PERMANENCE AND GLOBAL ASYMPTOTIC STABILITY OF A DELAYED PREDATOR-PREY MODEL WITH HASSELL-VARLEY TYPE FUNCTIONAL RESPONSE

K. WANG* AND Y. L. ZHU

Communicated by Nezam Mahdavi-Amiri

ABSTRACT. Here, a predator-prey model with Hassell-Varley type functional responses is studied. Some sufficient conditions are obtained for the permanence and global asymptotic stability of the system by using comparison theorem and constructing a suitable Lyapunov functional. Moreover, an example is illustrated to verify the results by simulation.

1. Introduction

The first differential equation of predator-prey model was introduced by Lotka [1] and Volterra [2]. After that, many more complicated but realistic predator-prey model have been formulated by ecologists and mathematicians. The dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. The most popular predator-prey model is the one with Michaelis-Menten type functional response (Freedman, 1980)[3]:

$$(1.1) \qquad \begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{K}) - \frac{cxy}{m+x}, \\ \frac{dy}{dt} = y(-d + \frac{fx}{m+x}), \\ x(0) > 0, y(0) > 0, \end{cases}$$

where, x and y denote the population of prey and predator at time t, respectively. The constants r, K, c, m, d and f are positive constants that stand for prey intrinsic growth rate (or the prey growth in absence of predators), carrying capacity, capturing rate (or the prey is fed upon by predators), half-saturation constant, predator death rate, maximal predator growth rate, respectively. In this system, the per capita rate of predator depends on the prey numbers only, which is not realistic in the real situation. There are growing explicit biological and physiological evidence (see [4-7], for more details) and in many situations, specially while predators have to search and share or compete for food, a more suitable and general predator-prey model should be based on the "ratio-dependent" theory. A more general but realistic predator-prey model with ratio-dependent type functional response was proposed by Arditi and Ainzburg [6] in the form:

$$(1.2) \qquad \begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{K}) - \frac{cxy}{my+x}, \\ \frac{dy}{dt} = y(-d + \frac{fx}{my+x}), \\ x(0) > 0, y(0) > 0. \end{cases}$$

The dynamics of system (1.2) is richer and more plausible than that of system (1.1). Many scholars have studied system (1.2), such as Arditi and Saiah [7], Berreta and Kuang [8], Jost et al.[9], Hsu et al.[10], Xiao and Ruan [11], Berezovskaya et al.[12] and Maiti et al.[13].

It was known that the functional response can dependent on the predator density in other ways. One of the more widely known one is due to Hassell and Varley (1969)[14]. A general predator-prey model with Hassell-Varley type functional response may take the following form:

$$(1.3) \qquad \begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{K}) - \frac{cxy}{my^\gamma+x}, \\ \frac{dy}{dt} = y(-d + \frac{fx}{my^\gamma+x}), \ \gamma \in (0, 1), \\ x(0) > 0, y(0) > 0, \end{cases}$$

where, γ is called Hassell-Varley constant. In a typical predator-prey interaction, where predators do not form groups, one can assume that

$\gamma = 1$, producing the so-called ratio-dependent predator-prey dynamics. For terrestrial predators that form a fixed number of tight groups, it is often reasonable to assume that $\gamma = 1/2$. For aquatic predator that form a fixed number of tights groups, $\gamma = 1/3$ maybe more appropriate.

Mathematically, system (1.1) or (1.2) can be viewed as an special case of system (1.3) with $\gamma = 0$ or $\gamma = 1$. A unified mechanistic approach was provided by Cosner et al.[15], where the functional response in system (1.3) was derived. Hsu et al.[16] studied system (1.3) and presented a systematic global qualitative analysis for it.

As was pointed by Kuang [17], any model of species dynamics without delays is an approximation at best. More detailed arguments on the importance and usefulness of time-delays in realistic models may also be found in the classical books of Macdonald [18], and Gopalsamy [19]. Many scholars studied the delay predator-prey system (Fan et al.[20], Xu et al.[21], Egami et al.[22]). Recently, Lu [23] investigated the influence of delays on the existence of positive periodic solution to a Lotka Volterra cooperative system.

Motivated by the above excellent works, here we consider a non-autonomous predator-prey model with the Hassell-Varley functional response and a delay in the prey specific growth term in the form:

(1.4)
$$
\begin{cases}
N_1'(t) = N_1(t) \left[a(t) - b(t)\, N_1(t - \tau(t)) - \dfrac{c(t)\, N_2(t)}{m\, N_2^{\gamma}(t) + N_1(t)} \right], \\
N_2'(t) = N_2(t) \left[-d(t) + \dfrac{r(t)\, N_1(t)}{m\, N_2^{\gamma}(t) + N_1(t)} \right] \gamma \in (0,1)
\end{cases}
$$

with the following initial conditions:

(1.5)
$$
\begin{cases}
N_1(\theta) = \varphi(\theta),\ \theta \in [-\tau^U, 0],\ \varphi(0) = \varphi_0 > 0, \\
N_2(\theta) = \psi(\theta),\ \theta \in [-\tau^U, 0],\ \psi(0) = \psi_0 > 0,
\end{cases}
$$

where, $\tau^U := \sup_{t \in [0,+\infty)} \{\tau(t)\}$, τ, a, b, c, d, r are all positive continuous bounded functions defined on $[0, +\infty)$, and m is a positive constant. Obviously, system (1.3) is a special case of system (1.4)-(1.5), if one let $\tau(t) \equiv 0$ in system (1.4)-(1.5). In [24], Wang have studied the existence of positive periodic solutions of system (1.4), by using the coincidence degree theorem.

