Bulletin of the Iranian Mathematical Society Vol. 37 *No. 3* (*2011*), *pp 217-233.*

SEMISTAR DIMENSION OF POLYNOMIAL RINGS AND PRÜFER-LIKE DOMAINS

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Communicated by Fariborz Azarpanah

ABSTRACT. Let D be an integral domain and $\star$ a semistar operation stable and of finite type on it. We define the semistar dimension (inequality) formula and discover their relations with $\star$-universally catenarian domains and $\star$-stably strong S-domains. As an application, we give new characterizations of $\star$-quasi-Prüfer domains and UMt domains in terms of dimension inequality formula (and the notions of universally catenarian domain, stably strong S-domain, strong S-domain, and Jaffard domain). We also extend Arnold's formula to the setting of semistar operations.

1. **Introduction**

All rings considered here are (commutative integral) domains (with 1). Throughout, D denotes a domain with quotient field K. In [22], Okabe and Matsuda introduced the concept of a semistar operation. Let D be an integral domain and $\star$ be a semistar operation on D.

In [24], we defined and studied the $\widetilde{\star}$-Jaffard domains and proved that every $\widetilde{\star}$-Noetherian and P$\star$MD of finite $\widetilde{\star}$-dimension is a $\widetilde{\star}$-Jaffard domain. In [25], we defined and studied two subclasses of $\widetilde{\star}$-Jaffard

MSC(2010): Primary: 13C15; Secondary: 13G05, 13A15.
Keywords: Semistar operation, Krull dimension, strong S-domain, Jaffard domain, quasi-Prüfer domain, UMt domain.
Received: 29 November 2009, Accepted: 6 March 2010.
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domains, namely the $\widetilde{\star}$-stably strong S-domains and $\widetilde{\star}$-universally catenarian domains and showed how these notions permit studies of $\widetilde{\star}$-quasi-Prüfer domains in the spirit of earlier works on quasi-Prüfer domains. The next natural step is to seek a semistar analogue of dimension (inequality) formula [15]. In Section 2, we define the $\widetilde{\star}$-dimension (inequality) formula and show that each $\widetilde{\star}$-universally catenarian domain satisfies the $\widetilde{\star}$-dimension formula and each $\widetilde{\star}$-stably strong S-domain satisfies the $\widetilde{\star}$-dimension inequality formula. In Section 3, we give new characterizations of $\star$-quasi-Prüfer domains and UMt domains in terms of the classical notions of dimension inequality formula, universally catenarian domain, stably strong S-domain, strong S-domain, and Jaffard domain. In the last section, we extend Arnold's formula to the setting of semistar operations (see Theorem 4.6).

To facilitate the reading, we first review some basic facts on semistar operations. Denote by $\overline{\mathcal{F}}(D)$, the set of all nonzero D-submodules of K, and by $\mathcal{F}(D)$, the set of all nonzero *fractional ideals* of D; i.e., $E \in \mathcal{F}(D)$ if $E \in \overline{\mathcal{F}}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [22], a *semistar operation on* D is a map $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following three properties hold: $\star_1$: $(xE)^\star = xE^\star$; $\star_2$: $E \subseteq F$ implies $E^\star \subseteq F^\star$; $\star_3$: $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$. Let $\star$ be a semistar operation on the domain D. For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \bigcup F^\star$, where the union is taken over all finitely generated $F \in f(D)$ with $F \subseteq E$. It is easy to see that $\star_f$ is a semistar operation on D, and $\star_f$ is called *the semistar operation of finite type associated with* $\star$. Note that $(\star_f)_f = \star_f$. A semistar operation $\star$ is said to be of *finite type* if $\star = \star_f$; in particular, $\star_f$ is of finite type. We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* of D, if $I^\star \cap D = I$; a *quasi-$\star$-prime* (ideal of D), if I is a prime quasi-$\star$-ideal of D; and a *quasi-$\star$-maximal* (ideal of D), if I is maximal in the set of all proper quasi-$\star$-ideals of D. Each quasi-$\star$-maximal ideal is a prime ideal. It was shown in [14, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi-$\star_f$-ideal of D is contained in a quasi-$\star_f$-maximal ideal of D. We denote by $\mathrm{QMax}^\star(D)$ (respectively $\mathrm{QSpec}^\star(D)$), the set of all quasi-$\star$-maximal ideals (respectively quasi-$\star$-prime ideals) of D.

If Δ is a set of prime ideals of a domain D, then there is an associated semistar operation on D, denoted by $\star_\Delta$, defined as follows:

$$E^{\star_\Delta} := \cap\{ED_P | P \in \Delta\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

If $\Delta = \emptyset$, then let $E^{\star_\Delta} := K$, for each $E \in \overline{\mathcal{F}}(D)$. When $\Delta := \mathrm{QMax}^{\star_f}(D)$, we set $\widetilde{\star} := \star_\Delta$. It has become standard to say that a semistar operation $\star$ is *stable*, if $(E \cap F)^\star = E^\star \cap F^\star$, for all $E, F \in \overline{\mathcal{F}}(D)$. All spectral semistar operations are stable [14, Lemma 4.1(3)]. In particular, for any semistar operation $\star$, we have that $\widetilde{\star}$ is a stable semistar operation of finite type [14, Corollary 3.9].

The most widely studied (semi)star operations on D have been the identity d_D, and v_D, $t_D := (v_D)_f$, and $w_D := \widetilde{v_D}$ operations, where $E^{v_D} := (E^{-1})^{-1}$, with $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$.

