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THE FEKETE-SZEGÖ COEFFICIENT FUNCTIONAL FOR TRANSFORMS OF ANALYTIC FUNCTIONS

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Abstract. Sharp bounds for the Fekete-Szegö coefficient functional associated with the k-th root transform $[f(z^k)]^{1/k}$ of normalized analytic functions $f(z)$ defined on the open unit disk in the complex plane are derived. A similar problem is investigated for functions $z/f(z)$ when f belongs to a certain class of functions.

1. Introduction

Let $\mathcal{A}$ be the class of all functions $f(z)$ analytic in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$ and $f'(0) = 1$. For a univalent function in the class $\mathcal{A}$, it is well known that the n-th coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient of normalized univalent functions readily yields the growth and distortion bounds for univalent functions. The Fekete-Szegö coefficient functional also arises naturally in the investigation of univalency of analytic functions. Several authors have investigated the Fekete-Szegö functional for functions in various subclasses of univalent

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and multivalent functions [1–5, 7–11, 14, 15, 19, 20, 22], and more recently by Choi et al. [6].

Recall that a function $f \in \mathcal{A}$ is starlike if $f(\Delta)$ is a starlike domain and convex if $f(\Delta)$ is a convex domain. The classes consisting of starlike and convex functions are usually denoted by S^* and C, respectively. The function $f(z)$ is subordinate to the function $g(z)$, written $f(z) \prec g(z)$, provided that there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. A function $f \in \mathcal{A}$ is starlike if and only if $\operatorname{Re} zf'(z)/f(z) > 0$, or equivalently if $zf'(z)/f(z) \prec (1+z)/(1-z)$. The superordinate function $\phi(z) = (1+z)/(1-z)$ is a convex function. Ma and Minda [13] have given a unified treatment of various subclasses consisting of starlike and convex functions for which either one of the quantities $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. In fact, they considered the analytic function φ with positive real part in the unit disk Δ, $\varphi(0) = 1$, $\varphi'(0) > 0$, where φ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The unified class $S^*(\varphi)$ introduced by Ma and Minda [13] consists of functions $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad (z \in \Delta).$$

They also investigated the corresponding class $C(\varphi)$ of convex functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z).$$

Ma and Minda [13] obtained subordination results, distortion, growth and rotation theorems. They also obtained estimates for the first few coefficients and determined bounds for the associated Fekete-Szegö functional. A function $f \in S^*(\varphi)$ is said to be starlike function with respect to φ, and a function $f \in C(\varphi)$ is a convex function with respect to φ.

Several authors [12, 16–18, 21] have studied the classes of analytic functions defined by using the expression $(zf'(z))/f(z)+\alpha(z^2f''(z))/f(z)$. The unified treatment of various subclasses of starlike and convex functions by Ma and Minda [13] motivates one to consider similar classes defined by subordination. Here, we consider the following classes of functions,

$$
\begin{aligned}
R_b(\varphi) &:= \left\{ f \in \mathcal{A} \,:\, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\},\\
S^*(\alpha,\varphi) &:= \left\{ f \in \mathcal{A} \,:\, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\},\\
L(\alpha,\varphi) &:= \left\{ f \in \mathcal{A} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},\\
M(\alpha,\varphi) &:= \left\{ f \in \mathcal{A} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},
\end{aligned}
$$

where $z \in \Delta$, $b \in \mathbb{C} \setminus \{0\}$ and $\alpha \geq 0$. Functions in the class $L(\alpha,\varphi)$ are called logarithmic α-convex functions with respect to φ and the functions in the class $M(\alpha,\varphi)$ are called α-convex functions with respect to φ. Some coefficient problems for functions f belonging to certain classes of p-valent functions were investigated in [4].

For a univalent function $f(z)$ of the form

$$
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}
$$

the k-th root transform is defined by:

$$
F(z) := [f(z^k)]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}
$$

In Section 2, sharp bounds for the Fekete-Szegö coefficient functional $\left|b_{2k+1} - \mu b_{k+1}^2\right|$ associated with the k-th root transform of the function f belonging to the above mentioned classes are derived. In Section 3, a similar problem is investigated for functions G, where $G(z) := z/f(z)$ and the function f belongs to the above mentioned classes.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, satisfying the condition $|w(z)| < 1$. The following two lemmas regarding the coefficients of functions in Ω are needed to prove our main results. Lemma 1.1 is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [13].

Lemma 1.1. [4] *If* $w \in \Omega$ *and*

$$w(z) := w_1 z + w_2 z^2 + \cdots \quad (z \in \Delta), \tag{1.3}$$

then,

$$|w_2 - t w_1^2| \le \begin{cases} -t & \text{if } t \le -1 \\ 1 & \text{if } -1 \le t \le 1 \\ t & \text{if } t \ge 1. \end{cases}$$

For $t < -1$ *or* $t > 1$*, equality holds if and only if* $w(z) = z$ *or one of its rotations. For* $-1 < t < 1$*, equality holds if and only if* $w(z) = z^2$ *or one of its rotations. Equality holds for* $t = -1$ *if and only if* $w(z) = z\frac{\lambda+z}{1+\lambda z}(0 \le \lambda \le 1)$ *or one of its rotations, while for* $t = 1$*, equality holds if and only if* $w(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \le \lambda \le 1)$ *or one of its rotations.*