The main objective of this paper is to obtain sufficient conditions for the permanence and global asymptotic stability of system (1.4)-(1.5). It

is interesting that the results obtained in this paper are based on the delay (or delay-dependent) which is different from the previous results that are delay-independent.

The organization of the paper is as follows: in the next section, we introduce some useful Definitions and lemmas. In Section 3, some sufficient condition, are established, by utilizing the comparison theorem, for the uniform persistence of system (1.4)-(1.5). In Section 4, by constructing a suitable Lyapunov functional for system (1.4)-(1.5), we investigate the global asymptotic stability of the system. Lastly, an example is given to show the feasibility of our results by simulation.

2. Definitions and Lemmas

Definition 2.1. *System (1.4) is said to be uniformly persistent if there exists a compact region $D \subseteq Int\, R_+^2$ such that every solution $(x(t), y(t))^\top$ of system (1.4) with the initial condition, (1.5) eventually enters and remains in region D, i.e., there exist positive constants m_i, M_i, $i = 1, 2$, and $T > 0$ such that $m_1 \leq x(t) \leq M_1$; $m_2 \leq y(t) \leq M_2$ for $t \geq T$,, for any positive solution $(x(t), y(t))$ of system (1.4).*

Definition 2.2. *For any two positive solutions $(\overline{x}(t), \overline{y}(t))$, $(x(t), y(t))$ of system (1.4), if they satisfy $\lim_{t \to +\infty}(|x(t) - \overline{x}(t)| + |y(t) - \overline{y}(t)|) = 0$, then we call system (1.4) is globally asymptotic stable.*

Lemma 2.3. *(See [25, 26][26]) If $a > 0$, $b > 0$, $\tau(t) \geq 0$ for $t \in R$, and $y'(t) \leq y(t)\,[b - a\,y(t - \tau(t))]$, then there exists a constant $T > 0$ such that $y(t) \leq b\,a^{-1}\exp\{b\,\tau^U\}$, for $t \geq T$.*

Lemma 2.4. *(See [25, 26]) If $a > 0$, $b > 0$, $\tau(t) \geq 0$ for $t \in R$, and $y'(t) \geq y(t)\,[b - a\,y(t - \tau(t))]$, and there exist constants $T > 0$, $M > 0$ such that $y(t) < M$, $for\, t \geq T$, then there exists a constant $T^* > T$ such that $y(t) \geq \min\left\{b\,a^{-1}\exp\{(b - aM)\tau^U\}, b\,a^{-1}\right\}$, for $t \geq T^*$.*

Lemma 2.5. *(See [27]) If $a > 0$, $b > 0$, $\beta > 0$ and $y'(t) \geq (\leq) y(t)\,[b - a\,y^\beta(t)]$, then*

$$\liminf_{t \to +\infty} y(t) \geq \left(b\,a^{-1}\right)^{1/\beta} \quad \left(\limsup_{t \to +\infty} y(t) \leq \left(b\,a^{-1}\right)^{1/\beta}\right).$$

Lemma 2.6. *If $a > 0$, $b > 0$, $\beta > 0$ and $y'(t) \geq (\leq) y(t)^{1-\beta} [b - a\, y^\beta(t)]$, then*

$$\liminf_{t \to +\infty} y(t) \geq \left(b\, a^{-1}\right)^{1/\beta} \quad \left(\limsup_{t \to +\infty} y(t) \leq \left(b\, a^{-1}\right)^{1/\beta}\right).$$

Proof. It follows from $y'(t) \leq y(t)^{1-\beta} [b - a\, y^\beta(t)]$ that

$$\frac{d(y^\beta(t))}{dt} \leq \beta(b - ay^\beta(t)),$$

which gives

$$\limsup_{t \to +\infty} y(t) \leq \left(b\, a^{-1}\right)^{1/\beta}.$$

Similarly, $y'(t) \geq y(t)^{1-\beta} [b - a\, y^\beta(t)]$ yields $\liminf\limits_{t \to +\infty} y(t) \geq \left(b\, a^{-1}\right)^{1/\beta}$. □

Before giving the main results, we give some useful notations for any continuous bounded function f defined on $[0, +\infty)$ as follows:

$$f^L := \inf_{t \in [0, +\infty)} \{f(t)\}, \ f^U := \sup_{t \in [0, +\infty)} \{f(t), \}$$

and

$$K_1 := \frac{a^U}{b^L} \exp\{a^U \tau^U\}, \ K_2 := \left(\frac{K_1 r^U}{m\, d^L}\right)^{1/\gamma},$$

$$D_1 := \left(a - m^{-1} K_2^{1-\gamma} c\right)^L, \ D_2 := (r - d)^L,$$

$$K_3 := \min\left\{\frac{D_1}{b^U} \exp\{(D_1 - b^U K_1)\tau^U\}, \frac{D_1}{b^U}\right\},$$

$$K_4 := \left(\frac{(r - d)^L K_3}{m\, r^U}\right)^{1/\gamma}, \ P_0 := (mK_4^\gamma + K_3)^2,$$

$$P_{00} := (mK_2^\gamma + K_1)^2, \ P_1 := (mK_2^\gamma + K_1)K_1.$$

3. Permanence

Theorem 3.1. *If $D_1 > 0$ and $D_2 > 0$, then system (1.4) is uniformly persistent.*

Proof. The first equation in system (1.4) leads to:

$$N_1'(t) \le N_1(t) \left[a^U - b^L N_1(t - \tau(t)) \right],$$

which together with Lemma 2.3 yields that there is a constant $T_1 \in R_+$ such that

(3.1) $$N_1(t) \le \frac{a^U}{b^L} \exp\{a^U \tau^U\} := K_1, \text{ for } t > T_1$$

