last section of this paper consists of some interesting examples.

The reader interested in the subject of symmetry classes of tensors will find a detailed introduction in [8] and [9]. For Lie algebras and their representations, one can see [1], [3] or [4]. For character theory of finite groups, see [5], and for representations of the symmetric group, see [2], [6] or [10].

2. Symmetry classes of tensors

In this section, we are going to review the notion of a symmetry class of tensors. The reader interested in the subject, can find a detailed introduction in [8] or [9].

Let V be an n-dimensional complex inner product space and G be a subgroup of the full symmetric group S_m. Let $V^{\otimes m}$ denote the tensor product of m copies of V and for any $\sigma \in G$, define the *permutation operator*

$$P_\sigma : V^{\otimes m} \to V^{\otimes m}$$

by

$$P_\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$

Suppose χ is a complex irreducible character of G and define the *symmetrizer*

$$S_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma.$$

The *symmetry class of tensors* associated with G and χ is the image of S_χ and it is denoted by $V_\chi(G)$. So,

$$V_\chi(G) = S_\chi(V^{\otimes m}).$$

For example, if we let $G = S_m$ and $\chi = \varepsilon$, the alternating character, then we get $\wedge^m V$, the m-th Grassman space over V and if $G = S_m$ and $\chi = 1$, the principal character, then we obtain $V^{(m)}$, the m-th symmetric power of V, as symmetry classes of tensors.

Several monographs and articles have been published on symmetry classes of tensors during last decade, see for example [8] and [9].

Let $v_1, \ldots, v_m$ be arbitrary vectors in V and define the *decomposable symmetrized tensor*

$$v_1 * v_2 * \cdots * v_m = S_\chi(v_1 \otimes v_2 \otimes \cdots \otimes v_m).$$

Let $\{e_1, \ldots, e_n\}$ be a basis of V and suppose Γ_n^m is the set of all m-tuples of integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $1 \le \alpha_i \le n$. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \Gamma_n^m$, we use the notation e_α^* for decomposable symmetrized tensor $e_{\alpha_1} * \cdots * e_{\alpha_m}$. It is clear that $V_\chi(G)$ is generated by all e_α^*; $\alpha \in \Gamma_n^m$. We define an action of G on Γ_n^m by

$$\alpha^\sigma = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(m)}),$$

for any $\sigma \in G$ and $\alpha \in \Gamma_n^m$. Given two elements $\alpha, \beta \in \Gamma_n^m$, we say that $\alpha \sim \beta$ if and only if α and β lie in the same orbit. Suppose Δ is a set of representatives of orbits of this action and let G_α denote the stablizer subgroup of α. Define

$$\Omega = \{\alpha \in \Gamma_n^m : [\chi, \ 1_{G_\alpha}] \ne 0\},$$

where $[\ ,\ ]$ denotes the inner product of characters (see [5]). It is well known that $e_\alpha^* \ne 0$, if and only if $\alpha \in \Omega$, see for example [9]. Suppose $\bar{\Delta} = \Delta \cap \Omega$. For any $\alpha \in \bar{\Delta}$, we have the cyclic subspace

$$V_\alpha^* = \langle\ e_{\alpha^\sigma}^* \ : \ \sigma \in G\ \rangle.$$

It is proved that we have the direct sum decomposition

$$V_\chi(G) = \sum_{\alpha \in \bar{\Delta}}^{\cdot} V_\alpha^*,$$

see [9] for a proof. It is also proved that

$$s_\alpha := \dim V_\alpha^* = \chi(1)[\chi, \ 1_{G_\alpha}],$$

and in particular, if χ is linear (a character with degree one), then $s_\alpha = 1$ and so the set

$$\{e_\alpha^* \ : \ \alpha \in \bar{\Delta}\}$$

is an orthogonal basis of $V_\chi(G)$. Also, in the case of linear character χ, we have $e^*_{\alpha\sigma} = \chi(\sigma^{-1})e^*_\alpha$. In the general case, let $\alpha \in \bar{\Delta}$ and suppose

$$e^*_{\alpha^{\sigma_1}}, \; e^*_{\alpha^{\sigma_2}}, \; \ldots, \; e^*_{\alpha^{\sigma_t}}$$

is a basis of V^*_α, with $\sigma_1 = 1$. Let

$$A_\alpha = \{\alpha^{\sigma_1}, \alpha^{\sigma_2}, \ldots, \alpha^{\sigma_t}\}.$$