For each quasi-$\star$-prime P of D, the $\star$-*height* of P (for short, $\star$-ht(P)) is defined to be the supremum of the lengths of the chains of quasi-$\star$-prime ideals of D, between prime ideal (0) (included) and P. Obviously, if $\star = d_D$ is the identity (semi)star operation on D, then $\star\text{-ht}(P) = \mathrm{ht}(P)$, for each prime ideal P of D. If the set of quasi-$\star$-prime of D is not empty, the $\star$-dimension of D is defined as follows:

$$\star\text{-}\dim(D) := \sup\{\star\text{-}\mathrm{ht}(P) | P \text{ is a quasi-}\star\text{-prime of } D\}.$$

If the set of quasi-$\star$-primes of D is empty, then pose $\star\text{-}\dim(D) := 0$. Thus, if $\star = d_D$, then $\star\text{-}\dim(D) = \dim(D)$, the usual (Krull) dimension of D. It is known (see [12, Lemma 2.11]) that

$$\widetilde{\star}\text{-}\dim(D) = \sup\{\mathrm{ht}(P) \mid P \text{ is a quasi-}\widetilde{\star}\text{-prime ideal of } D\}.$$

Let $\star$ be a semistar operation on a domain D. Recall from [12, Section 3] that D is said to be a $\star$-*Noetherian domain*, if D satisfies the ascending chain condition on quasi-$\star$-ideals. Also, recall from [16] that D is called a *Prüfer $\star$-multiplication domain* (for short, a P$\star$MD), if each finitely generated ideal of D is $\star_f$-*invertible*; i.e., if $(II^{-1})^{\star_f} = D^\star$, for all $I \in f(D)$. When $\star = v$, we recover the classical notion of PvMD; when $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain. Finally, recall from [7] that D is said to be a $\star$-*quasi-Prüfer domain*, in case, if Q is a prime ideal in $D[X]$, and $Q \subseteq P[X]$, for some $P \in \mathrm{QSpec}^\star(D)$, then $Q = (Q \cap D)[X]$. This notion is the semistar analogue of the classical notion of the quasi-Prüfer domains. By [7, Corollary 2.4], D is a $\star_f$-quasi-Prüfer domain if and only if D is a $\widetilde{\star}$-quasi-Prüfer domain.

2. The $\star$-dimension (Inequality) Formula

We begin with the following definition. Recall that if $D \subseteq T$ are domains, then $\mathrm{tr.deg.}_D(T)$ is defined as the *transcendence degree* of the quotient field of T over the quotient field of D. If P is a prime ideal of D, then $\mathbb{K}(P)$ is denoted to be the residue field of D in P; i.e., D_P/PD_P, which is canonically isomorphic to the field of quotients of the integral domain D/P.

Definition 2.1. *Let $D \subseteq T$ be an extension of domain and $\star$ and $\star'$ be semistar operation on D and T, respectively. We say that $D \subseteq T$ satisfies the* $(\star, \star')$-dimension formula (respectively $(\star, \star')$-dimension inequality formula)*, if for all $Q \in \mathrm{QSpec}^{\star'}(T)$ such that $(Q \cap D)^{\star} \subsetneq D^{\star}$,* $\star'\text{-}\operatorname{ht}(Q) + \operatorname{tr.deg.}_{\mathbb{K}(Q\cap D)}(\mathbb{K}(Q)) = \star\text{-}\operatorname{ht}(Q \cap D) + \operatorname{tr.deg.}_D(T)$ *(respectively*
$\star'\text{-}\operatorname{ht}(Q) + \operatorname{tr.deg.}_{\mathbb{K}(Q\cap D)}(\mathbb{K}(Q)) \leq \star\text{-}\operatorname{ht}(Q \cap D) + \operatorname{tr.deg.}_D(T)$). *The domain D is said to satisfy the* $\star$-dimension formula (respectively $\star$-dimension inequality formula)*, if for all finitely generated domain T over D, $D \subseteq T$ satisfies the $(\star, d_T)$-dimension formula (respectively $(\star, d_T)$-dimension inequality formula).*

If $\star = d_D$ and $\star' = d_T$, then these definitions coincide with the classical ones (see [15, 20]).

Proposition 2.2. *Let D be a domain and $\star$ a semistar operation on D. Then, the following conditions are equivalent:*

(1) *D satisfies the $\widetilde{\star}$-dimension formula (respectively $\widetilde{\star}$-dimension inequality formula).*
(2) *D_P satisfies the dimension formula, for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$ (respectively dimension inequality formula).*
(3) *D_M satisfies the dimension formula, for each $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$ (respectively dimension inequality formula).*

Proof. We only prove the case of dimension formula and the other case is the same.

(1) $\Rightarrow$ (2) Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Let T be a finitely generated domain over D_P so that there exist finitely many elements $\theta_1, \cdots, \theta_r \in T$ such that $T = D_P[\theta_1, \cdots, \theta_r]$. Set $T' = D[\theta_1, \cdots, \theta_r]$. Then, $T = T'_{D\backslash P}$ and T' is a finitely generated domain over D. Let Q be a prime ideal of T and set $qD_P := Q \cap D_P$, where $q(\subseteq P)$ is a prime ideal of D. Thus, there exists a prime ideal Q' of T' such that $Q' \cap (D\backslash P) = \emptyset$ and $Q = Q'T'_{D\backslash P}$. Thus, $Q' \cap D = q$. Since $q \subseteq P$, we have that q is a quasi-$\widetilde{\star}$-prime ideal

of D. Since $\widetilde{\star}$- $\mathrm{ht}(q) = \mathrm{ht}(q)$, then by the hypothesis we have:

$$\mathrm{ht}(Q') + \mathrm{tr.deg.}_{\mathbb{K}(q)}(\mathbb{K}(Q')) = \mathrm{ht}(q) + \mathrm{tr.deg.}_D(T').$$

Since $\mathrm{ht}(Q') = \mathrm{ht}(Q)$, we see that

$$\mathrm{ht}(Q) + \mathrm{tr.deg.}_{\mathbb{K}(q)}(\mathbb{K}(Q)) = \mathrm{ht}(q) + \mathrm{tr.deg.}_{D_P}(T).$$

(2) $\Rightarrow$ (3) is trivial.