Lemma 1.2. [10] *If* $w \in \Omega$*, then,*

$$|w_2 - t w_1^2| \le \max\{1;\ |t|\},$$

for any complex number t*. The result is sharp for the function* $w(z) = z^2$ *or* $w(z) = z$*.*

2. Coefficient bounds for the k-th root transformation

In the first theorem below, the bound for the coefficient functional $|b_{2k+1} - \mu b_{k+1}^2|$ corresponding to the k-th root transformation of starlike functions with respect to φ is given. Notice that the classes $S^*(\alpha, \varphi)$, $L(\alpha, \varphi)$ and $M(\alpha, \varphi)$ reduce to the class $S^*(\varphi)$ for appropriate choice of the parameters.

Although Theorem 2.1 is contained in the corresponding results for the classes $S^*(\alpha, \varphi)$, $L(\alpha, \varphi)$ and $M(\alpha, \varphi)$, it is stated and proved separately here because of its importance in its own right as well as to illustrate the main ideas.

Theorem 2.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$*, and*

$$\sigma_1 := \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right], \quad \sigma_2 := \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right].$$

If f given by (1.1) *belongs to $S^*(\varphi)$, and F is the k-th root transformation of f given by* (1.2), *then,*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \begin{cases} \dfrac{B_1^2}{2k^2}(1-2\mu)+\dfrac{B_2}{2k}, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2k}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\dfrac{B_1^2}{2k^2}(1-2\mu)-\dfrac{B_2}{2k}, & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{B_1}{2k}\max\left\{1;\ \left|\frac{B_1}{k}(1-2\mu)+\frac{B_2}{B_1}\right|\right\}.$$

Proof. If $f \in S^*(\varphi)$, then there is an analytic function $w \in \Omega$ of the form (1.3) such that

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{2.1}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (-a_2^2 + 2a_3)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \dots$$

and

$$\varphi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \dots,$$

then it follows from (2.1) that

$$a_2 = B_1 w_1 \tag{2.2}$$

and

$$a_3 = \frac{1}{2}[B_1 w_2 + (B_2 + B_1^2)w_1^2]. \tag{2.3}$$

For a function f given by (1.1), a computation shows that

$$[f(z^k)]^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right)z^{2k+1} + \cdots. \tag{2.4}$$

The equations (1.2) and (2.4) yield:

$$b_{k+1} = \frac{1}{k}a_2, \tag{2.5}$$

and

$$b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{2.6}$$

By using (2.2) and (2.3) in (2.5) and (2.6), it follows:

$$b_{k+1} = \frac{B_1 w_1}{k}$$

and

$$b_{2k+1} = \frac{1}{2k}\left[B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{k}\right],$$

and hence,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k}\left\{w_2 - \left[-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1}\right] w_1^2\right\}.$$

The first result is established by an application of Lemma 1.1.
If $-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \leq -1$, then,

$$\mu \leq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right] \quad (\mu \leq \sigma_1),$$

and Lemma 1.1 gives:

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1^2}{2k^2}(1-2\mu) + \frac{B_2}{2k}.$$

For $-1 \leq -\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \leq 1$, we have,

$$\frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right] \leq \mu \leq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right] \quad (\sigma_1 \leq \mu \leq \sigma_2),$$

and Lemma 1.1 yields:

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k}.$$

For $-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \geq 1$, we have,

$$\mu \geq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right] \quad (\mu \geq \sigma_2),$$

and it follows from Lemma 1.1 that

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq -\frac{B_1^2}{2k^2}(1-2\mu) - \frac{B_2}{2k}.$$

The second result follows by an application of Lemma 1.2:

$$\begin{aligned}\left|b_{2k+1}-\mu b_{k+1}^2\right| &= \frac{B_1}{2k}\left|w_2-\left[-\frac{B_1}{k}(1-2\mu)-\frac{B_2}{B_1}\right]w_1^2\right|\\ &\leq \frac{B_1}{2k}\max\left\{1;\ \left|\frac{B_1}{k}(1-2\mu)+\frac{B_2}{B_1}\right|\right\}.\end{aligned}$$

□

Remark 2.2.

(1) In view of the Alexander result that $f\in C(\varphi)$ if and only if $zf'\in S^*(\varphi)$, the estimate for $\left|b_{2k+1}-\mu b_{k+1}^2\right|$ for a function in $C(\varphi)$ can be obtained from the corresponding estimates in Theorem 2.1 for functions in $S^*(\varphi)$. The details are omitted here.

(2) For $k=1$, the k-th root transformation of f reduces to the given function f itself. Thus, the estimate given in equation (2.1) of Theorem 2.1 is an extension of the corresponding result for the Fekete-Szegö coefficient functional corresponding to functions starlike with respect to φ. Similar remarks apply to other results in this section.

The well-known Noshiro-Warschawski theorem states that a function $f\in\mathcal{A}$ with positive derivative in Δ is univalent. The class $R_b(\varphi)$ of functions defined in terms of the subordination of the derivative f' is closely associated with the class of functions with positive real part. The bound for the Fekete-Szegö functional corresponding to the k-th root transformation of functions in $R_b(\varphi)$ is given in Theorem 2.3.