Then, we define $\hat{\Delta} = \bigcup_{\alpha \in \bar{\Delta}} A_\alpha$. It is clear that

$$\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega,$$

and the set

$$\{e^*_\alpha \; : \quad \alpha \in \hat{\Delta}\}$$

is a basis of $V_\chi(G)$. Finally, we remind a formula for the dimension of symmetry classes. We have

$$\dim V_\chi(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)},$$

where $c(\sigma)$ denotes the number of disjoint cycles (including cycles of length one) in cycle decomposition of σ.

Remark 2.1. *Sometimes using the simple notation $|\alpha\rangle$ instead of e^*_α makes it more flexible to work with decomposable symmetrized tensors. In the forthcoming sections, we will use this kind of notation.*

3. Symmetry classes as $\mathfrak{sl}_n(\mathbb{C})$-modules

In this section, we define a Lie module structure on $V_\chi(G)$, so let L be a complex Lie algebra and suppose V is an L-module. For any $x \in L$, define

$$D(x) : V^{\otimes m} \to V^{\otimes m}$$

by

$$D(x)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = \sum_{i=1}^{m} v_1 \otimes \cdots \otimes xv_i \otimes \cdots \otimes v_m.$$

We know that $D(x)S_\chi = S_\chi D(x)$ and so $V_\chi(G)$ is invariant under $D(x)$. Suppose

$$D^*(x) = D(x) \downarrow_{V_\chi(G)},$$

where the down arrow denotes restriction.

Definition 3.1. *Define an action of Lie algebra L on $V_\chi(G)$ by*

$$x(v_1 * \cdots * v_m) = D^*(x)(v_1 * \cdots * v_m)$$
$$= \sum_{i=1}^{m} v_1 * \cdots * xv_i * \cdots * v_m.$$

Then, $V_\chi(G)$ becomes an L-module. In what follows, we will assume that $L = \mathfrak{sl}_n(\mathbb{C})$ and $V = \mathbb{C}^n$, the standard module for L. In [7], we studied irreducible constituents of $V_\chi(G)$ as well as their multiplicities. To give a summery of our results, it is necessary to introduce some notations.

A Cartan subalgebra for L is

$$H = \{diag(a_1, a_2, \ldots, a_n) : a_1 + a_2 + \cdots + a_n = 0\}$$

For any $1 \le i \le n$, define a linear functional

$$\mu_i : H \to \mathbb{C}$$

by

$$\mu_i(h) = a_i,$$

where $h = diag(a_1, a_2, \ldots, a_n)$, so we have

$$\mu_1 + \mu_2 + \cdots + \mu_n = 0$$

and hence $\mu_1, \mu_2, \ldots, \mu_{n-1}$ is a basis for H^*.

Now, let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the fundamental weights corresponding to H. It is easy to see that for any k ,

$$\lambda_k = \mu_1 + \mu_2 + \cdots + \mu_k.$$

Let $\alpha \in \Gamma_n^m$. We define a composition of m by $m(\alpha) = (m_1, m_2, \ldots, m_n)$, where m_i is the multiplicity of i in α. Suppose

$$\mu_\alpha = \mu_{\alpha_1} + \mu_{\alpha_2} + \cdots + \mu_{\alpha_m}.$$

So, we have

$$\mu_\alpha = m_1\mu_1 + m_2\mu_2 + \cdots + m_n\mu_n.$$

Also, we can see that

$$\mu_\alpha = (m_1 - m_2)\lambda_1 + (m_2 - m_3)\lambda_2 + \cdots + (m_l - m_n)\lambda_{n-1}.$$

In [7], we proved that $\mu_\alpha = \mu_\beta$, if and only if $m(\alpha) = m(\beta)$. So, for any $\alpha \in \Gamma_n^m$, we introduce a partition $\pi = \lambda(\alpha)$ which is just the multiplicity composition $m(\alpha)$ with a descending arrangement of entries. Hence, we can define $\mu_\pi = \mu_\alpha$ and this is well defined by the above observation. In fact, for any partition π of m, with height at most n, we can find $\alpha \in \bar\Delta$,

such that $\pi = \lambda(\alpha)$, so we can perform μ_π. Now, we are ready to restate the main result of [7].

