A SHARP MAXIMAL FUNCTION ESTIMATE FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATOR

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ABSTRACT. We establish a sharp maximal function estimate for some vector-valued multilinear singular integral operators. As an application, we obtain the \((L^p, L^q)\)-norm inequality for vector-valued multilinear operators.

1. Introduction and Results

We study the following singular integral operators.

Fix \(\varepsilon > 0\) and \(0 \leq \delta < n\). Let \(T: S \rightarrow S'\) be a linear operator and let there exists a locally integrable function \(K(x, y)\) on \(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}\) such that

\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy,
\]

for every bounded and compactly supported function \(f\), where \(K\) satisfies the Calderón-Zygmund type estimates:

\[
|K(x, y)| \leq C|x - y|^{-n+\delta}
\]


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and

$$|K(y,z) - K(z,x)| + |K(x,y) - K(x,z)| \leq C|y - z|^\varepsilon|x - z|^{-n-\varepsilon+\delta},$$

if $2|y - z| \leq |x - z|$. Let $m_j$ be the positive integers $(j = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and $A_j$ be the functions on $R^n$ $(j = 1, \ldots, l)$. For $1 < r < \infty$, the operator associated with $T$ is defined by

$$|T(f)(x)|_r = \left( \sum_{i=1}^{\infty} |T(f_i)(x)|^r \right)^{1/r},$$

where,

$$T_A(f_i)(x) = \int_{R^n} \prod_{j=1}^{l} \frac{R_{m_j+1}(A_j; x, y)}{|x - y|^m} K(x,y)f_i(y)dy,$$

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x - y)^\alpha$$

is the the $(m+1)$-th Taylor remainder of $A$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes any $n$-tuple index, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $D^\alpha = \partial^{|\alpha|+\cdots+n}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. We also use the notation:

$$|T(f)(x)|_r = \left( \sum_{i=1}^{\infty} |T(f_i)(x)|^r \right)^{1/r} \text{ and } |f|_r = \left( \sum_{i=1}^{\infty} |f_i(x)|^r \right)^{1/r}.$$

Suppose that $|T|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, for $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$.

Note that when $m = 0$, $T_A$ is just the vector-valued multilinear commutator of $T$ and $A$ (see [13]), while when $m > 0$, $T_A$ is non-trivial generalizations of the commutator.

It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [2-6]). Hu and Yang (see [9]) proved a variant sharp estimate for the multilinear singular integral operators. In [12], P´erez and Trujillo-Gonzalez proved a sharp estimate for the multilinear commutator (see also [10, 11, 13]).

Our main purpose of this paper is to prove a sharp maximal function inequality for the vector-valued multilinear singular integral operators when $D^\alpha A_j \in BMO(R^n)$, for all $\alpha$ with $|\alpha| = m_j$. As an application, we obtain the $(L^p, L^q)$-norm inequality for the vector-valued multilinear operators in Section 2.
First, we introduce some notation. Throughout this paper, $Q$ will denote a cube in $R^n$ with sides parallel to the axes. For any locally integrable function $f$, the sharp function of $f$ is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [8, 14])

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy,$$

where, $\approx$ means the equivalency up to multiplication by finite positive constants. We say that $f$ belongs to $BMO(R^n)$, if $f^\#$ belongs to $L^\infty(R^n)$ and $|||f^\#|||_{BMO} = ||f^\#||_{L^\infty}$. Let $M$ be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy.$$

We write $M_p(f) = (M(f^p))^{1/p}$, for $0 < p < \infty$. For $1 \leq p < \infty$ and $0 \leq \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1-\rho\delta/n}} \int_Q |f(y)|^p dy \right)^{1/p}.$$

We shall prove the following results.

**Theorem 1.1.** Let $1 < r < \infty$, $D^\alpha A_j \in BMO(R^n)$, for all $\alpha$ with $|\alpha| = m_j$, $j = 1, \cdots, l$. Then, there exists a constant $C > 0$ such that for any $f = \{f_i\} \in C_0^\infty(R^n)$, $1 < s < n/\delta$ and $\tilde{x} \in R^n$, we have

$$(|T_A(f)|_r)^\#(\tilde{x}) \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} |||D^\alpha_j A_j|||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

**Corollary 1.2.** Let $1 < r < \infty$, $D^\alpha A_j \in BMO(R^n)$, for all $\alpha$ with $|\alpha| = m_j$, $j = 1, \cdots, l$. Then, $|T_A|_r$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, for any $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, that is,

$|||T_A(f)|_r||_{L^q} \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} |||D^\alpha_j A_j|||_{BMO} \right) |||f|_r||_{L^p},$
2. Proof of Theorem

To prove the theorem, we need the following lemmas.