(3) $\Leftarrow$ (1) Suppose that T is a finitely generated domain over D. Let $Q \in \mathrm{Spec}(T)$ and set $P := Q \cap D$ such that $P^{\widetilde{\star}} \subsetneq D^{\widetilde{\star}}$. Thus, $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$. Let M be a quasi-$\widetilde{\star}$-maximal ideal of D containing P. Note that $T_{D\backslash M}$ is a finitely generated domain over D_M and that $Q \cap (D\backslash M) \neq \emptyset$. Thus, $QT_{D\backslash M} \in \mathrm{Spec}(T_{D\backslash M})$ and $PD_M = QT_{D\backslash M} \cap D_M$. Therefore, by the (3), we have

$$\mathrm{ht}(QT_{D\backslash M}) + \mathrm{tr.deg.}_{\mathbb{K}(PD_M)}(\mathbb{K}(QT_{D\backslash M}))$$
$$= \mathrm{ht}(PD_M) + \mathrm{tr.deg.}_{D_M}(T_{D\backslash M}).$$

Now, since $\widetilde{\star}$- $\mathrm{ht}(Q) = \mathrm{ht}(Q) = \mathrm{ht}(QT_{D\backslash M})$, $\mathrm{ht}(P) = \mathrm{ht}(PD_M)$, and $\mathrm{tr.deg.}_{\mathbb{K}(P)}(\mathbb{K}(Q)) = \mathrm{tr.deg.}_{\mathbb{K}(PD_M)}(\mathbb{K}(QT_{D\backslash M}))$ and that $\mathrm{tr.deg.}_D(T)$ $= \mathrm{tr.deg.}_{D_M}(T_{D\backslash M})$ and the proof is complete. □

Let D be an integral domain with quotient field K, let X and Y be two indeterminates over D and let $\star$ be a semistar operation on D. Set $D_1 := D[X]$, $K_1 := K(X)$ and take the following subset of $\mathrm{Spec}(D_1)$:

$$\Theta_1^{\star} := \{Q_1 \in \mathrm{Spec}(D_1) |\ Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^{\star_f} \subsetneq D^{\star}\}.$$

Set $\mathfrak{S}_1^{\star} := \mathcal{S}(\Theta_1^{\star}) := D_1[Y] \backslash (\bigcup\{Q_1[Y] | Q_1 \in \Theta_1^{\star}\})$ and

$$E^{\circlearrowleft_{\mathfrak{S}_1^{\star}}} := E[Y]_{\mathfrak{S}_1^{\star}} \cap K_1, \text{ for all } E \in \overline{\mathcal{F}}(D_1).$$

It is proved in [24, Theorem 2.1] that the mapping $\star[X] := \circlearrowleft_{\mathfrak{S}_1^{\star}}$: $\overline{\mathcal{F}}(D_1) \to \overline{\mathcal{F}}(D_1)$, $E \mapsto E^{\star[X]}$ is a stable semistar operation of finite type on $D[X]$; i.e., $\widetilde{\star[X]} = \star[X]$. It is also proved that $\widetilde{\star}[X] = \star_f[X] = \star[X]$, $d_D[X] = d_{D[X]}$ and $\mathrm{QSpec}^{\star[X]}(D[X]) = \Theta_1^{\star} \backslash \{0\}$. If $X_1, \cdots, X_r$ are indeterminates over D, for $r \geq 2$, we let

$$\star[X_1, \cdots, X_r] := (\star[X_1, \cdots, X_{r-1}])[X_r],$$

where, $\star[X_1, \cdots, X_{r-1}]$ is a stable semistar operation of finite type on $D[X_1, \cdots, X_{r-1}]$. For an integer r, put $\star[r]$ to denote $\star[X_1, \cdots, X_r]$ and $D[r]$ to denote $D[X_1, \cdots, X_r]$.

Following [25], the domain D is called $\star$-*catenary*, if for each pair $P \subset Q$ of quasi-$\star$-prime ideals of D, any two saturated chain of quasi-$\star$-prime ideals between P and Q have the same finite length. If for each $n \geq 1$, the polynomial ring $D[n]$ is $\star[n]$-catenary, then D is said to be $\star$-*universally catenarian*. Every P$\star$MD, which is $\widetilde{\star}$-LFD (that is, $\mathrm{ht}(P) < \infty$, for all $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$), is $\widetilde{\star}$-universally catenarian, by [25, Theorem 3.4].

Corollary 2.3. *Let D be an $\widetilde{\star}$-universally catenarian domain. Then, D satisfies the $\widetilde{\star}$-dimension formula.*

Proof. Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Hence, D_P is a universally catenarian domain, by [25, Lemma 3.3]. Thus, by [5, Corollary 4.8], D_P satisfies the dimension formula. Now, Proposition 2.2 completes the proof. $\square$

The domain D is called a $\star$-*strong S-domain*, if each pair of adjacent quasi-$\star$-prime ideals $P_1 \subset P_2$ of D extends to a pair of adjacent quasi-$\star[X]$-prime ideals $P_1[X] \subset P_2[X]$, of $D[X]$. If for each $n \geq 1$, the polynomial ring $D[n]$ is a $\star[n]$-strong S-domain, then D is said to be an $\star$-*stably strong S-domain*. Every $\widetilde{\star}$-Noetherian, $\widetilde{\star}$-quasi-Prüfer or $\widetilde{\star}$-universally catenarian domain is $\widetilde{\star}$-stably strong S-domain, by [25, corollaries 2.6 and 3.6].