On the other hand, the second equation in system (1.4), and (3.1) imply:

$$N_2'(t) \le N_2(t) \left(-d(t) + \frac{K_1}{m} r(t) N_2^{-\gamma}(t) \right)$$
$$\le N_2^{1-\gamma}(t) \left(\frac{K_1}{m} r^U - d^L N_2^\gamma(t) \right)$$

It follows from Lemma 2.6 that, for any given $\varepsilon > 0$, there is a $T_2 \in R_+$ such that

(3.2) $$N_2(t) \le \left(\frac{K_1 r^U}{m \, d^L} \right)^{1/\gamma} + \varepsilon := K_2, \text{ for } t > T_2.$$

Furthermore, from (3.2) and the first equation in system (1.4), we get

$$N_1'(t) \ge N_1(t) \left[a(t) - \frac{K_2^{1-\gamma}}{m} c(t) - b(t) N_1(t - \tau(t)) \right]$$
$$\ge N_1(t) \left[D_1 - b^U N_1(t - \tau(t)) \right].$$

It follows from Lemma 2.4 that there exists a $T_3 \in R_+$ such that

(3.3) $$N_1(t) \ge \min \left\{ \frac{D_1}{b^U} \exp\{(D_1 - b^U K_1)\tau^U\}, \frac{D_1}{b^U} \right\} := K_3, \text{ for } t \ge T_3.$$

Similarly, the second equation in system (1.4) yields:

$$N_2'(t) \ge N_2(t) \left[(r - d)^L - \frac{m \, r^U}{K_3} N_2^\gamma(t) \right].$$

It follows from the assumption and Lemma 2.5 that, for the above ε, there must be a constant $T_4 \in R_+$ such that

(3.4) $$N_2(t) \ge \left(\frac{(r - d)^L K_3}{m \, r^U} \right)^{1/\gamma} + \varepsilon := K_4 \text{ for } t \ge T_4.$$

Combination of (3.1)-(3.4), gives:

$$K_3 \le N_1(t) \le K_1 \;\; \text{and} \;\; K_4, \le N_2(t) \le K_2, \;\; \text{for } t > T,$$

where, $T = \max\{T_i, i = 1, 2, 3, 4\}$.

Now, we complete the proof of Theorem 3.1 is complete.   □

4. Global Attractivity

Theorem 4.1. *In addition to assumptions in Theorem 3.1, assume further that $\gamma \in (0,1)$ is a rational number and there is a positive constant α such that the following condition holds:*

$$[A_1] \quad \lim_{t \to +\infty} \inf\{g_1(t), g_2(t), K_2{}^{\gamma-1}g_2(t) - g_3(t)\} > 0, \; \text{where,}$$

$$\begin{aligned}
g_1(t) =& \alpha b(t) - \frac{\alpha K_2}{P_0}c(t) - \frac{mK_2^\gamma}{P_0}r(t) \\
& - K_1\left(\frac{b(\delta^{-1}(t))\phi(\delta^{-1}(t))}{1 - \tau'(\delta^{-1}(t))} + \frac{K_2}{P_0}c(t)\phi(t)\right) \\
& - \alpha\left(a(t) + K_1 b(t) + \frac{1}{m}K_2^{1-\gamma}c(t)\right)\int_t^{\delta^{-1}(t)} b(l)dl,
\end{aligned}$$

$$g_2(t) = \frac{mK_3}{P_{00}}r(t) - \frac{\alpha(mK_2^\gamma + K_1)}{P_0}c(t) - \frac{P_1}{P_0}c(t)\phi(t) \;\; and$$

$$g_3(t) = \frac{mK_2}{P_0}(\alpha + K_1\phi(t))c(t), \; \phi(t) = \delta^{-1}(t) - t \; and \; \delta(t) = t - \tau(t).$$

Then, system (1.4) is globally asymptotic stable.

Proof. For any two arbitrary positive solutions $(\overline{N}_1(t), \overline{N}_2(t))^\top$ and $(N_1(t), N_2(t))^\top$ of system (1.4), we get from Theorem 3.1, that

$$K_3 \le N_1(t), \; \overline{N}_1(t) \le K_1 \;\; \text{and} \;\; K_4 \le N_2(t), \; \overline{N}_2(t) \le K_2, \;\; \text{for } t > T.$$

Let

$$V_1(t) = \alpha \ln|\overline{N}_1(t) - N_1(t)| + \ln|\overline{N}_2(t) - N_2(t)|.$$

By directly calculating along the solution of system (1.4), we obtain:

(4.1)
$$D^+V_1(t)$$

$$= \alpha \operatorname{sgn}(\overline{N}_1(t) - N_1(t)) \left\{ -b(t) \left[\overline{N}_1(\delta(t)) - N_1(\delta(t)) \right] \right.$$

$$\left. - \left[\frac{c(t)\overline{N}_2(t)}{m\overline{N}_2^\gamma(t) + \overline{N}_1(t)} - \frac{c(t)N_2(t)}{m\,N_2^\gamma(t) + N_1(t)} \right] \right\}$$

$$+ r(t) \operatorname{sgn}(\overline{N}_2(t) - N_2(t)) \left\{ \frac{\overline{N}_1(t)}{m\overline{N}_2^\gamma(t) + \overline{N}_1(t)} - \frac{N_1(t)}{m\,N_2^\gamma(t) + N_1(t)} \right\}$$

$$= \alpha \operatorname{sgn}(\overline{N}_1(t) - N_1(t))$$

$$\times \left\{ -b(t)(\overline{N}_1(t) - N_1(t)) + b(t) \int_{\delta(t)}^t (\overline{N}_1'(s) - N_1'(s))\, ds \right\}$$

$$- \alpha\, c(t) \operatorname{sgn}(\overline{N}_1(t) - N_1(t)) \left[\frac{\overline{N}_2(t)}{m\overline{N}_2^\gamma(t) + \overline{N}_1(t)} - \frac{N_2(t)}{mN_2^\gamma(t) + N_1(t)} \right]$$

$$+ r(t) \operatorname{sgn}(\overline{N}_2(t) - N_2(t)) \left\{ \frac{\overline{N}_1(t)}{m\overline{N}_2^\gamma(t) + \overline{N}_1(t)} - \frac{N_1(t)}{mN_2^\gamma(t) + N_1(t)} \right\}.$$