Theorem 2.3. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. If f given by* (1.1) *belongs to $R_b(\varphi)$, and F is the k-th root transformation of f given by* (1.2)*, then,*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right|\leq\frac{|b|B_1}{3k}\max\left\{1;\left|\frac{3bB_1}{4}\left(\frac{1}{2}-\frac{1}{2k}+\frac{\mu}{k}\right)-\frac{B_2}{B_1}\right|\right\}.$$

Proof. If $f\in R_b(\varphi)$, then there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$ such that

$$1+\frac{1}{b}\left(f'(z)-1\right)=\varphi(w(z)). \tag{2.7}$$

Since

$$1 + \frac{1}{b}\left(f'(z) - 1\right) = 1 + \frac{2}{b}a_2 z + \frac{3}{b}a_3 z^2 + \dots$$

and

$$\varphi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \dots,$$

then it follows from (2.7) that

$$a_2 = \frac{bB_1 w_1}{2} \tag{2.8}$$

and

$$a_3 = \frac{b}{3}(B_1 w_2 + B_2 w_1^2). \tag{2.9}$$

By using (2.8) and (2.9) in (2.5) and (2.6), it follows:

$$b_{k+1} = \frac{bB_1 w_1}{2k}$$

and

$$b_{2k+1} = \frac{bB_1 w_2}{3k} + \frac{bB_2 w_1^2}{3k} - \frac{b^2 B_1^2 w_1^2}{8k} + \frac{b^2 B_1^2 w_1^2}{8k^2},$$

and hence,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{bB_1}{3k}\left\{ w_2 - \left[\frac{3bB_1}{4}\left(\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right) - \frac{B_2}{B_1}\right] w_1^2 \right\}. \tag{2.10}$$

Applying Lemma 1.2 yields:

$$\begin{aligned}\left|b_{2k+1} - \mu b_{k+1}^2\right| &= \frac{|b|B_1}{3k}\left| w_2 - \left[\frac{3bB_1}{4}\left(\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right) - \frac{B_2}{B_1}\right] w_1^2 \right| \\ &\leq \frac{|b|B_1}{3k} \max\left\{1; \left|\frac{3bB_1}{4}\left(\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right) - \frac{B_2}{B_1}\right|\right\}.\end{aligned}$$

□

Remark 2.4. When $k = 1$ and

$$\varphi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B \leq A \leq 1),$$

Theorem 2.3 reduces to a result in [9, Theorem 4, p. 894].

It is important to consider the special case of functions $f \in \mathcal{A}$ having positive derivatives, in particular, functions f satisfying the subordination $f'(z) \prec \varphi(z)$. The class of such functions is the special case of the class $R_b(\varphi)$, when $b = 1$. In fact, when $b = 1$, equation (2.10) becomes:

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{3k}\left\{w_2 - \left[\frac{3B_1}{4}\left(\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right) - \frac{B_2}{B_1}\right] w_1^2\right\}.$$

Lemma 1.1 now yields the following result.

Corollary 2.5. *If $f \in \mathcal{A}$ satisfies $f'(z) \prec \varphi(z)$, then,*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} \dfrac{-B_1^2}{4k}\left(\dfrac{1}{2} - \dfrac{1}{2k} + \dfrac{\mu}{k}\right) + \dfrac{B_2}{3k}, & \textit{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{3k}, & \textit{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_1^2}{4k}\left(\dfrac{1}{2} - \dfrac{1}{2k} + \dfrac{\mu}{k}\right) - \dfrac{B_2}{3k}, & \textit{if} \quad \mu \geq \sigma_2 \end{cases}$$

where,

$$\sigma_1 := \frac{4kB_2}{3B_1^2} - \frac{4k}{3B_1} - \frac{k}{2} + \frac{1}{2} \quad \textit{and} \quad \sigma_2 := \frac{4kB_2}{3B_1^2} + \frac{4k}{3B_1} - \frac{k}{2} + \frac{1}{2}.$$

The following result gives the bounds for the Fekete-Szegö coefficient functional corresponding to the k-th root transformation of functions in the class $S^*(\alpha, \varphi)$.

Theorem 2.6. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. Define σ_1, σ_2 and υ by:*

$$\sigma_1 := \frac{k(1+2\alpha)^2}{2B_1(1+3\alpha)}\left[\frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1} - \frac{B_1(k-1)(1+3\alpha)}{(1+2\alpha)^2} - 1\right],$$

$$\sigma_2 := \frac{k(1+2\alpha)^2}{2B_1(1+3\alpha)}\left[\frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1} - \frac{B_1(k-1)(1+3\alpha)}{(1+2\alpha)^2} + 1\right],$$

$$\upsilon := \frac{B_1}{(1+2\alpha)}\left[\frac{(k-1)(1+3\alpha)}{k(1+2\alpha)} + \frac{2\mu(1+3\alpha)}{k(1+2\alpha)} - 1\right] - \frac{B_2}{B_1}.$$