Corollary 2.4. *Let D be an $\widetilde{\star}$-stably strong S-domain. Then, D satisfies the $\widetilde{\star}$-dimension inequality formula.*

Proof. Use [25, Proposition 2.5] and [20, Theorem 1.6] and the same argument as in the proof of Corollary 2.3. $\square$

A valuation overring V of D is called a $\star$-*valuation overring of D* if $F^\star \subseteq FV$, for each $F \in f(D)$. Following [24], the $\star$-*valuative dimension* of D is defined as:

$$\star\text{-}\dim_v(D) := \sup\{\dim(V) | V \text{ is } \star\text{-valuation overring of } D\}.$$

Although Example 4.4 of [24] shows that $\star$-$\dim(D)$ is not always less than or equal to $\star$-$\dim_v(D)$, but it is observed in [24] that $\widetilde{\star}$-$\dim(D) \leq \widetilde{\star}$-$\dim_v(D)$. We say that D is a $\star$-*Jaffard domain*, if $\star\text{-}\dim(D) = \star\text{-}\dim_v(D) < \infty$. When $\star = d$, the identity operation, then d-Jaffard domain coincides with the classical Jaffard domain (cf. [1]). It is proved in [24] that D is a $\widetilde{\star}$-Jaffard domain if and only if

$$\star[X_1, \cdots, X_n]\text{-}\dim(D[X_1, \cdots, X_n]) = \widetilde{\star}\text{-}\dim(D) + n,$$

for each positive integer n.

Lemma 2.5. *For each domain D, we have*

$$\widetilde{\star}\text{-}\dim_v(D) = \sup\{\dim_v(D_P) | P \in \mathrm{QSpec}^{\widetilde{\star}}(D)\}.$$

Proof. We can assume that $\widetilde{\star}\text{-}\dim_v(D)$ is a finite number. Suppose that $n = \widetilde{\star}\text{-}\dim_v(D)$. Then, there exists a $\widetilde{\star}$-valuation overring V, with maximal ideal N, of D such that $\dim(V) = n$. Set $P := N \cap D$ so that V is a valuation overring of D_P. Hence, $n = \dim(V) \leq \dim_v(D_P) \leq \widetilde{\star}\text{-}\dim_v(D) = n$, where the second inequality is true, since each valuation overring of D_P is a $\widetilde{\star}$-valuation overring of D [17, Theorem 3.9]. □

In [1, Page 174], it is proved that a finite-dimensional domain satisfying the dimension inequality formula is a Jaffard domain. In the following result, we give the semistar analogue of the mentioned result.

Theorem 2.6. *Let D be a domain of finite $\widetilde{\star}$-dimension. If D satisfies the $\widetilde{\star}$-dimension inequality formula, then D is a $\widetilde{\star}$-Jaffard domain.*

Proof. Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Then D_P is a finite dimensional domain and satisfies the dimension inequality formula by Proposition 2.2. Consequently, D_P is a Jaffard domain, by [1]. Thus, using Lemma 2.5, we have

$$\begin{aligned}\widetilde{\star}\text{-}\dim(D) &= \sup\{\dim(D_P) | P \in \mathrm{QSpec}^{\widetilde{\star}}(D)\} \\ &= \sup\{\dim_v(D_P) | P \in \mathrm{QSpec}^{\widetilde{\star}}(D)\} \\ &= \widetilde{\star}\text{-}\dim_v(D).\end{aligned}$$

Thus, D is a $\widetilde{\star}$-Jaffard domain. □

Therefore, we have the following implications for finite $\widetilde{\star}$-dimensional domains:

$$\begin{array}{ccc}
\widetilde{\star}\text{-Noetherian or } \widetilde{\star}\text{-quasi-Prüfer} & & \text{P}\star\text{MD} \\
\Downarrow & & \Downarrow \\
\widetilde{\star}\text{-stably strong S-domain} & \Leftarrow & \widetilde{\star}\text{-universally catenary} \\
\Downarrow & & \Downarrow \\
\widetilde{\star}\text{-dimension inequality formula} & \Leftarrow & \widetilde{\star}\text{-dimension formula} \\
\Downarrow & & \\
\widetilde{\star}\text{-Jaffard} & &
\end{array}$$

Let D be a domain with quotient field K, let X be an indeterminate over D, let $\star$ be a semistar operation on D, and let P be a quasi-$\star$-prime ideal of D (or $P = 0$). Set

$$\mathcal{S}_P^\star := (D/P)[X] \backslash \bigcup \{(Q/P)[X] \mid Q \in \mathrm{QSpec}^{\star_f}(D) \text{ and } P \subseteq Q\}.$$

Clearly, $\mathcal{S}_P^\star$ is a multiplicatively closed subset of $(D/P)[X]$.

For all $E \in \overline{\mathcal{F}}(D/P)$, set

$$E^{\circlearrowleft_{\mathcal{S}_P^\star}} := E(D/P)[X]_{\mathcal{S}_P^\star} \cap (D_P/PD_P).$$

It is proved in [10, Theorem 3.2] that the mapping $\star/P := \circlearrowleft_{\mathcal{S}_P^\star} : \overline{\mathcal{F}}(D/P) \to \overline{\mathcal{F}}(D/P)$, $E \mapsto E^{\circlearrowleft_{\mathcal{S}_P^\star}}$ is a stable semistar operation of finite type on D/P; i.e., $\widetilde{\star/P} = \star/P$, $\mathrm{QMax}^{\star/P}(D/P) = \{Q/P \in \mathrm{Spec}(D/P) \mid Q \in \mathrm{QMax}^{\star_f}(D) \text{ and } P \subseteq Q\}$, $\widetilde{\star}/P = \star_f/P = \star/P$ and $d_D/P = d_{D/P}$.

Lemma 2.7. *A domain D is $\widetilde{\star}$-universally catenarian if and only if D/P is $(\star/P)$-universally catenarian, for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$.*

Proof. ($\Rightarrow$) Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. By [10, Theorem 3.2 (a)], $\star/P = \widetilde{\star/P}$. Hence, by [25, Proposition 3.2 and Lemma 3.3], D/P is $(\star/P)$-universally catenarian if and only if $(D/P)_{\mathcal{M}}$ is a universally catenarian domain, for each $\mathcal{M} \in \mathrm{QMax}^{\star/P}(D/P)$, that is, by [10, Theorem 3.2 (b)], if and only if D_M/PD_M is a universally catenarian domain, whenever P is a subset of $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. But, by [25, Proposition 3.2 and Lemma 3.3], D_M is a universally catenarian domain, for all $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. This, in turn, is immediate since any factor domain of a universally catenarian domain must be a universally catenarian domain.