Theorem 3.2.

$$V_\chi(G) = \sum_{\pi \vdash m}^{\cdot} L(\mu_\pi)^{\chi(1)[\chi, \chi_\pi]},$$

where χ_π is the corresponding character of π and $[\ ,\ ]$ denotes the inner product of characters in G.

As an special case, we let $G = S_m$ and $\chi = \chi_\pi$. We denote the corresponding symmetry class by $V_\pi(S_m)$. We have

$$V_\pi(S_m) = L(\mu_\pi)^{\chi_\pi(1)}.$$

Note that this last equality, affords a new method of constructing all the irreducible $\mathfrak{sl}_n(\mathbb{C})$-modules, namely, let $\lambda = \mu_\pi$ be any integral dominant weight of $\mathfrak{sl}_n(\mathbb{C})$. As in [9], we have

$$V_\pi(S_m) = \sum_{i=1}^{\chi_\pi(1)} V_\pi^i(S_m),$$

where $V_\pi^i(S_m)$ is defined as follows. Let

$$F : S_m \to GL_{\chi_\pi(1)}(\mathbb{C})$$

be the corresponding representation of χ_π, with $F(\sigma) = [a_{ij}(\sigma)]$. We introduce the partial symmetrizer S_π^i by,

$$S_\chi^i = \frac{\chi(1)}{m!} \sum_{\sigma \in S_m} a_{ii}(\sigma) P_\sigma.$$

Now, $V_\pi^i(S_m)$ is precisely the image of S_π^i. So, we have

$$L(\lambda) = V_\pi^i(S_m),$$

for all $1 \le i \le \chi_\pi(1)$. One of the most important consequences of this construction is the following dimension formula, which is much simpler than the one due to Weyl.

Corollary 3.3. *Let $L(\lambda)$ be an irreducible $\mathfrak{sl}_n(\mathbb{C})$-module with highest weight $\lambda = \mu_\pi$, in which π is a partition of some integer m. Then,*

$$\dim L(\lambda) = \frac{1}{m!} \sum_{\sigma \in S_m} \chi_\pi(\sigma) n^{c(\sigma)}.$$

4. **Monomial weights**

In this section, we fix the following notations. We denote by E_{ij}, the $n \times n$ matrix with (i, j)-entry equal to 1 and all other entries equal to 0. Then, we let $H_i = E_{ii} - E_{i+1,i+1}$ and $X_i = E_{i,i+1}$, for $1 \leq i \leq n - 1$. Also, we let $Y_i = X_i^T$. These are clearly the Chevalley generators of $\mathfrak{sl}_n(\mathbb{C})$. As in Section 2, we assume that $e_1, e_2, \ldots, e_n$ is the standard basis of $V = \mathbb{C}^n$. A simple computation on decomposable symmetrized tensors e_α^* shows that for any i, we have

$$H_i . e_\alpha^* = \mu_\alpha(H_i) e_\alpha^*$$

$$X_i . e_\alpha^* = \sum_{r=1}^m \delta_{i+1,\alpha_r} e_{\alpha - \epsilon_r}^*,$$

where ϵ_r is the m-tuple whose entries are zero except the r-th entry which is equal to 1. Note that a similar equation can be written for Y_i. From now on, we will denote e_α^* by the simpler notation $|\alpha\rangle$, so we have

$$H_i . |\alpha\rangle = \mu_\alpha(H_i) |\alpha\rangle$$

$$X_i . |\alpha\rangle = \sum_{r=1}^m \delta_{i+1,\alpha_r} |\alpha - \epsilon_r\rangle.$$

These equations describe the action of generators of $\mathfrak{sl}_n(\mathbb{C})$ on the basis elements $|\alpha\rangle$; $\alpha \in \hat{\Delta}$.