If f given by (1.1) *belongs to* $S^*(\alpha,\varphi)$, *and* F *is* k*-th root transformation of* f *given by* (1.2), *then*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \begin{cases} -\dfrac{B_1}{2k(1+3\alpha)}\upsilon, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2k(1+3\alpha)}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_1}{2k(1+3\alpha)}\upsilon, & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ *complex,*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{B_1}{2k(1+3\alpha)} \max\{1; |\upsilon|\}.$$

Proof. If $f \in S^*(\alpha,\varphi)$, then there is an analytic function $w(z) = w_1z + w_2z^2 + \cdots \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(w(z)).$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (-a_2^2 + 2a_3)z^2 + \cdots \tag{2.11}$$

and

$$\frac{\alpha z^2 f''(z)}{f(z)} = 2a_2\alpha z - (2a_2^2\alpha - 6a_3\alpha)z^2, \tag{2.12}$$

then equations (2.11) and (2.12) yield:

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + [2(1+3\alpha)a_3 - (1+2\alpha)a_2^2]z^2 + \cdots. \tag{2.13}$$

Since

$$\varphi(w(z)) = 1 + B_1w_1z + (B_1w_2 + B_2w_1^2)z^2 + \ldots,$$

then equation (2.13) gives:

$$a_2 = \frac{B_1w_1}{(1+2\alpha)} \tag{2.14}$$

and

$$a_3 = \frac{1}{2(1+3\alpha)}\left[B_1w_2 + B_2w_1^2 + \frac{B_1^2w_1^2}{(1+2\alpha)}\right]. \tag{2.15}$$

Using (2.14) and (2.15) in (2.5) and (2.6), we get

$$b_{k+1} = \frac{1}{k}\frac{B_1 w_1}{(1+2\alpha)},$$

and

$$b_{2k+1} = \frac{1}{2k(1+3\alpha)}\left(B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{(1+2\alpha)}\right) - \frac{B_1^2 w_1^2}{2k^2(1+2\alpha)^2}(k-1),$$

and hence,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k(1+3\alpha)}\left\{w_2 - \left[\frac{B_1}{(1+2\alpha)}\left(\frac{(k-1)(1+3\alpha)}{k(1+2\alpha)} + \frac{2\mu(1+3\alpha)}{k(1+2\alpha)} - 1\right) - \frac{B_2}{B_1}\right] w_1^2\right\}.$$

The first part of the result is established by applying Lemma 1.1.
If $\mu \leq \sigma_1$, then $\upsilon \leq -1$ and hence Lemma 1.1 yields:

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq -\frac{B_1}{2k(1+3\alpha)}\upsilon.$$

If $\sigma_1 \leq \mu \leq \sigma_2$, then $-1 \leq \upsilon \leq 1$ and hence Lemma 1.1 yields:

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k(1+3\alpha)}.$$

For $\mu \geq \sigma_2$, then $\upsilon \geq 1$ and hence Lemma 1.1 yields

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k(1+3\alpha)}\upsilon.$$

The second result follows by an application of Lemma 1.2.

Observe that Theorem 2.6 reduces to Theorem 2.1 when $\alpha = 0$.

Theorem 2.7. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ *and* $\beta = (1-\alpha)$. *Also, let*

$$\sigma_1 := \frac{k}{2(\alpha+3\beta)}\left[\frac{(\alpha+2\beta)^2}{B_1}\left(\frac{B_2}{B_1} - 1\right) - \frac{(\alpha+2\beta)^2 - 3(\alpha+4\beta)}{2} - \frac{(k-1)(\alpha+3\beta)}{k}\right],$$

$$\sigma_2 := \frac{k}{2(\alpha+3\beta)}\left[\frac{(\alpha+2\beta)^2}{B_1}\left(\frac{B_2}{B_1}+1\right)-\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}\right.$$
$$\left.-\frac{(k-1)(\alpha+3\beta)}{k}\right],$$

$$\upsilon := \frac{B_1}{(\alpha+2\beta)^2}\left[\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}+\frac{1}{k}(k-1)(\alpha+3\beta)\right.$$
$$\left.+\frac{2\mu}{k}(\alpha+3\beta)\right]-\frac{B_2}{B_1}.$$

If f given by (1.1) *belongs to $L(\alpha,\varphi)$, and F is k-th root transformation of f given by* (1.2)*, then,*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \le \begin{cases} -\dfrac{B_1}{2k(\alpha+3\beta)}\upsilon, & \text{if} \quad \mu\le\sigma_1, \\ \dfrac{B_1}{2k(\alpha+3\beta)}, & \text{if} \quad \sigma_1\le\mu\le\sigma_2, \\ \dfrac{B_1}{2k(\alpha+3\beta)}\upsilon, & \text{if} \quad \mu\ge\sigma_2, \end{cases} \tag{2.16}$$

and for μ complex,

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \le \frac{B_1}{2k(\alpha+3\beta)}\max\{1;|\upsilon|\}.$$

Proof. If $f\in L(\alpha,\varphi)$, then there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$ such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}=\varphi(w(z)). \tag{2.17}$$

We have,

$$\frac{zf'(z)}{f(z)} = 1+a_2z+(2a_3-a_2^2)z^2+(3a_4+a_2^3-3a_3a_2)z^3+\cdots,$$

and therefore,

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1+\alpha a_2z+\left(2\alpha a_3+\frac{\alpha^2-3\alpha}{2}a_2^2\right)z^2+\cdots. \tag{2.18}$$