($\Leftarrow$) It is enough to consider $P = 0$, since we have $\star/0 = \widetilde{\star}$. □

In [21, Corollary 14.D], it is proved that a Noetherian domain D is an universally catenarian domain if and only if D is catenary and D/P satisfies the dimension formula for each $P \in \mathrm{Spec}(D)$. In the following result, we give the semistar analogue of this result.

Theorem 2.8. *Let D be a $\widetilde{\star}$-Noetherian domain. Then, D is an $\widetilde{\star}$-universally catenarian domain if and only if D is $\widetilde{\star}$-catenary and D/P satisfies the $(\star/P)$-dimension formula, for each $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$.*

Proof. ($\Rightarrow$) Let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Then, D/P is $(\star/P)$-universally catenarian by Lemma 2.7. Hence D/P satisfies the $(\star/P)$-dimension formula by Corollary 2.3.

($\Leftarrow$) Let $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. It is enough to show that D_M is a universally catenarian domain. To this end, let PD_M be a prime ideal of D_M. Thus, P is a quasi-$\widetilde{\star}$-prime ideal of D. Since D/P satisfies the $(\star/P)$-dimension formula, then $(D/P)_{M/P} = D_M/PD_M$ satisfies the dimension formula, by Theorem 2.2. On the other hand, D_M is a Noetherian domain by [12, Proposition 3.8], and catenary, by [25, Proposition 3.2]. Consequently, D_M is a universally catenarian domain, by [21, Corollary 14.D]. □

Recall that the celebrated theorem of Ratliff [23, Theorem 2.6] says that a Noetherian ring R is universally catenarian if and only if $R[X]$ is catenarian. On the other hand, it is proved in [6, Theorem 1] that the Noetherian assumption in Ratliff's theorem can be replaced with the going-down condition by proving that for a going-down domain D, we have D is universally catenarian if and only if $D[X]$ is catenarian if and only if D is an LFD strong S-domain. As a semistar analogue, in [25, Theorem 3.7] we proved that if D is $\widetilde{\star}$-Noetherian, then $D[X]$ is $\star[X]$-catenary if and only if D is $\widetilde{\star}$-universally catenarian. In the last theorem of this section we treat the second case.

Let $D \subseteq T$ be an extension of domains. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. Following [9], we say that $D \subseteq T$ satisfies $(\star, \star')$-GD, if $P_0 \subset P$ are quasi-$\star$-prime ideals of D and Q is a quasi-$\star'$-prime ideal of T such that $Q \cap D = P$, then there exists a quasi-$\star'$-prime ideal Q_0 of T such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$. The integral domain D is said to be a *$\star$-going-down domain* (for short, a $\star$-GD *domain*), if for every overring T of D the extension $D \subseteq T$ satisfies $(\star, d_T)$-GD. These concepts are the semistar versions of the "classical" concepts of going-down property and the going-down domains (cf. [8]). It is known by [9, Propositions 3.5 and 3.2(e)] that every P$\star$MD and every integral domain D with $\star$-dim$(D) = 1$ are a $\star$-GD domain.

Theorem 2.9. *Let D be a $\widetilde{\star}$-GD domain. The following statements are equivalent:*

(1) *D is a $\widetilde{\star}$-LFD $\widetilde{\star}$-strong S-domain.*
(2) *D is $\widetilde{\star}$-universally catenarian.*
(3) *$D[X]$ is $\star[X]$-catenarian.*

Proof. (1) $\Rightarrow$ (2) holds by [25, Theorem 4.1] and (2) $\Rightarrow$ (3) is trivial. For (3) $\Rightarrow$ (1), let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Then, D_P is a going-down domain by [10, Proposition 2.5], and $D_P[X]$ is catenarian, by [25, Lemma 3.3]. Thus, D_P is a LFD strong S-domain, by [6, Theorem 1]. Hence, D is a $\widetilde{\star}$-LFD $\widetilde{\star}$-strong S-domain, by [25, Proposition 2.4]. □

3. Characterizations of $\star$-quasi-Prüfer Domains

In this section, we give some characterization of $\widetilde{\star}$-quasi-Prüfer domains. We need to recall the definition of $(\star, \star')$-linked overrings. Let D be a domain and T an overring of D. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. One says that T is *$(\star, \star')$-linked to D* (or that T is a *$(\star, \star')$-linked overring of D*), if $F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$, when F is a nonzero finitely generated ideal of D (cf. [11]). In particular, we are interested in the case $\star' = d_T$. We first recall the following characterization of $\widetilde{\star}$-quasi-Prüfer domains.

Theorem 3.1. ([25, Theorem 4.3]) *Let D be an integral domain. Suppose that $\widetilde{\star}$-dim(D) is finite. Consider the following statements:*

(1') *Each $(\star, \star')$-linked overring T of D is an $\widetilde{\star'}$-universally catenarian domain.*
(1) *Each $(\star, \star')$-linked overring T of D is an $\widetilde{\star'}$-stably strong S-domain.*
(2) *Each $(\star, \star')$-linked overring T of D is an $\widetilde{\star'}$-strong S-domain.*
(3) *Each $(\star, \star')$-linked overring T of D is an $\widetilde{\star'}$-Jaffard domain.*
(4) *Each $(\star, \star')$-linked overring T of D is an $\widetilde{\star'}$-quasi-Prüfer domain.*
(5) *D is an $\widetilde{\star}$-quasi-Prüfer domain.*

Then, $(1') \Rightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$.