It is clear that
(4.2a)

$$\operatorname{sgn}(\overline{N}_2(t) - N_2(t)) \left\{ \frac{\overline{N}_1(t)}{m\overline{N}_2^\gamma(t) + \overline{N}_1(t)} - \frac{N_1(t)}{m\,N_2^\gamma(t) + N_1(t)} \right\}$$

$$= m \operatorname{sgn}(\overline{N}_2(t) - N_2(t)) \frac{N_2^\gamma(t)(\overline{N}_1(t) - N_1(t)) + N_1(t)(N_2^\gamma(t) - \overline{N}_2^\gamma(t))}{(m\overline{N}_2^\gamma(t) + \overline{N}_1(t))(m\,N_2^\gamma(t) + N_1(t))}$$

$$\leq \frac{mN_2^\gamma(t)|\overline{N}_1(t) - N_1(t)|}{(m\overline{N}_2^\gamma(t) + \overline{N}_1(t))(m\,N_2^\gamma(t) + N_1(t))}$$

$$- \frac{mN_1(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|}{(m\overline{N}_2^\gamma(t) + \overline{N}_1(t))(m\,N_2^\gamma(t) + N_1(t))}$$

$$\leq \frac{mK_2^\gamma}{P_0}|\overline{N}_1(t) - N_1(t)| - \frac{mK_3}{P_{00}}|\overline{N}_2^\gamma(t) - N_2^\gamma(t)|$$

and

$(4.2b)$

$$-\text{sgn}(\overline{N}_1(t) - N_1(t)) \left\{ \frac{\overline{N}_2(t)}{m\overline{N}_2^{\gamma}(t)+\overline{N}_1(t)} - \frac{N_2(t)}{m\,N_2^{\gamma}(t)+N_1(t)} \right\}$$

$$= -\text{sgn}(\overline{N}_1(t) - N_1(t))$$

$$\times \left\{ \frac{m[\overline{N}_2(t)N_2^{\gamma}(t)-N_2(t)N_2^{\gamma}(t)+N_2(t)N_2^{\gamma}(t)-N_2(t)\overline{N}_2^{\gamma}(t)]}{(m\overline{N}_2^{\gamma}(t)+\overline{N}_1(t))(mN_2^{\gamma}(t)+N_1(t))} \right.$$

$$\left. + \frac{\overline{N}_2(t)N_1(t)-N_2(t)N_1(t)+N_2(t)N_1(t)-N_2(t)\overline{N}_1(t)}{(m\overline{N}_2^{\gamma}(t)+\overline{N}_1(t))(mN_2^{\gamma}(t)+N_1(t))} \right\}$$

$$= -\text{sgn}(\overline{N}_1(t) - N_1(t))$$

$$\times \left\{ \frac{m[N_2^{\gamma}(t)(\overline{N}_2(t)-N_2(t))+N_2(t)(N_2^{\gamma}(t)-\overline{N}_2^{\gamma}(t))]}{(m\overline{N}_2^{\gamma}(t)+\overline{N}_1(t))(mN_2^{\gamma}(t)+N_1(t))} \right.$$

$$\left. + \frac{N_1(t)(\overline{N}_2(t)-N_2(t))+N_2(t)(N_1(t)-\overline{N}_1(t))}{(m\overline{N}_2^{\gamma}(t)+\overline{N}_1(t))(mN_2^{\gamma}(t)+N_1(t))} \right\}$$

$$\leq \frac{mK_2^{\gamma}+K_1}{P_0}|\overline{N}_2(t) - N_2(t)| + \frac{mK_2}{P_0}|N_2^{\gamma}(t) - \overline{N}_2^{\gamma}(t)|$$

$$- \frac{K_2}{P_0}|N_1(t) - \overline{N}_1(t)|$$

and

$$\int_{\delta(t)}^{t} (\overline{N}_1'(s) - N_1'(s))\, ds$$

$$= \int_{\delta(t)}^{t} \left\{ \overline{N}_1(s) \left[a(s) - b(s)\overline{N}_1(\delta_1(s)) - \frac{c(s)\,\overline{N}_2(s)}{m\overline{N}_2^{\gamma}(s) + \overline{N}_1(s)} \right] \right.$$

$$\left. - N_1(s) \left[a(s) - b(s)N_1(\delta_1(s)) - \frac{c(s)\,N_2(s)}{m\,N_2^{\gamma}(s) + N_1(s)} \right] \right\} ds$$

$$= \int_{\delta(t)}^{t} (\overline{N}_1(s) - N_1(s)) \left[a(s) - b(s)N_1(\delta(s)) - \frac{c(s)\,N_2(s)}{m\,N_2^{\gamma}(s) + N_1(s)} \right] ds$$

$$+ \int_{\delta(t)}^{t} \overline{N}_1(s) \left\{ b(s) \left[N_1(\delta(s)) - \overline{N}_1(\delta(s)) \right] \right.$$

$$\left. + \left[\frac{c(s)\,N_2(s)}{m\,N_2^{\gamma}(s) + N_1(s)} - \frac{c(s)\overline{N}_2(s)}{m\overline{N}_2^{\gamma}(s) + \overline{N}_1(s)} \right] \right\} ds$$

