Bulletin of the Iranian Mathematical Society Vol. 37 *No. 3* (*2011*), *pp 235-248.*

A SHARP MAXIMAL FUNCTION ESTIMATE FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATOR

Z. XIAOSHA AND L. LANZHE*

Communicated by Mohammad Sal Moslehian

Abstract. We establish a sharp maximal function estimate for some vector-valued multilinear singular integral operators. As an application, we obtain the (L^p, L^q)-norm inequality for vector-valued multilinear operators.

1. Introduction and Results

We study the following singular integral operators.

Fix $\varepsilon > 0$ and $0 \leq \delta < n$. Let $T : S \to S'$ be a linear operator and let there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x, y) f(y) dy,$$

for every bounded and compactly supported function f, where K satisfies the Calderón-Zygmund type estimates:

$$|K(x, y)| \leq C|x - y|^{-n+\delta}$$

MSC(2000): Primary: 42B20; Secondary: 42B25.
Keywords: Vector-valued multilinear operator, singular integral operator, sharp maximal function estimate, BMO.
Received: 10 August 2009, Accepted: 6 March 2010.
∗Corresponding author

and

$$|K(y,x)-K(z,x)|+|K(x,y)-K(x,z)|\le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta},$$

if $2|y-z|\le|x-z|$. Let m_j be the positive integers ($j=1,\cdots,l$), $m_1+\cdots+m_l=m$ and A_j be the functions on R^n ($j=1,\cdots,l$). For $1<r<\infty$, the operator associated with T is defined by

$$|T_A(f)(x)|_r=\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)|^r\right)^{1/r},$$

where,

$$T_A(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy,$$

$$R_{m_j+1}(A_j;x,y)=A_j(x)-\sum_{|\alpha|\le m_j}\frac{1}{\alpha!}D^{\alpha}A_j(y)(x-y)^{\alpha}$$

is the the $(m+1)$-th Taylor remainder of A and $\alpha=(\alpha_1,\cdots,\alpha_n)$ denotes any n-tuple index, $x^{\alpha}=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $D^{\alpha}=\partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$. We also use the notation:

$$|T(f)(x)|_r=\left(\sum_{i=1}^{\infty}|T(f_i)(x)|^r\right)^{1/r}\quad\text{and}\quad|f|_r=\left(\sum_{i=1}^{\infty}|f_i(x)|^r\right)^{1/r}.$$

Suppose that $|T|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, for $1<p<n/\delta$ and $1/q=1/p-\delta/n$.

Note that when $m=0$, T_A is just the vector-valued multilinear commutator of T and A (see [13]), while when $m>0$, T_A is non-trivial generalizations of the commutator.

It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). Hu and Yang (see [9]) proved a variant sharp estimate for the multilinear singular integral operators. In [12], Pérez and Trujillo-Gonzalez proved a sharp estimate for the multilinear commutator (see also [10, 11, 13]). Our main purpose of this paper is to prove a sharp maximal function inequality for the vector-valued multilinear singular integral operators when $D^{\alpha}A_j\in BMO(R^n)$, for all α with $|\alpha|=m_j$. As an application, we obtain the (L^p,L^q)-norm inequality for the vector-valued multilinear operators in Section 2.

First, we introduce some notation. Throughout this paper, Q will denote a cube in R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [8, 14])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy,$$

where, $\approx$ means the equivalency up to multiplication by finite positive constants. We say that f belongs to $BMO(R^n)$, if $f^{\#}$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$

We write $M_p(f) = (M(f^p))^{1/p}$, for $0 < p < \infty$. For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

We shall prove the following results.

Theorem 1.1. *Let $1 < r < \infty$, $D^\alpha A_j \in BMO(R^n)$, for all α with $|\alpha| = m_j$, $j = 1, \cdots, l$. Then, there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(R^n)$, $1 < s < n/\delta$ and $\tilde{x} \in R^n$, we have*

$$(|T_A(f)|_r)^{\#}(\tilde{x}) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Corollary 1.2. *Let $1 < r < \infty$, $D^\alpha A_j \in BMO(R^n)$, for all α with $|\alpha| = m_j$, $j = 1, \cdots, l$. Then, $|T_A|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, for any $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, that is,*

$$|||T_A(f)|_r||_{L^q} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) |||f|_r||_{L^p},$$

2. **Proof of Theorem**

To prove the theorem, we need the following lemmas.