Proof. The implication $(1') \Rightarrow (1)$ holds, by [25, Corollary 3.6], and $(1) \Rightarrow (2)$ is trivial. For $(2) \Rightarrow (5)$, see proof of [25, Theorem 4.3 part $(3) \Rightarrow (6)$]. The implication $(5) \Rightarrow (1)$ holds, by [25, Corollary 2.6]. For $(4) \Leftrightarrow (5) \Leftrightarrow (6)$, see [24, Theorem 4.14]. □

Now, we have the following theorem, a result reminiscent of the well-known result of Ayache, et al. [4] (see also [15, Theorem 6.7.8]) for quasi-Prüfer domains.

Theorem 3.2. *Let D be an integral domain. Suppose that $\widetilde{\star}$-$\dim(D)$ is finite. Then, the following statements are equivalent.*

1. *Each $(\star, d_T)$-linked overring T of D is a stably strong S-domain.*
2. *Each $(\star, d_T)$-linked overring T of D is a strong S-domain.*
3. *Each $(\star, d_T)$-linked overring T of D is a Jaffard domain.*
4. *Each $(\star, d_T)$-linked overring T of D is a quasi-Prüfer domain.*
5. *D is an $\widetilde{\star}$-quasi-Prüfer domain.*

Proof. We only prove the equivalence of $(1) \Leftrightarrow (5)$ and the proofs of $(2) \Leftrightarrow (5)$ $(3) \Leftrightarrow (5)$, and $(4) \Leftrightarrow (5)$ are sammilar. The implication $(5) \Rightarrow (1)$ holds, by Theorem 3.1. For $(1) \Rightarrow (5)$, let $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. It is enough for us to show that D_P is a quasi-Prüfer domain, by [7, Theorem 2.16]. To this end, let T be an overring of D_P. Then, $T_{D\setminus P} = T$ and therefore T is $(\star, d_T)$-linked overring of D, by [11, Example 3.4 (1)]. Thus, by the hypothesis we have T is a stably strong S-domain. Therefore, D_P is a quasi-Prüfer domain, by [15, Theorem 6.7.8]. □

Theorem 3.3. *Let D be an integral domain. Suppose that $\widetilde{\star}$-$\dim(D)$ is finite. Then, the following statements are equivalent.*

1. *D is an $\widetilde{\star}$-quasi-Prüfer domain.*
2. *For each $(\star, \star')$-linked overring T of D, every extension of domains $T \subseteq S$, satisfies the $(\widetilde{\star'}, \widetilde{\star''})$-dimension inequality formula, where $\star'$ and $\star''$ are semistar operations on T and S, respectively.*

Proof. $(1) \Rightarrow (2)$ If D is an $\widetilde{\star}$-quasi-Prüfer domain and T is $(\star, \star')$-linked to D, then T is a $\widetilde{\star'}$-Jaffard domain, by [24, Theorem 4.14]. Let $Q \in \mathrm{QSpec}^{\widetilde{\star''}}(S)$ such that $(Q \cap T)^{\widetilde{\star'}} \subsetneq T^{\widetilde{\star'}}$ and set $q := Q \cap T$. Then, we have $q \in \mathrm{QSpec}^{\widetilde{\star'}}(T) \cup \{0\}$. Set $P := q \cap D$. Thus, we have $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup$

$\{0\}$. Therefore, D_P, and hence T_q, are quasi-Prüfer domains, by [7, Theorem 1.1]. In particular, T_q is a Jaffard domain. Thus, we have

$$\begin{aligned}\dim(S_Q) + \operatorname{tr.deg.}_{\mathbb{K}(q)}(\mathbb{K}(Q)) &\leq \dim_v(S_Q) + \operatorname{tr.deg.}_{\mathbb{K}(q)}(\mathbb{K}(Q)) \\ &\leq \dim_v(T_q) + \operatorname{tr.deg.}_T(S),\end{aligned}$$

where the first inequality holds, since $\dim(S_Q) \leq \dim_v(S_Q)$ and the second one due to [15, Lemma 6.7.3]. The conclusion follows easily from the fact that $\dim(T_q) = \dim_v(T_q)$.

(2) $\Rightarrow$ (1) Let T be an overring of D and $\star'$ be a semistar operation on T such that T is $(\star, \star')$-linked to D. Let (V, N) be any $\widetilde{\star'}$-valuation overring of T. Then, V is $(\star', d_V)$-linked to T, by [12, Lemma 2.7]. Set $Q := N \cap T$. Then, by assumption we have

$$\dim(V) \leq \dim(T_Q) - \operatorname{tr.deg.}_{\mathbb{K}(Q)}(\mathbb{K}(N)).$$

In particular, $\dim(V) \leq \dim(T_Q) \leq \widetilde{\star'}\text{-}\dim(T)$, and hence $\widetilde{\star'}\text{-}\dim_v(T) = \widetilde{\star'}\text{-}\dim(T)$, that is, T is a $\widetilde{\star'}$-Jaffard domain. Thus, D is an $\widetilde{\star}$-quasi-Prüfer domain, by [24, Theorem 4.14]. ☐

Corollary 3.4. *Let D be an integral domain. Suppose that $\widetilde{\star}$-$\dim(D)$ is finite. Then, the following statements are equivalent.*

(1) *D is an $\widetilde{\star}$-quasi-Prüfer domain.*
(2) *For each $(\star, d_T)$-linked overring T of D, every extension of domains $T \subseteq S$, satisfies the dimension inequality formula.*

Proof. (1) $\Rightarrow$ (2) holds by Theorem 3.3. For (2) $\Rightarrow$ (1), let $P \in \operatorname{QSpec}^{\widetilde{\star}}(D)$. It is enough to show that D_P is a quasi-Prüfer domain, by [7, Theorem 2.16]. To this end, let T be an overring of D_P. Then, $T_{D \setminus P} = T$ and therefore, T is $(\star, d_T)$-linked overring of D, by [11, Example 3.4 (1)]. If $T \subseteq S$ is any extension of domains, then $T \subseteq S$ satisfies the dimension inequality formula by the hypothesis. Therefore, D_P is a quasi-Prüfer domain, by [15, Theorem 6.7.4]. ☐

Recall that an integral domain, D is called a *UMt-domain*, if every upper to zero in $D[X]$ is a maximal t-ideal, which has been studied by several authors (see [7, 13, 19]). It is observed in [7, Corollary 2.4 (b)] that D is a w-quasi-Prüfer domain if and only if D is a UMt-domain. The following corollary is a new characterization of UMt domains.