$$(4.2c)$$

$$\leq \int_{\delta(t)}^{t} |\overline{N}_1(s) - N_1(s)| \left[a(s) + b(s)N_1(\delta(s)) + \frac{c(s)\,N_2(s)}{m\,N_2^{\gamma}(s) + N_1(s)} \right] ds$$

$$+ K_1 \int_{\delta(t)}^{t} b(s) \left| N_1(\delta(s)) - \overline{N}_1(\delta(s)) \right| ds$$

$$+ \frac{P_1}{P_0} \int_{\delta(t)}^{t} c(s)|\overline{N}_2(s) - N_2(s)| ds + \frac{K_1 K_2}{P_0} \int_{\delta(t)}^{t} c(s)|N_1(s) - \overline{N}_1(s)| ds$$

$$+ \frac{m K_1 K_2}{P_0} \int_{\delta(t)}^{t} c(s)|N_2^{\gamma}(s) - \overline{N}_2^{\gamma}(s)| ds$$

$$\leq \int_{\delta(t)}^{t} |\overline{N}_1(s) - N_1(s)| \left[a(s) + K_1 b(s) + \frac{1}{m} K_2^{1-\gamma} c(s) \right] ds$$

$$+ K_1 \int_{\delta(t)}^{t} b(s) \left| N_1(\delta(s)) - \overline{N}_1(\delta(s)) \right| ds$$

$$+ \frac{P_1}{P_0} \int_{\delta(t)}^{t} c(s)|\overline{N}_2(s) - N_2(s)| ds + \frac{K_1 K_2}{P_0} \int_{\delta(t)}^{t} c(s)|N_1(s) - \overline{N}_1(s)| ds$$

$$+ \frac{m K_1 K_2}{P_0} \int_{\delta(t)}^{t} c(s)|N_2^{\gamma}(s) - \overline{N}_2^{\gamma}(s)| ds.$$

Substitution of (4.2) into (4.1), gives:

$D^+V_1(t)$

$$\leq \quad -\alpha\, b(t)|\overline{N}_1(t) - N_1(t)| + \tfrac{mK_1K_2}{P_0}\int_{\delta(t)}^{t} c(s)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|ds$$

$$+\alpha\, b(t)\left\{\int_{\delta(t)}^{t}|\overline{N}_1(s) - N_1(s)|\left[a(s) + K_1b(s) + \frac{1}{m}K_2^{1-\gamma}c(s)\right]ds\right.$$

$$+K_1\int_{\delta(t)}^{t}b(s)\left|N_1(\delta(s)) - \overline{N}_1(\delta(s))\right|ds + \tfrac{\alpha K_2 c(t)}{P_0}|N_1(t) - \overline{N}_1(t)|$$

$$+\tfrac{P_1}{P_0}\int_{\delta(t)}^{t}c(s)|\overline{N}_2(t) - N_2(t)|ds + \tfrac{mK_2^\gamma r(t)}{P_0}|\overline{N}_1(t) - N_1(t)|$$

$$\left.+\tfrac{K_1K_2}{P_0}\int_{\delta(t)}^{t}c(s)|N_1(t) - \overline{N}_1(t)|ds\right\} - \tfrac{mK_3r(t)}{P_{00}}|\overline{N}_2(t) - N_2(t)|$$

$$+\tfrac{\alpha c(t)}{P_0}\left[(mK_2^\gamma + K_1)|\overline{N}_2(t) - N_2(t)| + mK_2|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|\right]$$

$$= - f_1(t)|\overline{N}_1(t) - N_1(t)| - f_2(t)|\overline{N}_2(t) - N_2(t)| + f_3(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|$$

$$+ \alpha\, b(t)\left\{\int_{\delta(t)}^{t}|\overline{N}_1(s) - N_1(s)|\left[a(s) + K_1b(s) + \frac{1}{m}K_2^{1-\gamma}c(s)\right]ds\right.$$

$$+ K_1\int_{\delta(t)}^{t}b(s)\left|N_1(\delta(s)) - \overline{N}_1(\delta(s))\right|ds$$

$$+ \frac{P_1}{P_0}\int_{\delta(t)}^{t}c(s)|\overline{N}_2(t) - N_2(t)|ds$$

$$+ \frac{mK_1K_2}{P_0}\int_{\delta(t)}^{t}c(s)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|ds$$

$$+ \frac{K_1K_2}{P_0}\int_{\delta(t)}^{t}c(s)|N_1(t) - \overline{N}_1(t)|ds\right\},$$

where,

$$f_1(t) = \alpha b(t) - \frac{\alpha K_2}{P_0}c(t) - \frac{mK_2^\gamma}{P_0}r(t),$$

$$f_2(t) = \frac{mK_3}{P_{00}}r(t) - \frac{\alpha(mK_2^\gamma + K_1)}{P_0}c(t),$$