**Lemma 2.1.** ([4]) Let \( A \) be a function on \( \mathbb{R}^n \) and \( D^\alpha A \in L^q(\mathbb{R}^n) \) for all \( \alpha \) with \( |\alpha| = m \) and some \( q > n \). Then

\[
|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} \left( \frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},
\]

where \( \tilde{Q} \) is the cube centered at \( x \) and having side length \( 5\sqrt{n}|x - y| \).

**Lemma 2.2.** ([1, 7]) Suppose that \( 1 < r < \infty \), \( 1 \leq s < p < n/\delta \) and \( 1/q = 1/p - \delta \). Then

\[
|||M_{\delta, s}(|f|_r)|||_{L^q} \leq C|||f|||_{L^p}.
\]

**Proof of Theorem 1.1.** It suffices to prove that for \( f = \{f_i\} \in C_0^\infty(\mathbb{R}^n) \) and some constant \( C_0 \), the following inequality holds:

\[
\frac{1}{|Q|} \int_Q ||T_A(f)(x)||_r - C_0|dx \leq C \prod_{j=1}^l \left( \sum_{|\alpha_j| = m_j} ||D^\alpha_j A_j||_{BMO} \right) M_{\delta, s}(|f|_r)(\tilde{x}).
\]

Without loss of generality, we assume \( l = 2 \). Fix a cube \( Q = Q(x_0, d) \) and \( \tilde{x} \in Q \). Let \( \tilde{Q} = 5\sqrt{n}Q \) and \( \tilde{A}_j(x) = A_j(x) - \sum_{|\alpha| = m_j} \frac{1}{\alpha!} (D^\alpha A_j) \hat{Q} x^\alpha \), where \( f_Q = |Q|^{-1} \int_Q f(x) dx \), then \( R_{m_j+1}(A_j; x, y) = R_{m_j+1}(\tilde{A}_j; x, y) \) and \( D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j) \hat{Q} \) for \( |\alpha| = m_j \) (see [4]). We split \( f =
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\[ g + h = \{ g_i \} + \{ h_i \} \] for \( g_i = f_i \chi_{\tilde{Q}} \) and \( h_i = f_i \chi_{\mathbb{R}^n \setminus \tilde{Q}} \). Write

\[
T_A(f_i)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}^{m_j+1}(\tilde{A}_j; x, y) K(x, y) f_i(y) dy
\]

\[
= \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}^{m_j+1}(\tilde{A}_j; x, y) K(x, y) h_i(y) dy
\]

\[
+ \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}^{m_j}(\tilde{A}_j; x, y) K(x, y) g_i(y) dy
\]

\[
- \sum_{|\alpha_1| = m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} R_{m_2}^{m_2}(\tilde{A}_2; x, y) (x - y)^{\alpha_1} D^{\alpha_1} \tilde{A}_1(y) K(x, y) g_i(y) dy
\]

\[
- \sum_{|\alpha_2| = m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} R_{m_1}^{m_1}(\tilde{A}_1; x, y) (x - y)^{\alpha_2} D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) dy
\]

\[
+ \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} (x - y)^{\alpha_1 + \alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) dy,
\]

then, by Minkowski’s inequality, we have

\[
\frac{1}{|Q|} \int_Q \left| |T_A(f)(x)|_r - |T_A(h)(x_0)|_r \right| dx
\]

\[
\leq \frac{1}{|Q|} \int_Q \left( \sum_{i=1}^\infty \left| T_A(f_i)(x) - T_A(h_i)(x_0) \right|^r \right)^{1/r} dx
\]

\[
\leq \frac{1}{|Q|} \int_Q \left( \sum_{i=1}^\infty \left| \int_{\mathbb{R}^n} \prod_{j=1}^2 R_{m_j}^{m_j+1}(\tilde{A}_j; x, y) K(x, y) g_i(y) dy \right|^r \right)^{1/r} dx
\]

\[
+ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^\infty \sum_{|\alpha_1| = m_1} \int_{\mathbb{R}^n} \frac{R_{m_2}^{m_2}(\tilde{A}_2; x, y)(x - y)^{\alpha_1}}{|x - y|^m} \right)^{1/r} dx
\]

\[
\times D^{\alpha_1} \tilde{A}_1(y) K(x, y) g_i(y) dy \right)^{1/r} dx
\]
\[
+ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left| \sum_{|\alpha_2|=m_2} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} \right| \right)^{1/r} dx
\]
\[
\times D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) dy \right)^{1/r} dx
\]
\[
+ \frac{C}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left| \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} \right| \right)^{1/r} dx
\]
\[
\times K(x, y) g_i(y) dy \right)^{1/r} dx
\]
\[
+ \frac{1}{|Q|} \int_Q \left( \sum_{i=1}^{\infty} \left| T_{\tilde{A}}(h_i)(x) - T_{\tilde{A}}(h_i)(x_0) \right| \right)^{1/r} dx
\]
\[
:= I_1 + I_2 + I_3 + I_4 + I_5.
\]