Corollary 3.5. *Let D be an integral domain. Suppose that w-$\dim(D)$ is finite. Then, the following statements are equivalent.*

(1) *Each (t_D, d_T)-linked overring T of D is a stably strong S-domain.*
(2) *Each (t_D, d_T)-linked overring T of D is a strong S-domain.*
(3) *Each (t_D, d_T)-linked overring T of D is a Jaffard domain.*
(4) *Each (t_D, d_T)-linked overring T of D is a quasi-Prüfer domain.*
(5) *For each (t_D, d_T)-linked overring T of D, every extension of domains $T \subseteq S$, satisfies the dimension inequality formula.*
(6) *D is a UMt domain.*

4. **Arnold's Formula**

Here, we extend some results of Arnold of the dimension of polynomial rings to the setting of the semistar operations. First, we give the following lemma as a new property of semistar valuative dimension.

Lemma 4.1. (*see* [24, Theorem 4.2]) *Let D be an integral domain and n be an integer. Then, the following statements are equivalent.*

(1) *Each $(\star, d_T)$-linked overring T of D has dimension at most n.*
(2) *Each $\widetilde{\star}$-valuation overring of D has dimension at most n.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For (2) $\Rightarrow$ (1), let T be a $(\star, d_T)$-linked overring of D and V be a valuation overring of T. Then, it is easy to see that V is $(\star, d_V)$-linked overring of D. Thus, by [12, Lemma 2.7], V is an $\widetilde{\star}$-valuation overring of D. Hence, $\dim(V) \leq n$. Consequently, $\dim(T) \leq \dim_v(T) \leq n$, as desired. $\square$

When $\star = d_D$, the equivalence of (1) and (3) of the following theorem is due to J. Arnold [2, Theorem 6].

Theorem 4.2. *Let D be an integral domain, and n be an integer. Then, the following statements are equivalent.*

(1) $\widetilde{\star}$-$\dim_v(D) = n$.
(2) $\star[n]$-$\dim(D[n]) = 2n$.
(3) $\star[r]$-$\dim(D[r]) = r + n$, *for all* $r \geq n - 1$.
(4) *Each $(\star, d_T)$-linked overring T of D has dimension at most n, and n is minimal.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [24, Theorem 4.5], and (3) $\Rightarrow$ (2) is trivial. For (1) $\Rightarrow$ (3), suppose that $\widetilde{\star}$-$\dim_v(D) = n$. Then, For all $r \geq n$, we have $\star[r]$-$\dim(D[r]) = \star[r]$-$\dim_v(D[r]) = r + \widetilde{\star}$-$\dim_v(D) = r + n$, by [24, Corollary 4.7 and Theorem 4.8]. Now, assume that $r = n - 1$. Since $\widetilde{\star}$-$\dim_v(D) = n$, there exists a quasi-$\widetilde{\star}$-prime ideal M of D such that $n = \dim_v(D_M)$, by Lemma 2.5. Thus, by [2, Theorem 6], we have $\dim(D_M[r]) = r + n$. Let $\mathcal{P} \in \mathrm{QSpec}^{\star[r]}(D[r])$ be such that $\star[r]$-$\dim(D[r]) = \mathrm{ht}(\mathcal{P})$. Set $P := \mathcal{P} \cap D$. Then by [24, Remark 2.3], we have $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{(0)\}$. Thus,

$$\begin{aligned} r + n &\leq \star[r]\text{-}\dim(D[r]) = \mathrm{ht}(\mathcal{P}) \\ &= \dim(D[r]_{\mathcal{P}}) = \dim(D_P[r]_{\mathcal{P}D_P[r]}) \\ &\leq \dim(D_P[r]) \leq \dim(D_M[r]) = r + n, \end{aligned}$$

where the first inequality holds, by [24, Theorem 3.1]. Hence, $\star[r]$-$\dim(D[r]) = r + n$ for all $r \geq n - 1$.

The equivalence (1) $\Leftrightarrow$ (4) follows from Lemma 4.1. $\square$

As an immediate consequence, we have the following results.

Corollary 4.3. *For each domain D, we have*

$$\widetilde{\star}\text{-}\dim_v(D) = \sup\{\dim(T) | T \textit{ is } (\star, d_T)\textit{-linked overring of } D\}.$$

One of the famous formulas in the dimension theory of commutative rings is the Arnold's formula [2, Theorem 5], which states as:

$$\dim(D[n]) = n + \sup\{\dim(D[\theta_1, \cdots, \theta_n]) | \{\theta_i\}_1^n \subseteq K\}.$$

Now, we prove the semistar analogue of Arnold's formula.