Similarly,

$$1+\frac{zf''(z)}{f'(z)} = 1+2a_2z+(6a_3-4a_2^2)z^2+\cdots,$$

and therefore,

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} = 1 + 2\beta a_2 z + \left(6\beta a_3 + 2(\beta^2-3\beta)a_2^2\right) z^2 + \cdots . \tag{2.19}$$

Thus, from (2.18) and (2.19),

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} = 1 + (\alpha+2\beta)a_2 z + \left[2(\alpha+3\beta)a_3 + \frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}a_2^2\right] z^2 + \cdots .$$

Since

$$\varphi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \dots ,$$

then it follows from (2.17) that

$$a_2 = \frac{B_1 w_1}{(\alpha+2\beta)} \tag{2.20}$$

and

$$a_3 = \frac{B_1 w_2 + B_2 w_1^2}{2(\alpha+3\beta)} - \frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]B_1^2 w_1^2}{4(\alpha+2\beta)^2(\alpha+3\beta)}. \tag{2.21}$$

Using (2.20) and (2.21) in (2.5) and (2.6), we get

$$b_{k+1} = \frac{1}{k}\frac{B_1 w_1}{k(\alpha+2\beta)},$$

and

$$b_{2k+1} = \frac{B_1 w_2}{2k(\alpha+3\beta)} + \frac{B_2 w_1^2}{2k(\alpha+3\beta)} - \frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]B_1^2 w_1^2}{4k(\alpha+2\beta)^2(\alpha+3\beta)} - \frac{B_1^2 w_1^2 (k-1)}{2k^2(\alpha+2\beta)^2},$$

and hence,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k(\alpha+3\beta)}\left\{w_2 - \sigma w_1^2\right\},$$

where,

$$\sigma := \frac{B_1}{(\alpha+2\beta)^2}\left[\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2} + \frac{1}{k}(k-1)(\alpha+3\beta) + \frac{2\mu}{k}(\alpha+3\beta)\right] - \frac{B_2}{B_1}.$$

The results now follow by using lemmas 1.1 and 1.2.

Remark 2.8. We note that if $k=1$, then inequality (2.16) is the result established in [19, Theorem 2.1, p. 3].

For the class $M(\alpha,\varphi)$, we now get the following coefficient bounds.

Theorem 2.9. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. Also, let*

$$\sigma_1 := \frac{k}{2(1+2\alpha)}\left[\frac{(1+\alpha)^2}{B_1}\left(\frac{B_2}{B_1}-1\right)+(1+3\alpha)-(k-1)(1+2\alpha)\right],$$

$$\sigma_2 := \frac{k}{2(1+2\alpha)}\left[\frac{(1+\alpha)^2}{B_1}\left(\frac{B_2}{B_1}+1\right)+(1+3\alpha)-(k-1)(1+2\alpha)\right],$$

$$\upsilon := \frac{B_1}{(1+\alpha)^2}\left[(k-1)(1+2\alpha)+\frac{2\mu}{k}(1+2\alpha)-(1+3\alpha)\right]-\frac{B_2}{B_1}.$$

If f given by (1.1) *belongs to $M(\alpha,\varphi)$, and F is k-th root transformation of f given by* (1.2)*, then,*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \begin{cases} -\dfrac{B_1}{2k(1+2\alpha)}\upsilon, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2k(1+2\alpha)}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_1}{2k(1+2\alpha)}\upsilon, & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{B_1}{2k(1+2\alpha)}\max\{1;|\upsilon|\}.$$

Proof. If $f \in M(\alpha,\varphi)$, then there is an analytic function $w(z)=w_1z+w_2z^2+\cdots \in \Omega$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=\varphi(w(z)). \tag{2.22}$$

Since

$$(1-\alpha)\frac{zf'(z)}{f(z)}=(1-\alpha)+a_2(1-\alpha)z+(1-\alpha)(-a_2^2+2a_3)z^2+\cdots \tag{2.23}$$

and

$$\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=\alpha+2a_2\alpha z+2\alpha(3a_3-2a_2^2)z^2, \tag{2.24}$$

then from equations (2.23) and (2.24), it follows:

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=1+(1+\alpha)a_2z$$
$$+[-(1+3\alpha)a_2^2+2(1+2\alpha)a_3]z^2+\cdots.$$

Since

$$\varphi(w(z))=1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\cdots,$$

then it follows from equation (2.22) that

$$a_2=\frac{B_1w_1}{(1+\alpha)} \tag{2.25}$$

and

$$a_3=\frac{1}{2(1+2\alpha)}\left[B_1w_2+B_2w_1^2+\frac{(1+3\alpha)B_1^2w_1^2}{(1+\alpha)^2}\right]. \tag{2.26}$$