Now, let us estimate \(I_1, I_2, I_3, I_4\) and \(I_5\), respectively. First, for \(x \in Q\) and \(y \in \tilde{Q}\), by Lemma 1, we get

\[
R_m(\tilde{A}; x, y) \leq C|x-y|^m \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO},
\]

thus, by the \((L^s, L^q)\)-boundedness of \(|T|_r\) with \(1 < s < n/\delta\) and \(1/q = 1/s - \delta/n\), we obtain

\[
I_1 \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |T(g)(x)|_r dx
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left( \frac{1}{|Q|} \int_Q |T(g)(x)|_q^q dx \right)^{1/q}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) |Q|^{-1/q} \left( \int_Q |f(x)|_s^s dx \right)^{1/s}
\]
\[
\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\]
For $I_2$, set $s = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/t = 1/p - \delta/n$, we get, by Hölder’s inequality

$$I_2 \leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1| = m_1} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{A}_1 g)(x)|_r \, dx$$

$$\leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1| = m_1} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 g)(x)|_r^t \, dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1| = m_1} \frac{1}{|Q|^{1/t}} \left( \int_{R^n} (|D^{\alpha_1} \tilde{A}_1(x)||g(x)|_r)^p \, dx \right)^{1/p}$$

$$\leq C \sum_{|\alpha_2| = m_2} ||D^{\alpha_2} A_2||_{BMO} \sum_{|\alpha_1| = m_1} \left( \frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|_{pq'}^r \, dx \right)^{1/pq'}$$

$$\times \left( \frac{1}{|Q|^{1-s\delta/n}} \int_{\tilde{Q}} |f(x)|_{pq} \, dx \right)^{1/pq}$$

$$\leq C \prod_{j=1}^2 \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} M_{\delta,s}(|f|_r)(\tilde{x}).$$

For $I_3$, similar to the proof of $I_2$, we get

$$I_3 \leq C \prod_{j=1}^2 \left( \sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).$$

Similarly, for $I_4$, set $s = pq_3$ for $1 < p < n/\delta$, $q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/t = 1/p - \delta/n$, we obtain

$$I_4 \leq C \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r \, dx$$

$$\leq C \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \left( \frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r^t \, dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \left|Q\right|^{-1/t} \left( \int_{R^n} (|D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x)||g(x)|_r)^p \, dx \right)^{1/p}$$

$$\leq C \sum_{|\alpha_1| = m_1, |\alpha_2| = m_2} \left( \frac{1}{|Q|} \int_{\tilde{Q}} \left|D^{\alpha_1} \tilde{A}_1(x)\right|_{pq_1} \, dx \right)^{1/pq_1}.$$
\begin{align*}
&\times \left( \frac{1}{|Q|} \int_Q |D^{\alpha_2} \tilde{A}_2(x)|^{p q_2} \, dx \right)^{1/p q_2} \left( \frac{1}{|Q|^{1-\delta/n}} \int_Q |f(x)|^{p q_3} \, dx \right)^{1/p q_3} \\
&\leq C \prod_{j=1}^2 \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|)(\tilde{x}).
\end{align*}

For $I_5$, we write
\begin{align*}
&T_\delta(h_i)(x) - T_\delta(h_i)(x_0) \\
&= \int_{R^n} \left( K(x,y) - \frac{K(x_0,y)}{|x_0-y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)h_i(y) \, dy \\
&+ \int_{R^n} \left( R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x_0,y) \right) \frac{R_{m_2}(\tilde{A}_2;x,y)}{|x_0-y|^m} K(x_0,y)h_i(y) \, dy \\
&+ \int_{R^n} \left( R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y) \right) \frac{R_{m_1}(\tilde{A}_1;x_0,y)}{|x_0-y|^m} K(x_0,y)h_i(y) \, dy \\
&- \sum_{|\alpha|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[ \frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y) \\
&- \frac{R_{m_2}(\tilde{A}_2;x_0,y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0,y) \right] D^\alpha_1 \tilde{A}_1(y)h_i(y) \, dy \\
&- \sum_{|\alpha|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[ \frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) \\
&- \frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] D^\alpha_2 \tilde{A}_2(y)h_i(y) \, dy \\
&+ \sum_{|\alpha|=m_1, |\beta|=m_2} \frac{1}{\alpha_1! \beta_2!} \int_{R^n} \left[ \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y) \\
&- \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] D^\alpha_1 \tilde{A}_1(y)D^\beta_2 \tilde{A}_2(y)h_i(y) \, dy \\
&= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
\end{align*}