Lemma 4.4. *Let D be an integral domain and n be an integer. Then,*

$$\star[n]\text{-}\dim(D[n]) = \sup\{\dim(D_M[n]) | M \in \mathrm{QMax}^{\widetilde{\star}}(D)\}.$$

Proof. If P is a quasi-$\widetilde{\star}$-prime ideal of D, and if $QD_P[n]$ is a non-zero prime ideal of $D_P[n](= D[n]_{D\setminus P})$, then $Q \cap D \subseteq P$, and hence $Q \in \mathrm{QSpec}^{\star[n]}(D[n])$, by [24, Remark 2.3]. Thus, the inequality $\geq$ is true. Now, let $Q \in \mathrm{QMax}^{\star[n]}(D[n])$ be such that $\star[n]$-$\dim(D[n]) = \mathrm{ht}(Q)$, and set $P := Q \cap D$. Then, by [24, Remark 2.3], we have $P \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup$

$\{0\}$. Thus,

$$\begin{aligned}\star[n]\text{-}\dim(D[n]) &= \mathrm{ht}(Q) = \dim(D[n]_Q) \\ &= \dim(D_P[n]_{QD_P[n]}) \leq \dim(D_P[n]) \\ &\leq \star[n]\text{-}\dim(D[n]).\end{aligned}$$

Therefore, the proof is complete. □

Corollary 4.5. *Let D be an integral domain and n be an integer. Then, there exist a quasi-$\widetilde{\star}$-maximal ideal M of D and a quasi-$\star[n]$-maximal ideal Q of $D[n]$ such that $M = Q \cap D$ and*

$$\star[n]\text{-}\dim(D[n]) = \mathrm{ht}(Q) = n + \mathrm{ht}(M[n]).$$

Proof. By Lemma 4.4, there exists a quasi-$\widetilde{\star}$-maximal ideal M of D such that $\star[n]\text{-}\dim(D[n]) = \dim(D_M[n])$. Thus, there exists a prime ideal Q of $D[n]$ such that $Q \cap (D\backslash M) = \emptyset$, $\dim(D_M[n]) = \mathrm{ht}(QD_M[n])$ and that $QD_M[n]$ is a maximal ideal of $D_M[n]$. Since $Q \cap D \subseteq M$, we have Q is a quasi-$\star[n]$-prime of $D[n]$, by [24, Remark 2.3], and since $\star[n]\text{-}\dim(D[n]) = \mathrm{ht}(Q)$, we have Q is a quasi-$\star[n]$-maximal ideal of $D[n]$. Set $PD_M := QD_M[n] \cap D_M$, for some $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$. Then, by [3, Corollary 2.9], we have $\mathrm{ht}(QD_M[n]) = n + \mathrm{ht}(PD_M[n])$ and that PD_M is a maximal ideal of D_M. Thus, we have $P = M$ and

$$\star[n]\text{-}\dim(D[n]) = \mathrm{ht}(QD_M[n]) = n + \mathrm{ht}(MD_M[n]) = n + \mathrm{ht}(M[n]),$$

which completes the proof. □

We are now ready to prove the semistar analogue of Arnold's formula.

Theorem 4.6. *Let D be an integral domain and n be a positive integer. Then,*

$$\star[n]\text{-}\dim(D[n]) = n + \sup\{\widetilde{\star}_\iota\text{-}\dim(D[\theta_1, \cdots, \theta_n]) | \{\theta_i\}_1^n \subseteq K\}.$$

where ι is the inclusion map of D in $D[\theta_1, \cdots, \theta_n]$,

Proof. Let $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$ and $\{\theta_i\}_1^n \subseteq K$. Let Q be a maximal ideal of $D_M[\theta_1, \cdots, \theta_n]$ such that $\dim(D_M[\theta_1, \cdots, \theta_n]) = \mathrm{ht}(Q)$. Let Q_0 be a prime ideal of $D[\theta_1, \cdots, \theta_n]$ such that $Q_0 \cap (D\backslash M) = \emptyset$ and $Q = Q_0 D_M[\theta_1, \cdots, \theta_n]$. Thus, Q_0 is a quasi-$\widetilde{\star}_\iota$-prime ideal of $D[\theta_1, \cdots, \theta_n]$, since $Q_0 \cap D \subseteq M$ [24, Remark 2.3]. Hence, we obtain that

$\dim(D_M[\theta_1,\cdots,\theta_n]) = \mathrm{ht}(Q) = \mathrm{ht}(Q_0) \leq \widetilde{\star}_\iota\text{-}\dim(D[\theta_1,\cdots,\theta_n])$. Using Lemma 4.4 and Arnold's formula [2, Theorem 5], we have

$$\star[n]\text{-}\dim(D[n]) = n + \sup\{\dim(D_M[\theta_1,\cdots,\theta_n])\},$$

where the supremum is taken over $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$ and $\{\theta_i\}_1^n \subseteq K$. Thus, $\star[n]\text{-}\dim(D[n]) \leq n + \sup\{\widetilde{\star}_\iota\text{-}\dim(D[\theta_1,\cdots,\theta_n]) | \{\theta_i\}_1^n \subseteq K\}$. Now, choose $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$ and $\{\theta_i\}_1^n \subseteq K$ such that $\star[n]\text{-}\dim(D[n]) = n + \dim(D_M[\theta_1,\cdots,\theta_n])$. Let Q' be a quasi-$\widetilde{\star}_\iota$-prime ideal of $D[\theta_1,\cdots,\theta_n]$ such that $\widetilde{\star}_\iota\text{-}\dim(D[\theta_1,\cdots,\theta_n]) = \mathrm{ht}(Q')$ and set $P' := Q' \cap D$. Thus,

$$\begin{aligned}\widetilde{\star}_\iota\text{-}\dim(D[\theta_1,\cdots,\theta_n]) &= \mathrm{ht}(Q') = \dim(D[\theta_1,\cdots,\theta_n]_{Q'}) \\ &= \dim(D_{P'}[\theta_1,\cdots,\theta_n]_{Q'D_{P'}[\theta_1,\cdots,\theta_n]}) \\ &\leq \dim(D_{P'}[\theta_1,\cdots,\theta_n]) \leq \dim(D_M[\theta_1,\cdots,\theta_n]).\end{aligned}$$

Hence, by the first part of the proof we have

$$\dim(D_M[\theta_1,\cdots,\theta_n]) = \widetilde{\star}_\iota\text{-}\dim(D[\theta_1,\cdots,\theta_n]).$$

Thus, we have $\star[n]\text{-}\dim(D[n]) = n + \widetilde{\star}_\iota\text{-}\dim(D[\theta_1,\cdots,\theta_n])$ to complete the proof. □

Acknowledgments

The author was supported in part by a grant from IPM (No. 88130034).

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