Lemma 2.1. *([4]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.2. *([1, 7]) Suppose that $1 < r < \infty$, $1 \leq s < p < n/\delta$ and $1/q = 1/p - \delta$. Then*

$$||M_{\delta,s}(|f|_r)||_{L^q} \leq C|||f|_r||_{L^p}.$$

Proof of Theorem 1.1. It suffices to prove that for $f = \{f_i\} \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q ||T_A(f)(x)|_r - C_0|dx \leq C \prod_{j=1}^{l} \Big(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{BMO} \Big) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Without loss of generality, we assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, where $f_Q = |Q|^{-1}\int_Q f(x)dx$, then $R_{m_j+1}(A_j; x, y) = R_{m_j+1}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$ (see [4]). We split $f =$

$g+h=\{g_i\}+\{h_i\}$ for $g_i=f_i\chi_{\tilde{Q}}$ and $h_i=f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$T_A(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy$$
$$=\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)h_i(y)dy$$
$$+\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy$$
$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy$$
$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy$$
$$+\sum_{\substack{|\alpha_1|=m_1,\\|\alpha_2|=m_2}}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy,$$

then, by Minkowski's inequality, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left||T_A(f)(x)|_r-|T_{\tilde{A}}(h)(x_0)|_r\right|dx\\
\leq\ &\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}|T_A(f_i)(x)-T_{\tilde{A}}(h_i)(x_0)|^r\right)^{1/r}dx\\
\leq\ &\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy\right|^r\right)^{1/r}dx\\
+\ &\frac{C}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}\right.\right.\\
&\left.\left.\times\ D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)dy\right|^r\right)^{1/r}dx
\end{aligned}$$

$$\begin{aligned}
&+ \quad \frac{C}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}\right.\right.\\
&\quad \times \left.\left. D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)dy\right|^r\right)^{1/r} dx\\
&+ \quad \frac{C}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|\sum_{\substack{|\alpha_1|=m_1,\\ |\alpha_2|=m_2}}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}\right.\right.\\
&\quad \times K(x,y)g_i(y)dy|^r)^{1/r} dx\\
&+ \quad \frac{1}{|Q|}\int_Q \left(\sum_{i=1}^{\infty}\left|T_{\tilde{A}}(h_i)(x)-T_{\tilde{A}}(h_i)(x_0)\right|^r\right)^{1/r} dx\\
&:= \quad I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_m(\tilde{A}_j;x,y) \le C|x-y|^m \sum_{|\alpha_j|=m} ||D^{\alpha_j}A_j||_{BMO},$$

thus, by the (L^s, L^q)-boundedness of $|T|_r$ with $1 < s < n/\delta$ and $1/q = 1/s - \delta/n$, we obtain

$$\begin{aligned}
I_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\frac{1}{|Q|}\int_Q |T(g)(x)|_r dx\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q |T(g)(x)|_r^q dx\right)^{1/q}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)|Q|^{-1/q}\left(\int_{\tilde{Q}} |f(x)|_r^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{BMO}\right)M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_2, set $s = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/t = 1/p - \delta/n$, we get, by Hölder's inequality

$$\begin{aligned}
I_2 &\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{A}_1 g)(x)|_r dx \\
&\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 g)(x)|_r^t dx\right)^{1/t} \\
&\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|^{1/t}}\left(\int_{R^n} (|D^{\alpha_1}\tilde{A}_1(x)||g(x)|_r)^p dx\right)^{1/p} \\
&\le C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{pq'} dx\right)^{1/pq'} \\
&\quad \times \left(\frac{1}{|Q|^{1-s\delta/n}}\int_{\tilde{Q}} |f(x)|_r^{pq} dx\right)^{1/pq} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^{\alpha}A_j||_{BMO}\right) M_{\delta,s}(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^{\alpha}A_j||_{BMO}\right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Similarly, for I_4, set $s = pq_3$ for $1 < p < n/\delta$,
$q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/t = 1/p - \delta/n$, we obtain

$$\begin{aligned}
I_4 &\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r dx \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r^t dx\right)^{1/t} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/t}\left(\int_{R^n} (|D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x)||g(x)|_r)^p dx\right)^{1/p} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{pq_1} dx\right)^{1/pq_1}
\end{aligned}$$

$$\times \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{pq_2}dx\right)^{1/pq_2} \left(\frac{1}{|Q|^{1-s\delta/n}}\int_{\tilde{Q}} |f(x)|_r^{pq_3}dx\right)^{1/pq_3}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO}\right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