By using (2.25) and (2.26) in (2.5) and (2.6), it follows:

$$b_{k+1}=\frac{B_1w_1}{k(1+\alpha)},$$

and

$$b_{2k+1}=\frac{1}{2k(1+2\alpha)}\left[B_1w_2+B_2w_1^2+\frac{(1+3\alpha)B_1^2w_1^2}{(1+\alpha)^2}\right]-\frac{B_1^2w_1^2(k-1)}{2k^2(1+\alpha)^2},$$

and hence,

$$b_{2k+1}-\mu b_{k+1}^2=\frac{B_1}{2k(1+\alpha)}\{w_2-\sigma w_1^2\},$$

where,

$$\sigma:=\frac{B_1}{(1+\alpha)^2}\left[(k-1)(1+2\alpha)+\frac{2\mu}{k}(1+2\alpha)-(1+3\alpha)\right]-\frac{B_2}{B_1}.$$

The results follow from lemmas 1.1 and 1.2. we have

$$\left|b_{2k+1}-\mu b_{k+1}^2\right|\le-\gamma\upsilon.$$

For $\sigma_1\le\mu\le\sigma_2$, then $-1\le\upsilon\le1$ and hence by Lemma 1.1, we have

$$\left|b_{2k+1}-\mu b_{k+1}^2\right|\le\gamma.$$

For $\mu\ge\sigma_2$, then $\upsilon\ge1$ and hence by Lemma 1.1, we have

$$\left|b_{2k+1}-\mu b_{k+1}^2\right|\le\gamma\upsilon.$$

Remark 2.10. When $k=1$ and $\alpha=1$, Theorem 2.9 reduces to a result in [13, Theorem 3, p. 164].

3. The Fekete-Szegö functional associated with $z/f(z)$

Here, bounds for the Fekete-Szegö coefficient functional associated with the function G defined by

$$G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{3.1}$$

where f belongs to one of the classes $S^*(\varphi)$, $R_b(\varphi)$, $S^*(\alpha,\varphi)$, $L(\alpha,\varphi)$ and $M(\alpha,\varphi)$, are investigated. Proofs of the results obtained here are similar to those given in Section 2, and hence the details are omitted.

The following result is for functions belonging to the class $S^*(\varphi)$.

Theorem 3.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *and*

$$\sigma_1 := \frac{1}{2} - \frac{2}{B_1^3} - \frac{B_2}{2B_1^2}, \quad \sigma_2 := \frac{1}{2} + \frac{2}{B_1^3} - \frac{B_2}{2B_1^2}.$$

If f given by (1.1) *belongs to* $S^*(\varphi)$, *and G is a function given by* (3.1), *then,*

$$|d_2 - \mu d_1^2| \leq \begin{cases} -\frac{1}{4}B_1B_2 - \frac{1}{4}B_1^3(2\mu - 1), & \text{if} \quad \mu \leq \sigma_1, \\ \frac{1}{2}B_1, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{1}{4}B_1B_2 + \frac{1}{4}B_1^3(2\mu - 1), & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$|d_2 - \mu d_1^2| \leq \frac{1}{2}B_1 \max\left\{1; \left|(1-\mu)B_1^2 - \frac{1}{2}(B_2 + B_1^2)\right|\right\}.$$

Proof. A computation yields:

$$\frac{z}{f(z)} = 1 - a_2 z + (a_2^2 - a_3)z^2 + \cdots. \tag{3.2}$$

Using (3.1), (3.2) yields:

$$d_1 = -a_2 \tag{3.3}$$

and

$$d_2 = a_2^2 - a_3. \tag{3.4}$$

By using (2.2) and (2.3) in (3.3) and (3.4), it follows:

$$d_1 = -B_1 w_1$$

and

$$d_2 = B_1^2 w_1^2 - \frac{1}{2}[B_1 w_2 + (B_2 + B_1^2) w_1^2],$$

and hence,

$$d_2 - \mu d_1^2 = -\frac{1}{2} B_1 \left\{ w_2 - \left[(1-\mu) B_1^2 - \frac{1}{2}(B_2 + B_1^2) \right] w_1^2 \right\}.$$

The result is established from an application of Lemma 1.1. The second result follows from Lemma 1.2:

$$\begin{aligned} \left| d_2 - \mu d_1^2 \right| &= \frac{1}{2} B_1 \left| w_2 - \left[(1-\mu) B_1^2 - \frac{1}{2}(B_2 + B_1^2) \right] w_1^2 \right| \\ &\le \frac{1}{2} B_1 \max \left\{ 1;\ \left| (1-\mu) B_1^2 - \frac{1}{2}(B_2 + B_1^2) \right| \right\}. \end{aligned}$$

□

For the class $R_b(\varphi)$, the following coefficient bound is obtained.

Theorem 3.2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* f *given by* (1.1) *belongs to* $R_b(\varphi)$, *and* G *is a function given by* (3.1), *then,*

$$\left| d_2 - \mu d_1^2 \right| \le \frac{|b| B_1}{3} \max \left\{ 1; \left| \frac{3}{4}(1-\mu) b B_1 - \frac{B_2}{B_1} \right| \right\}.$$

Proof. Using (2.8) and (2.9) in (3.3) and (3.4), it follows:

$$d_1 = -\frac{b B_1 w_1}{2}$$

and

$$d_2 = \frac{1}{4} b^2 B_1^2 w_1^2 - \frac{b}{3}(B_1 w_2 + B_2 w_1^2),$$

and hence,

$$d_2 - \mu d_1^2 = -\frac{b B_1}{3} \left\{ w_2 - \left[\frac{3}{4}(1-\mu) b B_1 - \frac{B_2}{B_1} \right] w_1^2 \right\}.$$

Lemma 1.2 gives:

$$\begin{aligned}\left|d_2-\mu d_1^2\right| &= \frac{|b|B_1}{3}\left|w_2-\left[\frac{3}{4}(1-\mu)bB_1-\frac{B_2}{B_1}\right]w_1^2\right|\\ &\leq \frac{|b|B_1}{3k}\max\left\{1;\left|\frac{3}{4}(1-\mu)bB_1-\frac{B_2}{B_1}\right|\right\}.\end{aligned}$$

□

For functions with positive derivative, the above theorem turns in to the following result.