$$f_3(t) = \frac{m\alpha K_2}{P_0}c(t).$$

Define further that

$$
\begin{aligned}
V_2(t) =& \alpha \int_t^{\delta^{-1}(t)} \int_{\delta(l)}^t b(l)[a(s) + K_1 b(s) \\
&+ \frac{1}{m} K_2^{1-\gamma} c(s)]|\overline{N}_1(s) - N_1(s)| ds dl \\
&+ K_1 \int_t^{\delta^{-1}(t)} \int_{\delta(l)}^t b(s)|N_1(\delta(s)) - \overline{N}_1(\delta(s))| ds dl \\
&+ \frac{P_1}{P_0} \int_t^{\delta^{-1}(t)} \int_{\delta(l)}^t c(s)|\overline{N}_2(s) - N_2(s)| ds dl \\
&+ \frac{mK_1 K_2}{P_0} \int_t^{\delta^{-1}(t)} \int_{\delta(l)}^t c(s)|N_2^\gamma(s) - \overline{N}_2^\gamma(s)| ds dl \\
&+ \frac{K_1 K_2}{P_0} \int_t^{\delta^{-1}(t)} \int_{\delta(l)}^t c(s)|N_1(s) - \overline{N}_1(s)| ds dl.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& V_2'(t) \\
=& \alpha \left[a(t) + K_1 b(t) + \frac{1}{m} K_2^{1-\gamma} c(t) \right] |\overline{N}_1(t) - N_1(t)| \int_t^{\delta^{-1}(t)} b(l) dl \\
& - \alpha \int_{\delta_1(t)}^t b(t) \left[a(s) + K_1 b(s) + \frac{1}{m} K_2^{1-\gamma} c(s) \right] |\overline{N}_1(s) - N_1(s)| ds \\
& + K_1 b(t)\phi(t)|N_1(\delta(t)) - \overline{N}_1(\delta(t))| \\
& - K_1 \int_{\delta_1(t)}^t b(s)|N_1(\delta(s)) - \overline{N}_1(\delta(s))| ds \\
& + \frac{P_1}{P_0} c(t)\phi(t)|\overline{N}_2(t) - N_2(t)| \\
& - \frac{P_1}{P_0} \int_{\delta(t)}^t c(s)|\overline{N}_2(s) - N_2(s)| ds \\
& + \frac{mK_1 K_2}{P_0} c(t)\phi(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)| \\
& - \frac{mK_1 K_2}{P_0} \int_{\delta(t)}^t c(s)|N_2^\gamma(s) - \overline{N}_2^\gamma(s)| ds \\
& + \frac{K_1 K_2}{P_0} c(t)\phi(t)|N_1(t) - \overline{N}_1(t)| \\
& - \frac{K_1 K_2}{P_0} \int_{\delta(t)}^t c(s)|N_1(s) - \overline{N}_1(s)| ds.
\end{aligned}
$$

Define

$$V_3(t) = K_1 \int_{\delta(t)}^t \frac{b(\delta^{-1}(s))\phi(\delta^{-1}(s))}{1 - \tau'(\delta^{-1}(s))} |\overline{N}_1(s) - N_1(s)| ds.$$

Then,

$$\begin{aligned} V_3'(t) =& K_1 \frac{b(\delta^{-1}(t))\phi(\delta^{-1}(t))}{1 - \tau'(\delta^{-1}(t))} |\overline{N}_1(t) - N_1(t)| \\ &- K_1 b(t)\phi(t)|N_1(\delta(t)) - \overline{N}_1(\delta(t))|. \end{aligned}$$

We choose the Lyapunov functional for the system (1.4) in the following form:

$$V(t) = V_1(t) + V_2(t) + V_3(t).$$

By simple calculations we get the Dini derivative along the system (1.4) as follows:

(4.2)

$$\begin{aligned} D^+V(t) =& V_1^+(t) + V_2'(t) + V_3'(t) \\ \leq\ & -f_1(t)|\overline{N}_1(t) - N_1(t)| - f_2(t)|\overline{N}_2(t) - N_2(t)| \\ & + f_3(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)| \\ & + \alpha\left(a(t) + K_1 b(t) + \frac{1}{m}K_2^{1-\gamma}c(t)\right)\int_t^{\delta^{-1}(t)} b(l)dl|\overline{N}_1(t) - N_1(t)| \\ & + K_1 \frac{b(\delta^{-1}(t))\phi(\delta^{-1}(t))}{1-\tau'(\delta^{-1}(t))}|\overline{N}_1(t) - N_1(t)| \\ & + \frac{K_1 K_2}{P_0}c(t)\phi(t)|N_1(t) - \overline{N}_1(t)| \\ & + \frac{P_1}{P_0}c(t)\phi(t)|\overline{N}_2(t) - N_2(t)| \\ & + \frac{mK_1 K_2}{P_0}c(t)\phi(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)| \\ \leq\ & -g_1(t)|\overline{N}_1(t) - N_1(t)| - g_2(t)|\overline{N}_2(t) - N_2(t)| \\ & + g_3(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|. \end{aligned}$$

Note that γ is a rational number, which yields that there exist two mutually prime numbers p and q with $p > q$ and $\gamma = \frac{q}{p}$, such that

$$(4.3) \quad \overline{N}_2{}^\gamma(t) - N_2{}^\gamma(t) = \left[\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right] \sum_{i=1}^{q} \overline{N}_2{}^{\frac{q-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t)$$

and

$$(4.4) \quad \overline{N}_2(t) - N_2(t) = \left[\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right] \sum_{i=1}^{p} \overline{N}_2{}^{\frac{p-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t).$$

Substitution of (4.4) into (4.3), yields:

$$D^+V(t)$$

$$\leq - g_1(t)|\overline{N}_1(t) - N_1(t)| - g_2(t)|\overline{N}_2(t) - N_2(t)| + g_3(t)|N_2^\gamma(t) - \overline{N}_2^\gamma(t)|$$

$$= - g_1(t)|\overline{N}_1(t) - N_1(t)| - \left\{ g_2(t) \sum_{i=1}^{p} \overline{N}_2{}^{\frac{p-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \right.$$

$$\left. - g_3(t) \sum_{i=1}^{q} \overline{N}_2{}^{\frac{q-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \right\} \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|$$

$$= - g_1(t)|\overline{N}_1(t) - N_1(t)| - g_2(t) \sum_{i=q}^{p} \overline{N}_2{}^{\frac{p-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|$$