By Lemma 1 and the following inequality (see [14]):
\begin{align*}
|b_{Q_2} - b_{Q_2}| &\leq C \log(|Q_2|/|Q_1|)\|b\|_{BMO} \text{ for } Q_1 \subset Q_2,
\end{align*}
we know, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$|R_m(\tilde{A}; x, y)| \leq C|x - y|^m \sum_{|\alpha| = m} (||D^\alpha A||_{BMO} + |(D^\alpha A)\tilde{Q}(x,y) - (D^\alpha A)\tilde{Q}|)$$

$$\leq Ck|x - y|^m \sum_{|\alpha| = m} ||D^\alpha A||_{BMO}.$$

Note that $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on $K$,

$$|I_5^{(1)}| \leq C \int_{R^n} \left( \frac{|x - x_0|}{|x_0 - y|^{m + n + 1 - \delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{m + n + \epsilon - \delta}} \right)$$

$$\times \prod_{j=1}^2 R_{m_j} (\tilde{A}_j; x, y) |h_i(y)| dy$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right)$$

$$\times \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} k^2 \left( \frac{|x - x_0|}{|x_0 - y|^{n + 1 - \delta}} + \frac{|x - x_0|^\epsilon}{|x_0 - y|^{n + \epsilon - \delta}} \right) |f_i(y)| dy$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\epsilon k}) \frac{1}{|2^k\tilde{Q}|^{1 - \delta/n}}$$

$$\int_{2^k\tilde{Q}} |f_i(y)| dy,$$

thus, by Minkowski’s inequality, we have

$$\left( \sum_{i=1}^\infty |I_5^{(1)}|^r \right)^{1/r}$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2 (2^{-k} + 2^{-\epsilon k}) \frac{1}{|2^k\tilde{Q}|^{1 - \delta/n}}$$

$$\int_{2^k\tilde{Q}} |f(y)|_r dy$$

$$\leq C \prod_{j=1}^2 \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta,1}(|f|_r)(\bar{x}).$$
For $I_5^{(2)}$, by the formula (see [4]):

$$R_m(\hat{A}; x, y) - R_m(\hat{A}; x_0, y) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \hat{A}; x, x_0)(x - y)^\beta$$

and Lemma 1, we have

$$|R_m(\hat{A}; x, y) - R_m(\hat{A}; x_0, y)| \leq C \sum_{|\beta| < m} \sum_{|\alpha| = m-|\beta|} |x - x_0|^{m-|\beta|}|x - y|^{|\beta|}||D^\alpha A||_{BMO},$$

thus

$$\left(\sum_{i=1}^{\infty} |I_5^{(2)}|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) \sum_{k=0}^{\infty} \int_{2^k+1\mathcal{Q}\setminus 2^k\mathcal{Q}} k \frac{|x - x_0|}{|x_0 - y|^{m+1-\delta}} |f(y)|_r dy$$

$$\leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta, 1}(|f|_r)(\hat{x}).$$

Similarly,

$$\left(\sum_{i=1}^{\infty} |I_5^{(3)}|^r \right)^{1/r} \leq C \prod_{j=1}^{2} \left( \sum_{|\alpha| = m_j} ||D^\alpha A_j||_{BMO} \right) M_{\delta, 1}(|f|_r)(\hat{x}).$$

For $I_5^{(4)}$, we get

$$\left(\sum_{i=1}^{\infty} |I_5^{(4)}|^r \right)^{1/r} \leq C \sum_{|\alpha_1| = m_1} \int_{R^n \setminus \mathcal{Q}} \left| \frac{(x - y)^{\alpha_1} K(x, y)}{|x - y|^m} - \frac{(x_0 - y)^{\alpha_1} K(x_0, y)}{|x_0 - y|^m} \right|$$

$$\times |R_{m_2}(\hat{A}_2; x, y)||D^{\alpha_1} \hat{A}_1(y)||h(y)|_r dy$$

$$+ C \sum_{|\alpha_1| = m_1} \int_{R^n \setminus \mathcal{Q}} |R_{m_2}(\hat{A}_2; x, y) - R_{m_2}(\hat{A}_2; x_0, y)|$$