For I_5, we write

$$T_{\tilde{A}}(h_i)(x) - T_{\tilde{A}}(h_i)(x_0)$$

$$= \int_{R^n}\left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m}\right)\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y)h_i(y)dy$$

$$+\int_{R^n}\Big(R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y)\Big)\frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy$$

$$+\int_{R^n}\Big(R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)\Big)\frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m}K(x_0,y)h_i(y)dy$$

$$- \sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)\right.$$

$$\left. -\frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m}K(x_0,y)\right] D^{\alpha_1}\tilde{A}_1(y)h_i(y)dy$$

$$- \sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y)\right.$$

$$\left. -\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m}K(x_0,y)\right] D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$

$$+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)\right.$$

$$\left. -\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right] D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)dy$$

$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 1 and the following inequality (see [14]):

$$|b_{Q_1} - b_{Q_2}| \leq C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$|R_m(\tilde{A};x,y)| \leq C|x-y|^m \sum_{|\alpha|=m}(||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|)$$
$$\leq Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on K,

$$|I_5^{(1)}| \leq C \int_{R^n} \left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}} \right)$$
$$\times \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|dy$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right)$$
$$\times \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} \right) |f_i(y)|dy$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}$$
$$\int_{2^k\tilde{Q}} |f_i(y)|dy,$$

thus, by Minkowski's inequality, we have

$$\left(\sum_{i=1}^{\infty} \left| I_5^{(1)} \right|^r \right)^{1/r}$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2(2^{-k} + 2^{-\varepsilon k}) \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}$$
$$\int_{2^k\tilde{Q}} |f(y)|_r dy$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,1}(|f|_r)(\tilde{x}).$$

For $I_5^{(2)}$, by the formula (see [4]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)| \le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|} ||D^\alpha A||_{BMO},$$

thus

$$\begin{aligned}&\left(\sum_{i=1}^\infty\left|I_5^{(2)}\right|^r\right)^{1/r}\\ &\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}|f(y)|_r dy\\ &\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,1}(|f|_r)(\tilde{x}).\end{aligned}$$

Similarly,

$$\left(\sum_{i=1}^\infty\left|I_5^{(3)}\right|^r\right)^{1/r}\le C\prod_{j=1}^2\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)M_{\delta,1}(|f|_r)(\tilde{x}).$$

For $I_5^{(4)}$, we get

$$\begin{aligned}&\left(\sum_{i=1}^\infty\left|I_5^{(4)}\right|^r\right)^{1/r}\\ &\le C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\ &\quad\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy\\ &+C\sum_{|\alpha_1|=m_1}\int_{R^n\setminus\tilde{Q}}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\ &\quad\times\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r dy\end{aligned}$$

$$\leq C \sum_{|\alpha|=m_2} ||D^\alpha A_2||_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k})$$
$$\times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{s'} dy \right)^{1/s'} \left(\frac{1}{|2^k \tilde{Q}|^{1-s\delta/n}} \int_{2^k \tilde{Q}} |f(y)|_r^s dy \right)^{1/s}$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty} \left| I_5^{(5)} \right|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

For $I_5^{(6)}$, taking $q_1, q_2 > 1$ such that $1/s + 1/q_1 + 1/q_2 = 1$, then

$$\left(\sum_{i=1}^{\infty} \left| I_5^{(6)} \right|^r \right)^{1/r}$$
$$\leq C \sum_{\substack{|\alpha_1|=m_1 \\ |\alpha_2|=m_2}} \int_{R^n \setminus \tilde{Q}} \left| \frac{(x-y)^{\alpha_1+\alpha_2} K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2} K(x_0,y)}{|x_0-y|^m} \right|$$
$$\times |D^{\alpha_1} \tilde{A}_1(y)||D^{\alpha_2} \tilde{A}_2(y)||f(y)|_r dy$$
$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-\varepsilon k}) \left(\frac{1}{|2^k \tilde{Q}|^{1-s\delta/n}} \int_{2^k \tilde{Q}} |f(y)|_r^s dy \right)^{1/s}$$
$$\times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{q_1} dy \right)^{1/q_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{A}_2(y)|^{q_2} dy \right)^{1/q_2}$$
$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Thus

$$|I_5| \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

This completes the proof of Theorem.