$$- \left\{ g_2(t) \sum_{i=1}^{q} \overline{N}_2{}^{\frac{p-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \right.$$

$$\left. - g_3(t) \sum_{i=1}^{q} \overline{N}_2{}^{\frac{q-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \right\} \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|$$

$$= - g_1(t)|\overline{N}_1(t) - N_1(t)| - g_2(t) \sum_{i=q}^{p} \overline{N}_2{}^{\frac{p-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|$$

$$- \left(g_2(t)\overline{N}_2{}^{\gamma-1}(t) - g_3(t)\right) \sum_{i=1}^{q} \overline{N}_2{}^{\frac{q-i}{p}}(t) N_2{}^{\frac{i-1}{p}}(t) \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|$$

$$\leq - g_1(t)|\overline{N}_1(t) - N_1(t)| - g_2(t) \sum_{i=q}^{p} \overline{N}_2{}^{\frac{p-i}{p}}(t)\, N_2{}^{\frac{i-1}{p}}(t) \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|$$

$$- \left(K_2{}^{\gamma-1} g_2(t) - g_3(t)\right) \sum_{i=1}^{q} \overline{N}_2{}^{\frac{q-i}{p}}(t) N_2{}^{\frac{i-1}{p}}(t) \left|\overline{N}_2{}^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right|,$$

which implies that there exist two positive constants β_1 and β_2 such that

$$D^+V(t) \leq -\beta_1|\overline{N}_1(t) - N_1(t)| - \beta_2|\overline{N}_2^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)| \text{ for } t > T_4.$$

Thus, integration of the above inequality from T_4 to t gives, for $t > T_4$,

$$V(t) + \beta_1 \int_{T_4}^t |\overline{N}_1(t) - N_1(t)| dt + \beta_2 \int_{T_4}^t \left|\overline{N}_2^{\frac{1}{p}}(t) - N_2^{\frac{1}{p}}(t)\right| d \leq V(0) < +\infty,$$

which together with Barbalat's Lemma [28] leads to $\lim_{t\to+\infty} |\overline{N}_1(t) - N_1(t)| = 0$ and $\lim_{t\to+\infty} |\overline{N}_2(t) - N_2(t)| = 0$. $\square$

Theorem 4.2. *If we replace the term $K_2{}^{\gamma-1}g_2(t) - g_3(t)$ in Theorem 4.1 by $\gamma^{-1}K_4{}^{(1-\gamma)/\gamma}g_2(t) - g_3(t)$, then the result still holds without the need for the assumption that γ is a rational number.*

Proof. The mean value theorem yields that there is a $\xi \in (K_4, K_2)$ such that

$$\begin{aligned}|\overline{N}_2(t) - N_2(t)| &= \gamma^{-1}\xi^{(1-\gamma)/\gamma}|\overline{N}_2\gamma(t) - N_2^\gamma(t)| \\ &\geq \gamma^{-1}K_4^{(1-\gamma)/\gamma}|\overline{N}_2^\gamma(t) - N_2^\gamma(t)|,\end{aligned}$$

which together with (4.3), gives:

$$\begin{aligned}D^+V(t) \leq &- g_1(t)|\overline{N}_1(t) - N_1(t)| \\ &- \left(\gamma^{-1}K_4^{(1-\gamma)/\gamma}g_2(t) - g_3(t)\right)|\overline{N}_2(t) - N_2(t)|.\end{aligned}$$

The remainder of the proof is similar to the previous proof of Theorem 4.1. $\square$

If $\tau(t) \equiv \tau$ is a positive constant, then Theorem 4.1 reduces to the following result.

Corollary 4.3. *In addition to the assumption in Theorem 3.1, assume further that $\gamma \in (0, 1)$ is a rational number and the following condition holds:*

$[A_2]$ $\quad D_3 := \liminf\limits_{t \to +\infty}\{g_1(t), K_2{}^{\gamma-1}g_2(t) - g_3(t)\} > 0$, where,

$$g_1(t) = \alpha b(t) - \frac{\alpha K_2}{P_0}c(t) - \frac{mK_2^{\gamma}}{P_0}r(t) - K_1\left(b(t+\tau) + \frac{\tau K_2}{P_0}c(t)\right)$$

$$- \alpha\left(a(t) + K_1 b(t) + \frac{1}{m}K_2^{1-\gamma}c(t)\right)\int_t^{t+\tau}(t)b(l)dl,$$

$$g_2(t) = \frac{mK_3}{P_{00}}r(t) - \frac{\alpha(mK_2^{\gamma} + K_1)}{P_0}c(t) - \frac{\tau P_1}{P_0}c(t) \quad and$$

$$g_3(t) = \frac{mK_2}{P_0}(\alpha + \tau K_1)c(t).$$

Then, system (1.4) is globally asymptotic stable.

Corollary 4.4. *If we replace the term $K_2{}^{\gamma-1}g_2(t) - g_3(t)$, in Corollary 4.3, by $\gamma^{-1}K_4{}^{(1-\gamma)/\gamma}g_2(t) - g_3(t)$, then the result still holds without the assumption that γ is a rational number.*

5. Simulation

Consider the following system:

$$(5.1) \quad \begin{cases} N_1'(t) = N_1(t)\left[5 - 0.1\sin t - 18N_1(t - 0.02) - \frac{(1 - 0.1\sin t)\,N_2(t)}{5\,N_2^{0.5}(t) + N_1(t)}\right], \\ N_2'(t) = N_2(t)\left[-5 - 0.11\sin t + \frac{(6 + 0.12\sin t)N_1(t)}{5N_2^{0.5}(t) + N_1(t)}\right] \end{cases}$$

Choose $\alpha = 2.08 \times 10^{-5}$. By simple calculations we get:

$$K_1 \approx 0.3137586504, K_2 \approx 5.223918756 \times 10^{-3},$$
$$K_3 \approx 0.2674010579, K_4 \approx 8.484344651 \times 10^{-5},$$
$$D_1 \approx 4.8953, \ D_2 = 0.99, \text{and } D_3 \approx 2.532623574.$$

Thus, Theorem 3.1 yields that the system is uniformly persistent, which together with Theorem 4.1 implies that it is also globally asymptotic stable. The following figure shows the dynamic behavior of the positive solution $(N_1(t), N_2(t))$ of the system (5.1) with initial values $(N_1(\theta), N_2(\theta)), \theta \in [-0.02, 0]$, verifying the above results.