But, in general $\alpha - \epsilon_r$ does not belong to $\hat{\Delta}$, and so we must express $|\alpha - \epsilon_r\rangle$ as a linear combination of basis elements. This is a very hard problem at this time, except in the case of linear characters. In other words, if χ is linear, then we can write $|\alpha - \epsilon_r\rangle$ in terms of basis elements. To do this, we first note that we have $\hat{\Delta} = \bar{\Delta}$ and there is a permutation $\sigma_r \in G$ such that

$$(\alpha - \epsilon_r)^{\sigma_r} \in \bar{\Delta},$$

except for the case $|\alpha - \epsilon_r\rangle = 0$. Hence, we have

$$|\alpha - \epsilon_r\rangle = \chi(\sigma_r) |(\alpha - \epsilon_r)^{\sigma_r}\rangle,$$

and the final result does not depend on the way we select σ_r. Hence, we have

$$X_i . |\alpha\rangle = \sum_{r=1}^m \delta_{i+1,\alpha_r} \chi(\sigma_r) |(\alpha - \epsilon_r)^{\sigma_r}\rangle.$$

Note that, if the permutation σ_r does not exist, then automatically $|\alpha - \epsilon_r\rangle = 0$.

Now, the question is this: for which groups $G \leq S_m$ and linear characters χ, the symmetry class $V_\chi(G)$ is an irreducible $\mathfrak{sl}_n(\mathbb{C})$-module? The answer comes from Theorem 3.2; the symmetry class is irreducible, if and only if χ^{S_m} (induced character) is irreducible. Let $\chi^{S_m} = \chi_\pi$, for some partition π of m. Then, obviously χ_π is a monomial character of S_m. If we let $\lambda = \mu_\pi$, then we call λ, a monomial integral dominant weight of $\mathfrak{sl}_n(\mathbb{C})$. In the next section, we will give some interesting examples of this kind of weights. At this stage, we summarize the results of the recent section in the following theorem.

Theorem 4.1. *Let $\lambda = \mu_\pi$ be a monomial integral dominant weight of $\mathfrak{sl}_n(\mathbb{C})$. Then, there is a subgroup $G \leq S_m$ and a linear character χ of G, such that*

$$L(\lambda) = V_\chi(G).$$

Further, $L(\lambda)$ has an orthonormal basis

$$E = \{|\alpha\rangle : \alpha \in \bar{\Delta}\}$$

such that

$$H_i.|\alpha\rangle = \mu_\alpha(H_i)|\alpha\rangle$$
$$X_i.|\alpha\rangle = \sum_{r=1}^{m} \delta_{i+1,\alpha_r}\chi(\sigma_r)|(\alpha - \epsilon_r)^{\sigma_r}\rangle,$$

where σ_r is any element of G with the property

$$(\alpha - \epsilon_r)^{\sigma_r} \in \bar{\Delta}.$$

5. **Examples**

In this final section, we give some examples of monomial $\mathfrak{sl}_n(\mathbb{C})$-modules. The first two examples are well-known.

Example 5.1. *Let $\pi = [1^m]$ be the alternating partition. We know that $\chi_\pi = \varepsilon$ is linear and so it is monomial. We have $\lambda = \lambda_m$ and hence $L(\lambda) = \bigwedge^m V$ is the corresponding fundamental module. In this case $\bar{\Delta} = Q_{m,n}$, the set of all strictly increasing sequences. Let $\alpha \in Q_{m,n}$, such that $\alpha_r = i + 1$. Suppose q_i denotes the number of terms α_j with*

$1 \leq j \leq r - 1$ and $\alpha_j > i + 1$. Now, it is clear that σ_r is equal to a product of q_i transpositions and so we have

$$X_i.|\alpha\rangle = \sum_{r=1}^{m} \delta_{i+1,\alpha_r}(-1)^{q_i}|(\alpha - \epsilon_r)^{\sigma_r}\rangle$$

$$= (-1)^{q_i}\delta_{1,m_{i+1}(\alpha)}|i + 1 \to i\rangle,$$

where $|i + 1 \to i\rangle$ is precisely $|\alpha - \epsilon_r\rangle$ after increasing re-arrangement of its entries.