Proof of Corollary 1.2. We choose $1 < s < p$ in Theorem and by using Lemma 2, we get

$$\begin{aligned} ||T_A(f)||_{L^q} &\leq C||(T_A(f))^{\#}||_{L^q} \\ &\leq C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)||M_{\delta,s}(|f|_r)||_{L^q} \\ &\leq C\prod_{j=1}^{l}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)|||f|_r||_{L^p}. \end{aligned}$$

This finishes the proof.

3. Applications

In this section we apply Theorem 1.1 and Corollary 1.2 to the Calderón-Zygmund singular integral operator and fractional integral operator.

Application 1. Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [8, 14]). The operator related to T is defined by

$$|T_A(f)(x)|_r = \left(\sum_{i=1}^{\infty}|T_A(f_i)(x)|^r\right)^{1/r},$$

where,

$$T_A(f_i)(x) = \int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy.$$

Then, the theorem and corollary in the paper hold for the operator.

Application 2. Fractional integral operator with rough kernel.

For $0 < \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by (see [6, 9])

$$T^\delta f(x) = \int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n-\delta}}f(y)dy.$$

The operator related to T_δ is defined by

$$|T_A^\delta(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T_A^\delta(f_i)(x)|^r\right)^{1/r},$$

where,

$$T_A^\delta(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^{m+n-\delta}}\Omega(x-y)f_i(y)dy,$$

Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}}\Omega(x')d\sigma(x') = 0$ and $\Omega \in Lip_\varepsilon(S^{n-1})$, for some $0 < \varepsilon \leq 1$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x-y|^\varepsilon$. Then, the theorem and corollary hold for the operator. When $\Omega \equiv 1$, T is the Riesz potentials.

Acknowledgments

The authors express their deep gratitude to the referee for his/her comments and suggestions. This research is Supported by Scientific Research Fund of Hunan Provincial Education Department 09C057.

References

[1] S. Chanillo, A note on commutators, *Indiana Univ. Math. J.* **31** (1982) 7-16.

[2] J. Cohen, A sharp estimate for a multilinear singular integral in R^n, *Indiana Univ. Math. J.* **30** (1981) 693-702.

[3] J. Cohen and J. Gosselin, On multilinear singular integrals on R^n, *Studia Math.* **72** (1982) 199-223.

[4] J. Cohen and J. Gosselin, A BMO estimate for multilinear singular integrals, *Illinois J. Math.* **30** (1986) 445-464.

[5] R. Coifman and Y. Meyer, Wavelets, *Calderón-Zygmund and Multilinear Operators*, Cambridge Studies in Advanced Mathematics, 48, Cambridge University Press, Cambridge, 1997.

[6] Y. Ding and S. Z. Lu, Weighted boundedness for a class rough multilinear operators, *Acta Math. Sin. (Engl. Ser.)* **17** (2001) 517-526.

[7] C. Fefferman and E. M. Stein, Some maximal inequalities, *Amer. J. Math.* **93** (1971) 107-115.

[8] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies, 116, North-Holland Publishing Co., Amsterdam, 1985.

[9] G. E. Hu and D. C. Yang, A variant sharp estimate for multilinear singular integral operators, *Studia Math.* **141** (2000) 25-42.

[10] C. Pérez, Endpoint estimates for commutators of singular integral operators, *J. Funct. Anal.* **128** (1995) 163-185.

[11] C. Pérez and G. Pradolini, Sharp weighted endpoint estimates for commutators of singular integrals, *Michigan Math. J.* **49** (2001) 23-37.

[12] C. Pérez and R. Trujillo-González, Sharp weighted estimates for multilinear commutators, *J. London Math. Soc. (2)* **65** (2002) 672-692.

[13] C. Pérez and R. Trujillo-González, Sharp weighted estimates for vector-valued singular integral operators and commutators, *Tohoku Math. J. (2)* **55** (2003) 109-129.

[14] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton NJ, 1993.

Zhou Xiaosha and Liu Lanzhe
Department of Mathematics Changsha University of Science and Technology, P.O. Box Changsha 410077,
P. R. of China.
Email: zhouxiaosha57@126.com, lanzheliu@163.com