$$\times \frac{|(x_0 - y)^{\alpha_1} K(x_0, y)|}{|x_0 - y|^m} |D^{\alpha_1} \hat{A}_1(y)||h(y)|_r dy.$$
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\[
\leq C \sum_{|\alpha|=m_2} \|D^\alpha A_2\|_{BMO} \sum_{|\alpha|=m_1} \sum_{k=1}^\infty k(2^{-k} + 2^{-\varepsilon k}) \\
\times \left( \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{s'} dy \right)^{1/s'} \left( \frac{1}{|2^k \tilde{Q}|^{1-s\delta/n}} \int_{2^k \tilde{Q}} |f(y)|^s dy \right)^{1/s} \\
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\]

Similarly,

\[
\left( \sum_{i=1}^\infty |I_5^{(5)}|^r \right)^{1/r} \leq C \prod_{j=1}^2 \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\]

For \(I_5^{(6)}\), taking \(q_1, q_2 > 1\) such that \(1/s + 1/q_1 + 1/q_2 = 1\), then

\[
\left( \sum_{i=1}^\infty |I_5^{(6)}|^r \right)^{1/r} \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{\mathbb{R}^n \setminus \tilde{Q}} \frac{|(x-y)^{\alpha_1+\alpha_2} K(x,y) - (x_0-y)^{\alpha_1+\alpha_2} K(x_0,y)|}{|x-y|^m} \frac{1}{|x_0-y|^m} \left| D^{\alpha_1} \tilde{A}_1(y) \right| \left| D^{\alpha_2} \tilde{A}_2(y) \right| |f(y)|_r dy \\
\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^\infty k(2^{-k} + 2^{-\varepsilon k}) \left( \frac{1}{|2^k \tilde{Q}|^{1-s\delta/n}} \int_{2^k \tilde{Q}} |f(y)|^s dy \right)^{1/s} \\
\times \left( \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{q_1} dy \right)^{1/q_1} \left( \frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{A}_2(y)|^{q_2} dy \right)^{1/q_2} \\
\leq C \prod_{j=1}^2 \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\]

Thus

\[
|I_5| \leq C \prod_{j=1}^2 \left( \sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,s}(|f|_r)(\tilde{x}).
\]

This completes the proof of Theorem.
Proof of Corollary 1.2. We choose $1 < s < p$ in Theorem and by using Lemma 2, we get
\[
||T_A(f)||_{L^q} \leq C||\langle T_A(f) \rangle \rangle||_{L^q} \\
\leq C \prod_{j=1}^l \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) ||M_{\delta,s}(|f|_r)||_{L^q} \\
\leq C \prod_{j=1}^l \left( \sum_{|\alpha|=m_j} ||D^\alpha A_j||_{BMO} \right) |||f||_{L^p}.
\]
This finishes the proof.

3. Applications

In this section we apply Theorem 1.1 and Corollary 1.2 to the Calderón-Zygmund singular integral operator and fractional integral operator.

Let $T$ be the Calderón-Zygmund operator (see [8, 14]). The operator related to $T$ is defined by
\[
|T_A(f)(x)|_r = \left( \sum_{i=1}^\infty |T_A(f_i)(x)|^r \right)^{1/r},
\]
where,
\[
T_A(f_i)(x) = \int_{R^n} \prod_{j=1}^l R_{m_j+1}(A_j; x, y) \frac{|y|^m}{|x-y|^m} K(x, y) f_i(y) dy.
\]
Then, the theorem and corollary in the paper hold for the operator.

Application 2. Fractional integral operator with rough kernel.
For $0 < \delta < n$, let $T_{\delta}$ be the fractional integral operator with rough kernel defined by (see [6, 9])
\[
T^\delta f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} f(y) dy.
\]
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The operator related to $T_\delta$ is defined by

$$|T_A^\delta(f)(x)|_r = \left( \sum_{i=1}^{\infty} |T_A^\delta(f_i)(x)|^r \right)^{1/r},$$

where,

$$T_A^\delta(f_i)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{l} R_{m_j+1}(A_j; x, y) |x - y|^{m+n-\delta} \Omega(x - y) f_i(y) dy,$$

$\Omega$ is homogeneous of degree zero on $\mathbb{R}^n$, $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$ and $\Omega \in \text{Lip}_\varepsilon(S^{n-1})$, for some $0 < \varepsilon \leq 1$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M |x - y|^\varepsilon$. Then, the theorem and corollary hold for the operator. When $\Omega \equiv 1$, $T$ is the Riesz potentials.

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References


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