Example 5.2. Now, let $\pi = [m]$, the trivial partition. Then, $\chi_\pi = 1$ and so it is monomial. We have $\lambda = m\lambda_1$ and $L(\lambda) = V^{(m)}$. In this case $\bar{\Delta} = G_{m,n}$, the set of all increasing sequences. If $\alpha \in G_{m,n}$ and $\alpha_r = i + 1$, then after a re-arrangement of its entries, $\alpha - \epsilon_r$ becomes an element of $G_{m,n}$. We denote this new element by $(\alpha - \epsilon_r)^{inc}$. So, we have

$$X_i.|\alpha\rangle = \sum_{r=1}^{m} \delta_{i+1,\alpha_r}|(\alpha - \epsilon_r)^{inc}\rangle.$$

Example 5.3. Let $m = 3$ and $\pi = [2, 1]$. Then, We have $\lambda = 2\mu_1 + \mu_2 = \lambda_1 + \lambda_2$. Also, the character χ_π has degree 3 and it is monomial. Let

$$G = < (1\ 2\ 3) >$$

and ω be a primitive third root of unity. The subgroup G has a character χ with values 1, ω and ω^2, such that

$$\chi^{S_3} = \chi_\pi.$$

Suppose

$$\Delta_1 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 < \alpha_2, \alpha_3\},$$
$$\Delta_2 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 = \alpha_2 \leq \alpha_3\}.$$

Then, we have $\Delta = \Delta_1 \cup \Delta_2$. Also, we have $\alpha \in \Omega$ if and only if $G_\alpha = 1$, because $\ker \chi = 1$. So, we have

$$\Omega = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 \neq \alpha_2,\ \alpha_1 \neq \alpha_3\}.$$

Hence, $\bar{\Delta} = \Delta_1 \cup \bar{\Delta}_2$, where

$$\bar{\Delta}_2 = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 = \alpha_2 \lneq \alpha_3\}.$$

Now, it is easy to check that each of the following implications are true:

1) If $\alpha_1 = i + 1$, then $\sigma_1 = 1$.
2) If $\alpha_2 = i + 1$ and $\alpha \in \Delta_1$, then $\sigma_2 = 1$.
3) If $\alpha_2 = i + 1$ and $\alpha \in \bar{\Delta}_2$, then $\sigma_2 = (1\ 2\ 3)$.

4) If $\alpha_3 = i + 1$, $\alpha \in \Delta_1$ and $\alpha_3 = \alpha_1 + 1$, then $\sigma_3 = (1\ 3\ 2)$.

5) If $\alpha_3 = i + 1$, $\alpha \in \Delta_1$ and $\alpha_3 > \alpha_1 + 1$, then $\sigma_3 = 1$.

6) If $\alpha_3 = i + 1$, $\alpha \in \bar{\Delta}_2$ and $\alpha_3 = \alpha_1 + 1$, then the corresponding term is zero.

7) If $\alpha_3 = i + 1$, $\alpha \in \bar{\Delta}_2$ and $\alpha_3 > \alpha_1 + 1$, then $\sigma_3 = 1$.

In what follows, we denote the tensor $|\alpha\rangle$ by $|\alpha_1, \alpha_2, \alpha_3\rangle$. Summarizing all the above facts, we have

1) If $\alpha \in \Delta_1$ and $\alpha_3 = \alpha_1 + 1$, then

$$
\begin{aligned}
X_i.|\alpha_1, \alpha_2, \alpha_3\rangle &= \delta_{i+1,\alpha_1}|\alpha_1 - 1, \alpha_2, \alpha_3\rangle + \delta_{i+1,\alpha_2}|\alpha_1, \alpha_2 - 1, \alpha_3\rangle \\
&+ \delta_{i+1,\alpha_3}\omega^2|\alpha_1, \alpha_1, \alpha_2\rangle.
\end{aligned}
$$

2) If $\alpha \in \Delta_1$ and $\alpha_3 > \alpha_1 + 1$, then

$$
\begin{aligned}
X_i.|\alpha_1, \alpha_2, \alpha_3\rangle &= \delta_{i+1,\alpha_1}|\alpha_1 - 1, \alpha_2, \alpha_3\rangle + \delta_{i+1,\alpha_2}\omega|\alpha_1, \alpha_2 - 1, \alpha_3\rangle \\
&+ \delta_{i+1,\alpha_3}|\alpha_1, \alpha_2, \alpha_3 - 1\rangle.
\end{aligned}
$$