Corollary 3.3. *If $f\in\mathcal{A}$ satisfies $f'(z)\prec\varphi(z)$, then*

$$\left|d_2-\mu d_1^2\right| \leq \begin{cases} \frac{1}{4}(1-\mu)B_1^2-\frac{B_2}{3}, & \text{if} \quad \mu\leq\sigma_1,\\ \frac{B_1}{3}, & \text{if} \quad \sigma_1\leq\mu\leq\sigma_2,\\ -\frac{1}{4}(1-\mu)B_1^2+\frac{B_2}{3}, & \text{if} \quad \mu\geq\sigma_2 \end{cases}$$

where,

$$\sigma_1 := 1-\frac{4}{3B_1}-\frac{4B_2}{3B_1^2} \quad \text{and} \quad \sigma_2 := 1+\frac{4}{3B_1}-\frac{4B_2}{3B_1^2}.$$

The following result gives the coefficient bounds for the class $S^*(\alpha,\varphi)$.

Theorem 3.4. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. Define σ_1, σ_2, υ and γ by:*

$$\sigma_1 := (1+2\alpha)\left[1+\frac{(1+2\alpha)B_2}{2(1+3\alpha)B_1^2}+\frac{1}{2(1+3\alpha)}-\frac{(1+2\alpha)}{2(1+3\alpha)B_1}\right],$$

$$\sigma_2 := (1+2\alpha)\left[1+\frac{(1+2\alpha)B_2}{2(1+3\alpha)B_1^2}+\frac{1}{2(1+3\alpha)}+\frac{(1+2\alpha)}{2(1+3\alpha)B_1}\right],$$

$$\upsilon := \frac{2B_1(1+3\alpha)}{(1+2\alpha)^2}-\frac{2\mu(1+3\alpha)B_1}{(1+2\alpha)^2}+\frac{B_1}{(1+2\alpha)}+\frac{B_2}{B_1}, \quad \gamma := -\frac{B_1}{2(1+3\alpha)}.$$

If f given by (1.1) *belongs to* $S^*(\alpha,\varphi)$*, and G is a function given by* (3.1)*, then,*

$$\left|d_2-\mu d_1^2\right| \leq \begin{cases} \dfrac{B_1}{2(1+3\alpha)}\upsilon, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2(1+3\alpha)}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\dfrac{B_1}{2(1+3\alpha)}\upsilon, & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|d_2-\mu {d_1}^2\right| \leq \frac{B_1}{2(1+3\alpha)} \max\left\{1;|\upsilon|\right\}.$$

Proof. By using the relations (2.14) and (2.15) in (3.3) and (3.4), it follows:

$$d_1=-\frac{B_1 w_1}{(1+2\alpha)},$$

and

$$d_2=\frac{B_1^2 w_1^2}{(1+2\alpha)^2}-\frac{1}{2(1+3\alpha)}\left(B_1 w_2+B_2 w_1^2+\frac{B_1^2 w_1^2}{(1+2\alpha)}\right),$$

and hence,

$$\begin{aligned} d_2-\mu d_1^2 = -\frac{B_1}{2(1+3\alpha)}\Bigg\{ w_2 - \Bigg[& \frac{2B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{2\mu(1+3\alpha)B_1}{(1+2\alpha)^2} \\ & + \frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1}\Bigg] w_1^2 \Bigg\}. \end{aligned}$$

The result is established by an application of Lemma 1.1. The second result follows by an application of Lemma 1.2:

$$\begin{aligned} \left|d_2-\mu d_1^2\right| &= \frac{B_1}{2(1+3\alpha)}\Bigg| w_2 - \Bigg[\frac{2B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{2\mu(1+3\alpha)B_1}{(1+2\alpha)^2} \\ &\quad + \frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1}\Bigg] w_1^2 \Bigg| \\ &\leq \frac{B_1}{2(1+3\alpha)} \max\left\{1;|\upsilon|\right\}. \end{aligned}$$

□

For the class $L(\alpha,\varphi)$, we now get the following coefficient bounds.