Acknowledgments

References

[1] A. Lotka, *The Elements of Physical Biology*, Williams and Wilkins, Baltimore, 1925.

[2] V. Volterra, Variazioni e fluttuazioni del numero di individui in specie animali conviventi, *Mem. Accd. Lincei.* **2** (1926) 31-113.

[3] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel dekker, Inc., New York, 1980.

[4] N.G. Hairston, F. E. Smith and L. B. Slobodkin, Community structure, population control and competition, *American Naturalist* **94** (1960) 421-425.

[5] M. L. Rosenzweig, Paradox of enrichment: destabilization of exploitation ecosystems in ecological time, *Science* **171** (1971) 385-387.

[6] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics: ratio-dependence, *J. Theoret. Biol.* **139** (1989) 311-326.

[7] R. Arditi and H. Saiah, Empirical evidence of the role of heterogeneity in ratio-dependent consumption, *Ecology* **73** (1992) 1544-1551.

[8] E. Berreta and Y. Kuang, Global analysis in some delayed ratio-dependent predator-prey systems, *Nonlinear Anal.* **32** (1998) 381-408.

[9] C. Jost, O. Arino and R. Arditi, About deterministic extinction in ratio-dependent predator-prey models, *Bull. Math. Biol.* **61** (1999) 19-32.

[10] S. B. Hsu, T.-W. Hwang and Y. Kuang, Rich dynamics of a ratio-dependent one-prey two-predator model, *J. Math. Biol.* **43** (2001) 377-396.

[11] D. Xiao and S. Ruan, Global dynamics of a ratio-dependent predator-prey system, *J. Math. Biol.* **43** (2001) 268-290.

[12] F. Berezovskaya, G. Karev and R. Arditi, Parametric analysis of the ratio-dependent predator-prey model, *J. Math. Biol.* **43** (2001) 221-246.

[13] A. Maiti, M. M. Jana and G. P. Samanta, Deterministic and stochastic analysis of a ratio-dependent predator-prey system with delay, *Nonlinear Anal. Model. Control* **12** (2007) 383-398.

[14] M. P. Hassell and G. C. Varley, New inductive population model for insect parasites and its bearing on biological control, *Nature* **223** (1969) 1133-1137.

[15] C. Cosner, D. L. DeAngelis, J. S. Ault and D. B. Olson, Effects of spatial grouping on the functional response of predators, *Theoretical Population Biology* **56** (1999) 65-75.

[16] S. B. Hsu, T. W. Hwang and Y. Kuang, Global dynamics of a predator-prey model with Hassell-Varley type functional response, *Discrete Contin. Dyn. Syst. Ser. B* **10** (2008) 857-871.

[17] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, 191. Academic Press, Inc., Boston, MA, 1993.

[18] N. Macdonald, *Biological Delay Systems: Linear Stability Theory*, Cambridge University Press, Cambridge, 1989.

[19] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers Group, Dordrecht, 1992.

[20] M. Fan, K. Wang and D. Jiang, Existence and global attractivity of positive periodic solutions of periodic n-species Lotka-Volterra competition systems with several deviating arguments, *Math. Biosci.* **160** (1999) 47-61.

[21] R. Xu, M. A. J. Chaplain and F. A. Davidson, Periodic solutions for a delayed predator-prey model of prey dispersal in two-patch environments, *Nonlinear Anal. Real World Appl.* **5** (2004) 183-206.

[22] C. Egami and N. Hirano, Periodic solutions in a class of periodic delay predator-prey systems, *Yokohama Math. J.* **51** (2004) 45-61.

[23] S. P. Lu, On the existence of positive periodic solutions to a Lotka Volterra cooperative population model with multiple delays, *Nonlinear Anal.* **68** (2008) 1746-1753.

[24] K. Wang, Periodic solutions to a delayed predator-prey model with Hassell-Varley type functional response, *Nonlinear Anal. Real World Appl.* **12** (2011) 137-145.

[25] X. Z. He, Stability and delays in a predator-prey system, *J. Math. Anal. Appl.* **198** (1996) 355-370.

[26] F. D. Chen, Permanence and global stability of nonautonomous Lotka-Volterra system with predator-prey and deviating arguments, *Appl. Math. Comput.* **173** (2006) 1082-1100.

[27] C. X. Shen, Permanence and global attractivity of the food-chin system with Holling IV type functional response, *Appl. Math. Comput.* **194** (2007) 179-185.

[28] I. Barbălat, Systèmes d'équations différentielles d'oscillations non linéaires, (French), *Rev. Math. Pures Appl.* **4** (1959) 267-270.

Kai Wang
Department of Mathematics, Southeast University, Nanjing, 210096, PR CHINA and Institute of Dynamical Sytems, School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu, 233030, PR CHINA
Email: wangkai050318@163.com; kaiwang2011@gmail.com

Yanling Zhu
Institute of Applied Mathematics, School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu, 233030, PR CHINA
Email: zhuyanling99@yahoo.com.cn