3) If $\alpha \in \bar{\Delta}_2$ and $\alpha_3 = \alpha_1 + 1$, then

$$
X_i.|\alpha_1, \alpha_2, \alpha_3\rangle = \delta_{i+1,\alpha_1}|\alpha_1 - 1, \alpha_2, \alpha_3\rangle + \delta_{i+1,\alpha_2}\omega|\alpha_1 - 1, \alpha_3, \alpha_1\rangle.
$$

4) If $\alpha \in \bar{\Delta}_2$ and $\alpha_3 > \alpha_1 + 1$, then

$$
\begin{aligned}
X_i.|\alpha_1, \alpha_2, \alpha_3\rangle &= \delta_{i+1,\alpha_1}|\alpha_1 - 1, \alpha_2, \alpha_3\rangle + \delta_{i+1,\alpha_2}\omega|\alpha_1 - 1, \alpha_3, \alpha_1\rangle \\
&+ \delta_{i+1,\alpha_3}|\alpha_1, \alpha_2, \alpha_3 - 1\rangle.
\end{aligned}
$$

Example 5.4. *For $m = 4$, there are 3 non-linear monomial characters. In this example, we study the partition $\pi = [3, 1]$. It is easy to see that*

$$
G = \langle (1\ 2), (1\ 3)(2\ 4)\rangle,
$$

which is isomorphic to the dihedral group of order 8. Its conjugacy classes are as follows,

$$
\begin{aligned}
K_1 &= 1 \\
K_2 &= \{(1\ 2), (3\ 4)\} \\
K_3 &= \{(1\ 2)(3\ 4)\} \\
K_4 &= \{(1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \\
K_5 &= \{1\ 3\ 2\ 4), (1\ 4\ 2\ 3)\}.
\end{aligned}
$$

Suppose χ is the linear character of G with values $1, 1, 1, -1, -1$, respectively on $K_1, \ldots, K_5$. One can see that

$$
\chi^{S_4} = \chi_\pi.
$$

As in the previous example, we can determine Δ and $\bar{\Delta}$; for Δ, we have

$$\Delta = \{\alpha \in \Gamma_n^4 : \alpha_1 \leq \alpha_2, \alpha_3 \leq \alpha_4, \alpha_1 \leq \alpha_3\}.$$

Let

$$\bar{\Delta}_1 = \{\alpha : \alpha_1 < \alpha_2, \alpha_3 < \alpha_4, \alpha_1 \leq \alpha_3, \{\alpha_1, \alpha_2\} \neq \{\alpha_3, \alpha_4\}\},$$
$$\bar{\Delta}_2 = \{\alpha : \alpha_1 = \alpha_2 \leq \alpha_3 < \alpha_4\},$$
$$\bar{\Delta}_3 = \{\alpha : \alpha_1 < \alpha_2, \alpha_1 \leq \alpha_3 = \alpha_4\},$$
$$\bar{\Delta}_4 = \{\alpha : \alpha_1 = \alpha_2 < \alpha_3 = \alpha_4\}.$$

Hence, we have

$$\bar{\Delta} = \bar{\Delta}_1 \cup \bar{\Delta}_2 \cup \bar{\Delta}_3 \cup \bar{\Delta}_4.$$

Now, for all $\alpha \in \bar{\Delta}$, we can compute $X_i.|\alpha\rangle$ easily. For example,

$$\begin{aligned} X_2.|2,2,3,3\rangle &= |2,2,2,3\rangle + |2,2,3,2\rangle \\ &= |2,2,2,3\rangle + \chi((3\ 4))|2,2,2,3\rangle \\ &= 2|2,2,2,3\rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} X_3.|1,4,4,4\rangle &= |1,3,4,4\rangle + |1,4,3,4\rangle + |1,4,4,3\rangle \\ &= |1,4,3,4\rangle + 2|1,4,3,4\rangle. \end{aligned}$$

Example 5.5. *Using GAP program, we see that in the case $m = 5$ and $\pi = [3,1,1]$, the character χ_π is monomial. The corresponding weight is $\lambda = 2\lambda_1 + \lambda_3$. The subgroup G is a non-abelian group of order 20, generated by $(1\ 2\ 3\ 4\ 5)$ and $(2\ 3\ 5\ 4)$. The character χ, has values $1, i, -i, -1, 1$. As in the previous examples, one can compute $\bar{\Delta}$ and determine the required basis.*

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