Theorem 3.5. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ *and* $\beta = (1-\alpha)$. *Also, let*

$$\sigma_1 := 1 - \frac{(\alpha+2\beta)^2 B_2}{2(\alpha+3\beta)B_1^2} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)](\alpha+2\beta)}{4(\alpha+3\beta)} - \frac{1}{B_1},$$

$$\sigma_2 := 1 - \frac{(\alpha+2\beta)^2 B_2}{2(\alpha+3\beta)B_1^2} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)](\alpha+2\beta)}{4(\alpha+3\beta)} + \frac{1}{B_1},$$

$$\upsilon := B_1 - \frac{(\alpha+2\beta)^2 B_2}{2(\alpha+3\beta)B_1} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)]B_1(\alpha+2\beta)}{4(\alpha+3\beta)} - \mu B_1.$$

If f *given by* (1.1) *belongs to* $L(\alpha,\varphi)$, *and* G *is a function given by* (3.1), *then,*

$$|d_2 - \mu d_1^2| \le \begin{cases} \dfrac{B_1 \upsilon}{(\alpha+2\beta)^2}, & \text{if} \quad \mu \le \sigma_1, \\ \dfrac{B_1}{(\alpha+2\beta)^2}, & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ -\dfrac{B_1 \upsilon}{(\alpha+2\beta)^2}, & \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

and for μ *complex,*

$$|d_2 - \mu {d_1}^2| \le \frac{B_1}{(\alpha+2\beta)^2} \max\{1; |\upsilon|\}.$$

Proof. Using (2.20) and (2.21) in (3.3) and (3.4), it follows:

$$d_1 = -\frac{B_1 w_1}{(\alpha+2\beta)},$$

and

$$d_2 = \frac{B_1^2 w_1^2}{(\alpha+2\beta)^2} - \frac{B_2 w_1 + B_2 w_1^2}{2(\alpha+3\beta)} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)]B_1^2 w_1^2}{4(\alpha+2\beta)^2(\alpha+3\beta)}.$$

Hence,

$$d_2 - \mu d_1^2 = -\frac{B_1}{(\alpha+2\beta)^2}\{w_2 - \sigma w_1^2\},$$

where,

$$\sigma := B_1 - \frac{(\alpha+2\beta)^2 B_2}{2(\alpha+3\beta)B_1} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)]B_1(\alpha+2\beta)}{4(\alpha+3\beta)} - \mu B_1.$$

The result now follows from Lemma 1.1. The second result follows by an application of Lemma 1.2:

$$\begin{aligned}\left|d_2-\mu d_1^2\right| &= \frac{B_1}{(\alpha+2\beta)^2}\left|w_2-\sigma w_1^2\right| \\ &\le \frac{B_1}{(\alpha+2\beta)^2}\max\{1;|\sigma|\}.\end{aligned}$$

□

Finally, for the class $M(\alpha,\varphi)$, the following coefficient bounds are obtained.

Theorem 3.6. *Let* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. *Also, let*

$$\sigma_1 := 1-\frac{B_2(1+\alpha)^2}{2B_1^2(1+2\alpha)}-(1+3\alpha)-\frac{(1+\alpha)^2}{2B_1(1+2\alpha)},$$

$$\sigma_2 := 1-\frac{B_2(1+\alpha)^2}{2B_1^2(1+2\alpha)}-(1+3\alpha)+\frac{(1+\alpha)^2}{2B_1(1+2\alpha)},$$

$$\upsilon := \frac{2(1+2\alpha)B_1}{(1+\alpha)^2}-\frac{B_2}{B_1}-\frac{2(1+2\alpha)(1+3\alpha)B_1}{(1+\alpha)^2}-\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}.$$

If f given by (1.1) *belongs to* $M(\alpha,\varphi)$, *and G is a function given by* (3.1), *then,*

$$\left|d_2-\mu d_1^2\right| \le \begin{cases} \dfrac{B_1}{2(1+2\alpha)}\upsilon, & \text{if} \quad \mu\le\sigma_1, \\ \dfrac{B_1}{2(1+2\alpha)}, & \text{if} \quad \sigma_1\le\mu\le\sigma_2, \\ -\dfrac{B_1}{2(1+2\alpha)}\upsilon, & \text{if} \quad \mu\ge\sigma_2, \end{cases}$$

and for μ complex,

$$\left|d_2-\mu {d_1}^2\right| \le \frac{B_1}{2(1+2\alpha)}\max\{1;|\upsilon|\}.$$

Proof. Using (2.25) and (2.26) in (3.3) and (3.4), it follows:

$$d_1=-\frac{B_1w_1}{(1+\alpha)},$$

and

$$d_2 = \frac{B_1^2 w_1^2}{(1+\alpha)^2} - \frac{1}{2(1+2\alpha)} \left[B_1 w_2 + B_2 w_1^2 + \frac{(1+3\alpha) B_1^2 w_1^2}{(1+\alpha)^2} \right],$$

and hence,

$$d_2 - \mu d_1^2 = -\frac{B_1}{2(1+2\alpha)^2} \left\{ w_2 - \sigma w_1^2 \right\},$$

where,

$$\sigma := \frac{2(1+2\alpha)B_1}{(1+\alpha)^2} - \frac{B_2}{B_1} - \frac{2(1+2\alpha)(1+3\alpha)B_1}{(1+\alpha)^2} - \frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}.$$

The results follows from Lemma 1.1. The second result follows by an application of Lemma 1.2:

$$\begin{aligned} \left| d_2 - \mu d_1^2 \right| &= \frac{B_1}{2(1+2\alpha)^2} \left| w_2 - \sigma w_1^2 \right| \\ &\le \frac{B_1}{2(1+2\alpha)} \max \left\{ 1; |\sigma| \right\}. \end{aligned}$$

□

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